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# SPHERICAL FUNCTIONS ON A SEMISIMPLE LIE GROUP, I.\*

By HARISH-CHANDRA.<sup>1</sup>

**1. Introduction.** Let  $G$  be a connected semisimple Lie group with finite center and  $K$  a maximal compact subgroup of  $G$ . A complex-valued function  $f$  on  $G$  is called a spherical function if  $f(k_1 x k_2) = f(x)$  for  $k_1, k_2 \in K$  and  $x \in G$ . Let  $dk$  denote the normalized Haar measure of  $K$ . A spherical function  $f \neq 0$  is said to be elementary (see Godement [4, p. 497]) if it is continuous and if

$$\int_K f(xky) dk = f(x)f(y)$$

for  $x, y$  in  $G$ . It can be shown that every such function is analytic and, in fact, it is possible to give an alternative characterization of elementary spherical functions as follows. Let  $\mathfrak{S}$  be the algebra of all differential operators on  $G$  which are invariant under left translations by elements of  $G$  and right translations by elements of  $K$ . Then a spherical function  $f$  of class  $C^\infty$  is elementary if and only if  $f(1) = 1$  and  $f$  is an eigenfunction of every differential operator in  $\mathfrak{S}$ . Finally, there exists a simple integral formula (see [5(d), Theorem 5]) for any such function.

It follows from the Plancherel formula for the factor space  $G/K$  (see [5(m), p. 204]) that an "arbitrary" spherical function can be "expanded" in terms of the elementary ones. However the fundamental measure appearing in this formula had, so far, not been satisfactorily related to the group structure of  $G$ . If we agree to ignore certain technical complications, the situation may roughly be described as follows. A certain class of elementary spherical functions (of positive-definite type) can be parameterized by a space  $E/W$ . Here  $E$  is a finite-dimensional real Euclidean space,  $W$  is a finite group of linear transformations in  $E$  and  $E/W$  denotes the quotient space obtained by identifying points in  $E$  which are congruent under  $W$ . Let  $\phi_\lambda$  denote the elementary spherical function which corresponds to a point  $\lambda \in E$  so that  $\phi_{s\lambda} = \phi_\lambda$  ( $s \in W$ ). For any continuous spherical function  $f$  with compact support, put

$$\tilde{f}(\lambda) = \int f(x) \phi_\lambda(x^{-1}) dx$$

\* Received July 16, 1957.

<sup>1</sup> John Simon Guggenheim Fellow.

where  $dx$  is the Haar measure of  $G$ . Then the Plancherel formula asserts<sup>2</sup> the existence of a unique positive measure  $d\mu$  on  $E$  (which is invariant under  $W$ ) such that

$$\int |f(x)|^2 dx = \int |\tilde{f}(\lambda)|^2 d\mu$$

for all such  $f$ . The problem is to determine this measure  $d\mu$ . Let  $d\lambda$  denote the Euclidean measure on  $E$ . In this paper we shall give an asymptotic expansion for  $\phi_\lambda$  on  $G$ . The leading terms of this expansion involve a certain coefficient  $c(\lambda)$ , which, considered as a function of  $\lambda$ , is analytic on  $E$  except on certain hyperplanes (see Lemmas 37 and 52). In any case, the reciprocal  $c^{-1}$  is analytic on  $E$  and it will be shown in another paper that<sup>3</sup>  $d\mu = |c(\lambda)|^{-2} d\lambda$  (if  $dx$  and  $d\lambda$  are suitably normalized). On the other hand the Fourier transform<sup>2</sup> of  $c$  is a distribution on  $E$  which is given by a simple formula in which the group structure of  $G$  enters in a very direct manner (see Theorem 5).

The above outline shows that our problem can be divided into three more or less distinct parts: (1) the asymptotic formula for  $\phi_\lambda$ , (2) the investigation of the function  $c$  and (3) the proof of the relation  $d\mu = |c(\lambda)|^{-2} d\lambda$ . Only the first two of these questions will be taken up in this paper. For (1) we consider the system of differential equations  $D\phi_\lambda = \chi_\lambda(D)\phi_\lambda$  ( $D \in \mathfrak{S}$ ) where  $\chi_\lambda(D)$  is the eigenvalue corresponding to the operator  $D$ . Actually the equation  $\omega\phi_\lambda = \chi_\lambda(\omega)\phi_\lambda$ , corresponding to the Casimir operator  $\omega$ , plays a predominant role in this discussion. In fact, this single equation, together with some general properties of  $\phi_\lambda$ , permits us to derive the asymptotic formula for  $\phi_\lambda$ .

Now in order to obtain more information about  $c$ , we have to investigate  $\phi_\lambda$  as a function of  $\lambda$ , in the neighborhood of infinity on  $G$ . It turns out that one can select a polynomial function  $\pi$  on  $E$  such that  $b = \pi c$  is everywhere analytic on  $E$ . Moreover every derivative of  $b$  is majorized on  $E$  by a suitable polynomial function. We shall see in another paper that  $|b(s\lambda)| = |b(\lambda)|$  for  $s \in W$ .

The contents of this paper are as follows. In Section 2 we collect some elementary facts which are obtained by considering the finite-dimensional representations of  $G$ . Section 3 is devoted to deriving the consequences of two unpublished lemmas of Chevalley. These results, which are of an

<sup>2</sup> This is a simplified version of the true picture and therefore is not entirely accurate.

<sup>3</sup> This is reminiscent of a result of Weyl [8(a), p. 266] on ordinary differential equations.

algebraic nature, will be used constantly during this and the next paper of this series. In Section 4 we define a homomorphism  $\gamma$  of  $\mathfrak{S}$  onto the algebra  $J$  consisting of those polynomial functions on  $E$  which are invariant under  $W$ . Let  $l$  be the rank of symmetric Riemannian space  $G/K$ . Then it is possible to select a connected abelian Lie subgroup  $A_p$  of  $G$  of dimension  $l$  such that  $G = KA_pK$ . Obviously a spherical function  $f$  is completely determined by its restriction  $\bar{f}$  on  $A_p$ . In Sections 5 and 6 we investigate the relationship<sup>4</sup> between  $\overline{Df}$  and  $\bar{f}$  for any  $D \in \mathfrak{S}$ . It turns out that one can define a differential operator  $\delta'(D)$  on an open dense subset  $A_p'$  of  $A_p$  such that  $\overline{Df} = \delta'(D)\bar{f}$  on  $A_p$ . Moreover there is an intimate connection between  $\delta'(D)$  and  $\gamma(D)$  and this permits us to prove (see the corollary of Theorem 2) that if  $w$  is the order of the group  $W$ , there cannot exist more than  $w$  linearly independent analytic functions on any open connected subset of  $A_p'$ , which are all eigenfunctions of  $\delta'(D)$  with the same eigenvalues, for every  $D \in \mathfrak{S}$ . Therefore if we could somehow find  $w$  such functions, corresponding to the eigenvalues  $\chi_\lambda(D)$ , on a connected component  $A_p^+$  of  $A_p'$ ,  $\phi_\lambda$  would be expressible on  $A_p^+$  as a linear combination of these. In order to do this we compute the operator  $\delta'(\omega)$  and, starting from the equation  $\delta'(\omega)\phi = \chi_\lambda(\omega)\phi$ , give a method of constructing the required functions  $\phi_\lambda^{(i)}$  ( $1 \leq i \leq w$ ). The asymptotic behaviour of the  $\phi_\lambda^{(i)}$  is obvious from their construction and so in this way, we get an asymptotic formula for  $\phi_\lambda$  on  $A_p^+$ .

In order to make further progress, it is necessary to make a closer study of the function  $\phi_0$  corresponding to  $\lambda = 0$ . Theorem 3 contains the main result on  $\phi_0$ . It is possible to derive from this certain important consequences (see Lemma 45 and Theorems 4 and 5) and actually to obtain explicit formulae for  $c$  and its Fourier transform. Section 12 is devoted to a deeper study of the function  $c$  and the principal result is given in Lemma 52. In Section 13, we give some explicit calculations for the case  $l = 1$ . These will be required in the next paper of this series. The case when  $G$  is complex is especially simple and therefore it is discussed separately in Section 14. Certain simple lemmas in analysis, which are often needed during this paper, are collected together in the Appendix (§15).

Some of the results of this paper have been announced in a short note [5(n)].

**2. Preliminary lemmas.** Let  $R$  and  $C$  be the fields of real and complex numbers respectively and  $G$  a connected semisimple Lie group and  $\mathfrak{g}_0$  its Lie algebra over  $R$ . Define  $\mathfrak{k}_0$  and  $\mathfrak{p}_0$  as usual (see [5(c), p. 187]) and let  $K$  be

<sup>4</sup> The results of Sections 5 and 6 should be compared with Theorem 1 of [5(k)].

the analytic subgroup of  $G$  corresponding to  $\mathfrak{k}_0$ . Since  $K$  contains the center of  $G$  (see [7]) and since we shall be concerned in this paper primarily with functions on  $G/K$ ,  $G$  can be replaced by any connected group locally isomorphic to it. Let  $G_C$  be a simply connected complex-analytic group corresponding to the complexification  $\mathfrak{g}$  of  $\mathfrak{g}_0$ . Then we may assume that  $G$  is the real analytic subgroup of  $G_C$  which corresponds to  $\mathfrak{g}_0$ . This permits us to identify the finite-dimensional representations of  $G$  with those of  $\mathfrak{g}_0$  and thus also with the complex representations of  $G_C$  and  $\mathfrak{g}$ . If  $\pi$  is such a representation on a vector space  $V$ , one can always introduce the structure of a Hilbert space in  $V$  in such a way that  $\pi(X)$  becomes skew-Hermitian for  $X \in \mathfrak{u} = \mathfrak{k}_0 + (-1)^{\frac{1}{2}}\mathfrak{p}_0$ . We shall always tacitly assume that such a structure has been defined. Also observe that  $K$  is now compact.

Let  $\mathfrak{h}_{\mathfrak{p}_0}$  be a maximal abelian subspace of  $\mathfrak{p}_0$  and  $\mathfrak{h}_0$  a Cartan subalgebra of  $\mathfrak{g}_0$  containing  $\mathfrak{h}_{\mathfrak{p}_0}$ . Then  $\mathfrak{h}_0 = \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{h}_{\mathfrak{k}_0}$  where  $\mathfrak{h}_{\mathfrak{k}_0} = \mathfrak{h}_0 \cap \mathfrak{k}_0$ . Complexify  $\mathfrak{k}_0, \mathfrak{p}_0, \mathfrak{h}_0, \mathfrak{h}_{\mathfrak{p}_0}, \mathfrak{h}_{\mathfrak{k}_0}$  to  $\mathfrak{k}, \mathfrak{p}, \mathfrak{h}, \mathfrak{h}_{\mathfrak{p}}, \mathfrak{h}_{\mathfrak{k}}$  respectively in  $\mathfrak{g}$  and introduce compatible orders (see [4(1), §2]) in the spaces of real-valued linear functions on  $\mathfrak{h}_{\mathfrak{p}_0} + (-1)^{\frac{1}{2}}\mathfrak{h}_{\mathfrak{k}_0}$  and  $\mathfrak{h}_{\mathfrak{p}_0}$ . Let  $P$  denote the set of all positive roots of  $\mathfrak{g}$  (with respect to  $\mathfrak{h}$ ) under this order. Consider the set  $\Sigma$  of all linear functions  $\lambda \neq 0$  on  $\mathfrak{h}_{\mathfrak{p}}$  which are restrictions of some  $\alpha$  in  $P$ . Then every element in  $\Sigma$  is positive under the above order.

LEMMA 1. *Let  $\pi$  be an irreducible finite-dimensional representation of  $\mathfrak{g}$  such that the zero representation of  $\mathfrak{k}$  occurs in the reduction<sup>\*</sup> of  $\pi$  with respect to  $\mathfrak{k}$ . Then if  $\lambda$  is the highest weight of  $\pi$ ,  $\lambda$  is identically zero on  $\mathfrak{h}_{\mathfrak{k}}$ .*

Let  $\psi_0 \neq 0$  be a vector in the representation space  $V$  belonging to the weight  $\lambda$ . For any root  $\alpha$ , define  $X_\alpha$  as usual (see [5(c), p. 188]) and put  $n = \sum_{\alpha \in P_+} CX_\alpha$  where  $P_+$  is the set of those roots  $\alpha \in P$  whose restriction on  $\mathfrak{h}_{\mathfrak{p}}$  is not zero. Let  $A_{\mathfrak{p}}$  and  $N$  be the analytic subgroups of  $G$  corresponding to  $\mathfrak{h}_{\mathfrak{p}_0}$  and  $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}_0$  respectively. Then  $G = KA_{\mathfrak{p}}N$  (see Iwasawa [6]). It is clear that  $\pi(n)\psi_0 = \psi_0$  for  $n \in N$  and therefore the vector space  $C\psi_0$  is invariant under  $\pi(A_{\mathfrak{p}}N)$ . Hence  $V$  is spanned by vectors of the form  $\pi(k)\psi_0$  ( $k \in K$ ). Put  $E = \int_K \pi(k)dk$  where  $dk$  is the normalized Haar measure of  $K$ . Then it follows that  $EV = CE\psi_0$ . Now we can, by hypothesis, select a unit vector  $\psi$  in  $V$  which is invariant under  $K$ . Then  $\psi = E\psi = cE\psi_0$  where  $c \in C$ . But  $\pi(X)E = E\pi(X) = 0$  if  $X \in \mathfrak{k}$  and therefore  $\lambda(H)E\psi_0 = E\pi(H)\psi_0 = 0$  for  $H \in \mathfrak{h}_{\mathfrak{k}}$ . Since  $\psi \neq 0$ , this implies that  $\lambda(H) = 0$ .

\* It follows from Lemmas 2 and 3 of [5(c)] that  $\pi(\mathfrak{k})$  is fully reducible.

It is well known that the exponential mapping is univalent and regular on  $\mathfrak{p}_0$ . Hence  $A_p$  is simply connected and, for any  $x \in G$ , there exists a unique element  $H(x) \in \mathfrak{h}_{\mathfrak{p}_0}$  such that  $x \in K(\exp H(x))N$  (see [6]). Moreover  $x \rightarrow H(x)$  is an analytic mapping of  $G$  into  $\mathfrak{h}_{\mathfrak{p}_0}$  (see [5(c), Lemma 26]).

Let  $\mathfrak{F}_0$  denotes the set of all linear functions  $\lambda$  of  $\mathfrak{h}_p$  with the property that there exists a representation  $\pi$  of  $\mathfrak{g}$  satisfying the conditions of Lemma 1, whose highest weight coincides on  $\mathfrak{h}_p$  with  $\lambda$ .

LEMMA 2. *Let  $\Lambda$  be the highest weight of any irreducible finite-dimensional representation  $\pi$  of  $\mathfrak{g}$  and let  $\lambda$  denote the restriction of  $\Lambda$  on  $\mathfrak{h}_p$ . Then  $2\lambda \in \mathfrak{F}_0$ .*

Select a unit vector  $\psi$  in the representation space  $V$  belonging to the weight  $\Lambda$  and put<sup>6</sup>

$$\phi(x) = \int_K |\pi(xk)\psi|^2 dk \quad (x \in G).$$

Since  $\pi(k)$  is unitary ( $k \in K$ ) and  $\pi(n)\psi = \psi$  ( $n \in N$ ), it is clear that  $|\pi(x)\psi| = e^{\Lambda(H(x))}$ . Hence  $\phi(x) = \int_K e^{2\Lambda(H(xk))} dk$ . Also it follows from its definition that  $\phi$  is spherical, that is,  $\phi(k_1 x k_2) = \phi(x)$  ( $k_1, k_2 \in K; x \in G$ ). Let  $\phi_y$  ( $y \in G$ ) denote the right translate of  $\phi$  by  $y$  so that  $\phi_y(x) = \phi(xy)$  ( $x \in G$ ). Select an orthonormal base  $\psi_0 = \psi, \psi_1, \dots, \psi_r$  for  $V$  and let  $\pi(x)\psi_i = \sum_{0 \leq j \leq r} \psi_j a_{ji}(x)$  ( $0 \leq i \leq r$ ) where  $a_{ji}$  are analytic functions on  $G$ . Then

$$\pi(xk)\psi = \sum_j \psi_j a_j(xk) = \sum_{j,i} \psi_j a_{ji}(x) a_i(k)$$

where  $a_i = a_{i0}$ . Therefore

$$|\pi(xk)\psi|^2 = \sum_j \left| \sum_i a_{ji}(x) a_i(k) \right|^2$$

and<sup>7</sup>

$$\phi(x) = \sum_{0 \leq i, j, m \leq r} a_{ji}(x) (\text{conj } a_{jm}(x)) c_{im} \quad (x \in G)$$

where  $c_{im}$  are certain constants. Let  $U'$  be the linear space spanned over  $C$  by the  $(r+1)^2$  functions  $\sum_{0 \leq j \leq r} a_{ji}(\text{conj } a_{jm})$  ( $0 \leq i, m \leq r$ ). Then it is clear that  $\phi_y \in U'$ . Let  $U$  be the subspace of  $U'$  spanned by all  $\phi_y$  ( $y \in G$ ). We define a representation  $\tau$  of  $G$  on  $U$  as follows. If  $\phi' \in U$ ,  $\tau(y)\phi'$  is the function  $x \rightarrow \phi'(xy)$  on  $G$ . It is clear that  $\tau(k)\phi = \phi_k = \phi$  ( $k \in K$ ). Moreover  $\phi \neq 0$  since  $\phi(1) = 1$  and therefore the trivial representation of  $K$

<sup>6</sup> We denote the scalar product of two elements  $\phi_1, \phi_2$  in  $V$  in the usual way by  $(\phi_1, \phi_2)$ . Similarly  $|\phi_1|$  denotes the norm of  $\phi_1$ .

<sup>7</sup>  $\text{conj } c$  denotes the conjugate of a complex number  $c$ .

occurs in the reduction of  $\tau$ . Also as we shall see below (corollary to Lemma 3),  $\tau$  is irreducible.

Now we may obviously assume that each  $\psi_i$  belongs to a weight  $\Lambda_i$  ( $0 \leq i \leq r$ ,  $\Lambda_0 = \Lambda$ ) of  $\pi$ . Then  $a_{ji}(x \exp H) = a_{ji}(x) e^{\Lambda_i(H)}$  ( $H \in \mathfrak{h}_{p_0}$ ). Moreover the weights being real<sup>a</sup> linear functions,

$$\sum_j a_{ji}(xh) \text{conj } a_{jm}(xh) = \sum_j a_{ji}(x) (\text{conj } a_{jm}(x)) \exp\{\Lambda_i(H) + \Lambda_m(H)\}$$

where  $h = \exp H$  ( $x \in G, H \in \mathfrak{h}_{p_0}$ ). Let  $\mu$  be the highest weight of  $\tau$ . Then we conclude from the above formula that  $\mu$  coincides on  $\mathfrak{h}_{p_0}$  with  $\Lambda_i + \Lambda_m$  for some  $i$  and  $m$ . Since  $\Lambda_i + \Lambda_m \leq 2\Lambda_0 = 2\Lambda$ , it follows from the compatibility of our orders that  $\bar{\mu} \leq 2\lambda$ , where  $\bar{\mu}$  is the restriction of  $\mu$  on  $\mathfrak{h}_p$ . On the other hand<sup>b</sup>

$$\begin{aligned} \phi(x) &= \int_K |\pi(xk)\psi|^2 dk \\ &= \sum_{0 \leq i, j, m \leq r} a_{ji}(x) (\text{conj } a_{jm}(x)) \int_K (\psi_i, \pi(k)\psi) \text{conj}(\psi_m, \pi(k)\psi) dk. \end{aligned}$$

Now<sup>c</sup>  $a_{ji}(\exp H) = \delta_{ji} e^{\Lambda_j(H)}$  ( $H \in \mathfrak{h}_{p_0}$ ). Hence  $\phi(\exp H) = \sum_{0 \leq i \leq r} e^{2\Lambda_i(H)} c_i$  where

$$c_i = \int_K |(\psi_i, \pi(k)\psi)|^2 dk \geq 0.$$

Let  $f_0, f_1, \dots, f_p$  be a base for  $U$  over  $C$  such that each  $f_i$  belongs to a weight  $\mu_i$  of the representation  $\tau$  and  $\mu_0 = \mu$ . Then  $\phi = \sum_{0 \leq i \leq p} b_i f_i$  ( $b_i \in C$ ) and therefore

$$\tau(\exp H)\phi = \sum_{0 \leq i \leq p} b_i e^{\mu_i(H)} f_i \quad (H \in \mathfrak{h}_{p_0}).$$

This shows that  $\phi(\exp H) = \sum_i b_i f_i(1) e^{\mu_i(H)}$ . Let  $\bar{\mu}_i$  denote the restriction of  $\mu_i$  on  $\mathfrak{h}_p$ . Then if we note that

$$c_0 = \int_K |(\psi, \pi(k)\psi)|^2 dk > 0$$

and recall that the exponentials of distinct linear functions on  $\mathfrak{h}_p$  are linearly independent over  $C$  (see [5(b), Lemma 41]), it follows from a comparison of the above two formulas for  $\phi(\exp H)$  that  $2\lambda = \bar{\mu}_i$  for some  $i$ . But  $\mu \geq \mu_i$  and therefore  $\bar{\mu} \geq \bar{\mu}_i = 2\lambda$ . However we have seen above that  $\bar{\mu} \leq 2\lambda$  and so  $\bar{\mu} = 2\lambda$ . This proves Lemma 2.

<sup>a</sup> A linear function on  $\mathfrak{h}$  or  $\mathfrak{h}_p$  is said to be real, if it takes only real values on  $\mathfrak{h}_{p_0} + (-1)^i \mathfrak{h}_{l_0}$  or  $\mathfrak{h}_{p_0}$  respectively.

<sup>b</sup> As usual  $\delta_{i,j} = 1$  or 0 according as  $i = j$  or not. Sometimes it will be convenient to write  $\delta_{i,j}$  instead of  $\delta_{i,j}$ .

Let  $\mathfrak{B}$  be the universal enveloping algebra of  $\mathfrak{g}$ . We regard elements of  $\mathfrak{B}$  as left-invariant differential operators on  $G$  (see [5(g), § 4]). From now on we shall make use of the notational conventions described in footnote 1 of [5(k)]. Define  $n = \sum_{a \in P_+} C X_a$  as before and let  $\mathfrak{S}_p$  be the subalgebra of  $\mathfrak{B}$  generated by  $(1, \mathfrak{h}_p)$ . We denote the centralizer of  $\mathfrak{f}$  in  $\mathfrak{B}$  by  $I_{\mathfrak{g}}$ .

LEMMA 3. *For any  $b \in \mathfrak{B}$ , there exists a unique element  $u_b \in \mathfrak{S}_p$  such that  $b - u_b \in \mathfrak{f}\mathfrak{B} + \mathfrak{B}n$ . The degree of  $u_b$  does not exceed that of  $b$ . Moreover if  $\nu$  is any linear function on  $\mathfrak{h}_p$  and*

$$g(x) = \int_K e^{\nu(H(xk))} dk \quad (x \in G),$$

then  $qg = \chi_\nu(u_q)g$  for  $q \in I_{\mathfrak{g}}$ . Here  $\chi_\nu$  is the homomorphism of  $\mathfrak{S}_p$  into  $C$  such that  $\chi_\nu(1) = 1$  and  $\chi_\nu(H) = \nu(H)$  ( $H \in \mathfrak{h}_p$ ).

In order to prove the first statement, it is sufficient to show that  $\mathfrak{B}$  is the direct sum of  $\mathfrak{S}_p$  and  $\mathfrak{f}\mathfrak{B} + \mathfrak{B}n$ . Let  $\mathfrak{X}$  and  $\mathfrak{N}$  be the subalgebras of  $\mathfrak{B}$  generated by  $(1, \mathfrak{f})$  and  $(1, n)$  respectively. Consider the linear mapping of the tensor product  $\mathfrak{X} \otimes \mathfrak{S}_p \otimes \mathfrak{N}$  into  $\mathfrak{B}$  which maps  $x \otimes h \otimes n$  onto  $xhn$  ( $x \in \mathfrak{X}, h \in \mathfrak{S}_p, n \in \mathfrak{N}$ ). Then our first two assertions follow immediately from Lemma 12 of [5(c)] if we recall that  $\mathfrak{g}$  is the direct sum of  $\mathfrak{f}, \mathfrak{h}_p$  and  $\mathfrak{n}$ .

Now it is obvious from the definition of  $H(x)$  that  $H(xn) = H(x)$  ( $n \in N$ ) and  $^{10} H(xh) = H(x) + \log h$  ( $h \in A_p$ ). Therefore if  $F(x) = e^{\nu(H(x))}$  ( $x \in G$ ), it follows that  $F(x; b) = 0$  for  $b \in \mathfrak{B}n$  and  $F(x; u) = \chi_\nu(u)F(x)$  ( $u \in \mathfrak{S}_p$ ). Now put  $^{11} F(x; k) = F(xk)$  ( $x \in G, k \in K$ ). Then if  $q \in I_{\mathfrak{g}}$ , it is clear that  $F(x; q; k) = F(xk; q)$ . Hence

$$g(x; q) = \int_K F(x; q; k) dk = \int_K F(xk; q) dk.$$

But  $q = b_1 + b_2 + u_q$  where  $b_1 \in \mathfrak{f}\mathfrak{B}$  and  $b_2 \in \mathfrak{B}n$ . Hence

$$F(xk; q) = F(xk; b_1) + \chi_\nu(u_q)F(xk).$$

However if  $b \in \mathfrak{B}$ , it is obvious that  $\int_K F(xkk_1; b) dk$  is actually independent of  $k_1 \in K$ . Therefore  $\int_K F(xk; b') dk = 0$  for any  $b' \in \mathfrak{f}\mathfrak{B}$  and hence

$$g(x; q) = \int_K F(xk; q) dk = \chi_\nu(u_q) \int_K F(xk) dk = \chi_\nu(u_q)g(x).$$

<sup>10</sup>  $\log h$  denotes the unique element  $H \in \mathfrak{h}_{p_0}$  such that  $\exp H = h$ .

<sup>11</sup> Here we follow the mode of writing introduced in [5(k), § 2].

**COROLLARY.** *The representation  $\tau$  defined during the proof of Lemma 2 is irreducible.*

We denote the corresponding representation of  $\mathfrak{B}$  on  $U$  also by  $\tau$ . Then it is clear that  $\tau(b)\phi' = b\phi'$  for any  $\phi' \in U$  and  $b \in \mathfrak{B}$ . Hence  $U$  consists of functions of the form  $b\phi$  ( $b \in \mathfrak{B}$ ). Let  $E$  denote the projection  $\int_K \tau(k)dk$ . Then  $EU = E\tau(\mathfrak{B})\phi$ . However since  $\tau(k)\phi = \phi$  ( $k \in K$ ), it follows that  $Eb\phi = q\phi$  where  $^{12} q = \int_K b^k dk \in I_{\mathfrak{B}}$  ( $b \in \mathfrak{B}$ ). But

$$\phi(x) = \int_K e^{2\Lambda(H(xk))} dk$$

and therefore  $EU = C\phi$  from Lemma 3. Hence the trivial representation of  $K$  occurs exactly once in the reduction of  $U$  under  $\tau(K)$ . Now let  $U_1$  be any subspace of  $U$  which is invariant and irreducible under  $\tau$  and such that  $EU_1 \neq \{0\}$ . (It is obvious that such a space exists.) Then  $\phi \in EU_1 \subset U_1$  and therefore  $U = \tau(\mathfrak{B})\phi \subset U_1$ . This proves that  $U_1 = U$  and therefore  $U$  is irreducible.

**LEMMA 4.** *Suppose  $u$  is an element in  $\mathfrak{S}_{\mathfrak{p}}$  such that  $\chi_{\lambda}(u) = 0$  for all  $\lambda \in \mathfrak{F}_0$ . Then  $u = 0$ .*

Let  $L = \dim \mathfrak{h}$ . Then (see [5(b), Theorem 1]) there exist  $L$  linearly independent linear functions  $\Lambda_1, \dots, \Lambda_L$  on  $\mathfrak{h}$  with the property that  $m_1\Lambda_1 + \dots + m_L\Lambda_L$  is the highest weight of an irreducible finite-dimensional representation of  $\mathfrak{g}$  whenever  $m_1, \dots, m_L$  are nonnegative integers. Let  $\lambda_i$  denote the restriction of  $2\Lambda_i$  on  $\mathfrak{h}_{\mathfrak{p}}$ . Then if  $l = \dim \mathfrak{h}_{\mathfrak{p}}$ , it is obvious that we can choose  $l$  linearly independent elements, say  $\lambda_1, \dots, \lambda_l$ , among  $\lambda_1, \dots, \lambda_L$ . Then if  $m_1, \dots, m_l$  are nonnegative integers, it follows from Lemma 2 that  $m_1\lambda_1 + \dots + m_l\lambda_l \in \mathfrak{F}_0$ . Our assertion is now an immediate consequence of Lemma 32 of [5(b)].

**LEMMA 5.** *Let  $\pi$  be an irreducible representation of  $G$  on a finite-dimensional vector space  $V$ . Suppose  $\phi$  is a unit vector in  $V$  such that  $\pi(k)\phi = \phi$  for all  $k \in K$ . Then*

$$(\phi, \pi(x)\phi) = \int_K e^{\lambda(H(xk))} dk \quad (x \in G)$$

where  $\lambda$  is the highest weight of  $\pi$ .

<sup>12</sup> For any  $x \in G$ ,  $b \rightarrow b^x$  ( $b \in \mathfrak{B}$ ) denotes the automorphism of  $\mathfrak{B}$  which coincides with  $\text{Ad}(x)$  on  $\mathfrak{g}$ .

Let  $\psi$  be a unit vector in  $V$  belonging to the weight  $\lambda$ . Since  $G = KA_pN$ , it is clear that  $V$  is spanned by vectors of the form  $\pi(k)\psi$  ( $k \in K$ ). Put  $E = \int_K \pi(k) dk$ . Then  $E\pi(k)\psi = E\psi$  and therefore  $EV = CE\psi$ . But  $\phi = E\psi \in EV$  and so  $\phi = cE\psi$  for some  $c \in C$ . It is obvious that  $E$  is self-adjoint and  $E^2 = E$ . Hence

$$(\phi, \pi(x)\phi) = |c|^2 (\psi, E\pi(x)E\psi) \quad (x \in G).$$

Now let  $k \in K$  and  $x \in G$ . Then  $xk = k'(\exp H(xk))n$  where  $k' \in K$  and  $n \in N$ . Hence

$$E\pi(xk)\psi = e^{\lambda(H(xk))} E\pi(k')\psi = e^{\lambda(H(xk))} E\psi.$$

Therefore

$$E\pi(x)E\psi = \int_K E\pi(xk)\psi dk = \int_K e^{\lambda(H(xk))} dk E\psi.$$

Since  $|c|^2 |E\psi|^2 = |\phi|^2 = 1$ , this implies that

$$(\phi, \pi(x)\phi) = |c|^2 (\psi, E\pi(x)E\psi) = \int_K e^{\lambda(H(xk))} dk.$$

**3. Some results of Chevalley and their consequences.** Let  $M$  be the centralizer and  $M'$  the normalizer of  $\mathfrak{h}_{p_0}$  in  $K$ . Then  $W = M'/M$  is a finite group (see [5(j), p. 619]) which operates as a group of linear transformations on  $\mathfrak{h}_p$  in the obvious way. This linear representation of  $W$  being faithful, we can identify  $W$  with the corresponding linear group. We shall call  $W$  the little Weyl group of  $\mathfrak{g}_0$  with respect to  $^{13} \mathfrak{h}_{p_0}$ .

Put  $B(X, Y) = \text{sp}(\text{ad } X \text{ ad } Y)$  ( $X, Y \in \mathfrak{g}$ ) where  $X \rightarrow \text{ad } X$  denotes the adjoint representation of  $\mathfrak{g}$ . Then the quadratic form  $B(X, X)$  is positive-definite on  $\mathfrak{p}_0$ . Therefore it defines a Euclidean metric on  $\mathfrak{p}_0$  and hence also on  $\mathfrak{h}_{p_0}$ . Define  $\Sigma$  as in the beginning of Section 2. Then for each  $\alpha \in \Sigma$ , there exists a unique element  $H_\alpha \in \mathfrak{h}_{p_0}$  such that  $B(H, H_\alpha) = \alpha(H)$  for every  $H \in \mathfrak{h}_{p_0}$ . Let  $s_\alpha$  denote the linear transformation in  $\mathfrak{h}_{p_0}$  which corresponds to the reflexion in the hyperplane  $\alpha = 0$ . Then  $s_\alpha$  is given by  $s_\alpha H = H - 2\{\alpha(H)/\alpha(H_\alpha)\}H_\alpha$  ( $H \in \mathfrak{h}_p$ ). It is known (see Cartan [2]) that  $s_\alpha \in W$ . The following lemma has been proved by Chevalley.<sup>14</sup>

**LEMMA 6 (Chevalley).** *Let  $H_0$  be any element in  $\mathfrak{h}_p$  and  $W'$  the sub-*

<sup>13</sup> It is seen without difficulty that  $M'A_p$  and  $MA_p$  respectively are the normalizer and centralizer of  $\mathfrak{h}_{p_0}$  in  $G$ . Hence  $W = M'A_p/MA_p$  is independent of our choice of  $\mathfrak{k}_0$ , so long as  $\mathfrak{h}_{p_0}$  remains fixed.

<sup>14</sup> I am grateful to Professor Chevalley for showing me his unpublished results.

group consisting of those elements  $s \in W$  which leave  $H_0$  fixed. Then  $W'$  is generated by the reflexions  $s_\alpha$  corresponding to those  $\alpha \in \Sigma$  for which  $\alpha(H_0) = 0$ .

COROLLARY.  $W$  is generated by  $s_\alpha$  ( $\alpha \in \Sigma$ ).

This follows by taking  $H_0 = 0$ . (This corollary had been verified by Cartan [2] in the case of classical Lie algebras.)

Let  $S(\mathfrak{p})$  and  $S(\mathfrak{h}_\mathfrak{p})$  denote the symmetric algebras over  $\mathfrak{p}$  and  $\mathfrak{h}_\mathfrak{p}$  respectively. Then  $S(\mathfrak{p}) \supset S(\mathfrak{h}_\mathfrak{p})$  and we can identify them with the algebras of polynomial functions on  $\mathfrak{p}$  and  $\mathfrak{h}_\mathfrak{p}$  respectively by means of the non-degenerate bilinear form  $B(X, Y)$  (see [5(k), p. 93]). Moreover  $\text{Ad}(k)\mathfrak{p} = \mathfrak{p}$  ( $k \in K$ ) and therefore  $K$  operates on  $\mathfrak{p}$  and therefore also on  $S(\mathfrak{p})$ . Similarly  $W$  operates on  $S(\mathfrak{h}_\mathfrak{p})$ . Let  $I(\mathfrak{p})$  and  $I(\mathfrak{h}_\mathfrak{p})$  be the subalgebras consisting of those elements of  $S(\mathfrak{p})$  and  $S(\mathfrak{h}_\mathfrak{p})$  which are invariant under  $K$  and  $W$  respectively.

LEMMA 7 (Chevalley<sup>14</sup>). For any  $p \in I(\mathfrak{p})$  let  $\bar{p}$  denote the restriction of the polynomial function  $p$  on  $\mathfrak{h}_\mathfrak{p}$ . Then  $\bar{p} \in I(\mathfrak{h}_\mathfrak{p})$  and  $p \rightarrow \bar{p}$  is an isomorphism of  $I(\mathfrak{p})$  onto  $I(\mathfrak{h}_\mathfrak{p})$ .

Now fix an element  $H_0$  in  $\mathfrak{h}_\mathfrak{p}$  and define  $W'$  as in Lemma 6. Put  $J = I(\mathfrak{h}_\mathfrak{p})$  and let  $J'$  denote the algebra of those elements in  $S(\mathfrak{h}_\mathfrak{p})$  which are invariant under  $W'$ .

LEMMA 8. Let<sup>15</sup>  $r = [W : W']$ . Then there exist  $r$  homogeneous elements  $v_1 = 1, v_2, \dots, v_r$  in  $J'$  such that  $J' = \sum_{1 \leq i \leq r} Jv_i$ . Moreover  $v_1, \dots, v_r$  are linearly independent over the quotient field of  $J$ .

Put  $S = S(\mathfrak{h}_\mathfrak{p})$  and let  $C(S)$ ,  $C(J')$  and  $C(J)$  denote the quotient fields of  $S$ ,  $J'$  and  $J$  respectively. Then  $C(S)/C(J)$  is normal and its Galois group is  $W$  (see Chevalley [3(b), p. 781]). By the same argument  $W'$  is the Galois group of  $C(S)/C(J')$ . Hence  $[C(J') : C(J)] = [W : W'] = r$ . Let  $S_d$  denote the space of homogeneous elements in  $S$  of degree  $d$ . Then it is obvious that  $J = \sum_{d \geq 0} J_d$  and  $J' = \sum_{d \geq 0} J'_d$  where  $J_d = S_d \cap J$  and  $J'_d = S_d \cap J'$ . Put  $J_+ = \sum_{d \geq 1} J_d$ ,  $J'_+ = \sum_{d \geq 1} J'_d$ . We claim that  $J' \cap (SJ_+) = J'J_+$ . For if  $u = \sum_{1 \leq i \leq m} p_i u_i \in J'$  ( $p_i \in S, u_i \in J_+$ ), it is obvious that

$$u = u^s = \sum_i p_i^s u_i \quad (s \in W')$$

and therefore  $u = \sum_i q_i u_i \in J'J_+$  where  $q_i = w'^{-1} \sum_{s \in W'} p_i^s$  and  $w'$  is the order of  $W'$ . Hence

<sup>15</sup>  $[W : W']$  stands for the index of  $W'$  in  $W$ .

$$S/SJ_+ \supset (J' + SJ_+)/SJ_+ \cong J'/J' \cap SJ_+ = J'/J'J_+.$$

Moreover Chevalley has shown [3(b), Theorem (B)] that  $\dim S/SJ_+ = w$  (where  $w$  is the order of  $W$ ) and the natural representation of  $W$  on  $S/SJ_+$  is equivalent to the regular representation of  $W$ . Hence in the reduction of this representation with respect to  $W'$ , the trivial representation of  $W'$  occurs exactly  $r = [W : W']$  times. Let  $S'$  be the set of those elements  $u \in S$  whose residue class mod  $SJ_+$  is left fixed by  $W'$ . Then  $\dim S'/SJ_+ = r$ . Moreover  $u - u^s \in SJ_+$  for any  $u \in S'$  and  $s \in W'$  and therefore  $u \equiv \sum_{s \in W'} u^s$  mod  $SJ_+$ . This shows that  $S' = J' + SJ_+$  and hence  $S'/SJ_+ \cong J'/J'J_+$ . Therefore  $\dim J'/J'J_+ = r$  and so we can select  $r$  homogeneous elements  $v_1 = 1, v_2, \dots, v_r$  in  $J'$  such that their residue classes mod  $J'J_+$  form a base for  $J'/J'J_+$  over  $C$ . Let  $V = \sum_{1 \leq i \leq r} C v_i$ . We shall prove that  $J' = JV$ .

It is clear that  $J' = V + J_+ J'$  and from this it follows by induction that  $J' = JV + J_+^m J'$  for any integer  $m \geq 1$ . In particular  $J_d' \subset JV + J_+^m J'$  for any  $d \geq 0$ . But then by choosing  $m > d$ , it is obvious that  $J_d' \subset JV$  and therefore  $J' = \sum_{d \geq 0} J_d' \subset JV$ . This proves that  $J' = JV$ . Moreover since  $C(J')/C(J)$  is a finite algebraic extension, it is clear that  $C(J') = C(J)J' = C(J)V$ . Therefore since  $[C(J') : C(J)] = r$ ,  $v_1, \dots, v_r$  must be linearly independent over  $C(J)$ .

The above proof depended only on the fact that  $W$  and  $W'$  are finite groups generated by reflexions. Hence if we replace  $(W, W')$  by  $(W, 1)$  and  $(W', 1)$  respectively, we get the following corollary.

COROLLARY. *There exist homogeneous elements  $u_1 = 1, u_2, \dots, u_w$  and  $u'_1 = 1, u'_2, \dots, u'_w$  in  $S$  such that*

$$S = \sum_{1 \leq i \leq w} J u_i, \quad S = \sum_{1 \leq j \leq w'} J' u'_j.$$

Moreover  $(u_1, \dots, u_w)$  are linearly independent over  $C(J)$  and  $(u'_1, \dots, u'_w)$  over  $C(J')$ .

Let  $m$  denote the centralizer of  $\mathfrak{h}_p$  in  $\mathfrak{g}$  and  $p(H)$  the determinant of the linear transformation on  $\mathfrak{g}/m$  defined by  $\text{ad } H$  ( $H \in \mathfrak{h}_p$ ). Then  $p \in S$  and it is obvious that  $p^s = p$  ( $s \in W$ ). Moreover  $p(H) = \pm \prod_{\alpha \in P_+} \alpha(H)^2$ . Hence if  $\beta \in \Sigma$  and  $s \in W$ , it follows that either  $s\beta$  or  $-s\beta$  lies in  $\Sigma$ . Put  $\langle H, H' \rangle = B(H, H')$  ( $H, H' \in \mathfrak{h}_p$ ) and identify  $\mathfrak{h}_p$  with its dual by means of the bilinear form  $B$ . We shall say that two nonzero linear functions  $\lambda, \mu$  on  $\mathfrak{h}_p$  are equivalent, if they are linearly dependent. Then if  $\alpha$  and  $\beta$  are equivalent elements in  $\Sigma$ ,  $\beta = c\alpha$  where  $c$  is a positive real number.<sup>10</sup>

<sup>10</sup> Actually it is known (see Cartan [2]) that  $c = \frac{1}{2}, 1$  or  $2$ .

Let  $l = \dim \mathfrak{h}_p$ . Then we can select (see [5(1), Lemma 1])  $l$  linearly independent elements  $\alpha_1, \dots, \alpha_l \in \Sigma$  such that every  $\alpha$  in  $\Sigma$  can be written in the form  $\alpha = m_1\alpha_1 + \dots + m_l\alpha_l$  where  $m_i$  are nonnegative integers. Let  $q$  be the number of distinct equivalence classes in  $\Sigma$ . Extend  $(\alpha_1, \dots, \alpha_l)$  to a maximal set  $(\alpha_1, \dots, \alpha_q)$  of inequivalent elements in  $\Sigma$  and put  $\pi = \alpha_1\alpha_2 \dots \alpha_q \in S$ . Since  $p^s = p$  ( $s \in W$ ), it is obvious that  $\pi^s = \epsilon(s)\pi$  where  $\epsilon(s)$  is a real number. But the mapping  $s \rightarrow \epsilon(s)$  is obviously a homomorphism of the finite group  $W$  into the multiplicative group of nonzero real numbers and therefore  $\epsilon(s) = \pm 1$ . We shall see presently that  $\epsilon(s_\alpha) = -1$  for  $\alpha \in \Sigma$ .

Put  $s_i = s_{\alpha_i}$ ,  $1 \leq i \leq l$ .

LEMMA 9.  $W$  is generated by  $(s_1, \dots, s_l)$ . Moreover for any  $\alpha \in \Sigma$ , we can choose  $s \in W$  and an index  $i$  ( $1 \leq i \leq l$ ) such that  $\alpha$  is equivalent to  $s\alpha_i$ .

Let  $W_1$  be the subgroup of  $W$  generated by  $(s_1, \dots, s_l)$  and  $\alpha$  a given element in  $\Sigma$ . Then  $\alpha = m_1\alpha_1 + \dots + m_l\alpha_l$  where  $m_1, \dots, m_l$  are nonnegative integers. Put  $m(\alpha) = m_1 + \dots + m_l$ . We shall prove by induction on  $m(\alpha)$  that  $\alpha = cs\alpha_i$  for some  $s \in W_1$ , some  $i$  ( $1 \leq i \leq l$ ) and a suitable positive number  $c$ . This is obvious if  $m(\alpha) = 1$ . So now suppose that  $m(\alpha) \geq 2$  and  $\alpha$  is not equivalent to any  $\alpha_i$  ( $1 \leq i \leq l$ ). Since  $\langle \alpha, \alpha \rangle$  is positive,  $\langle \alpha, \alpha_j \rangle$  is positive for some  $j$ . Moreover since  $\alpha$  is not equivalent to  $\alpha_j$ ,  $m_i > 0$  for some  $i \neq j$ . But  $s_j\alpha = \alpha - 2\{\langle \alpha, \alpha_j \rangle / \langle \alpha_j, \alpha_j \rangle\}\alpha_j$  and  $-s_j\alpha$  cannot lie in  $\Sigma$  since  $m_i > 0$ . Hence  $s_j\alpha \in \Sigma$  and  $m(\alpha) > m(s_j\alpha)$ . Our assertion now follows by applying the induction hypothesis to  $s_j\alpha$ . Now if  $\alpha = cs\alpha_i$ , it is clear that  $s_\alpha = ss_i s^{-1} \in W_1$ . Since  $s_\alpha$  ( $\alpha \in \Sigma$ ) generate  $W$ , this proves that  $W_1 = W$ .

LEMMA 10.  $\epsilon(s_\alpha) = -1$  for  $\alpha \in \Sigma$ .

We first claim that  $\pi^{s_i} = -\pi$  ( $1 \leq i \leq l$ ). It is sufficient to prove this for  $i = 1$ . Obviously  $s_1\alpha_1, s_1\alpha_2, \dots, s_1\alpha_q$  are all inequivalent and therefore  $s_1\alpha_j = c_j\alpha'_j$  ( $1 \leq j \leq q$ ) where  $\alpha_j \rightarrow \alpha'_j$  is a permutation of the set  $(\alpha_1, \alpha_2, \dots, \alpha_q)$  and  $c_j$  are nonzero real numbers. Now suppose  $j \neq 1$ . Then if  $\alpha_j = m_1\alpha_1 + \dots + m_l\alpha_l$ ,  $m_k > 0$  for some  $k \neq 1$ . On the other hand  $s_1\alpha_j = \alpha_j + m\alpha_1$  for some integer  $m$ . Hence  $-s_1\alpha_j$  cannot lie in  $\Sigma$ . Therefore  $s_1\alpha_j \in \Sigma$  and  $c_j > 0$ . However  $s_1\alpha_1 = -\alpha_1$  and so this proves that  $\epsilon(s_1) = \prod_{1 \leq j \leq q} c_j < 0$ . Hence  $\epsilon(s_1) = -1$ . Now let  $\alpha \in \Sigma$ . Then, as we have seen during the proof of Lemma 9,  $s_\alpha = ss_i s^{-1}$  for some  $s \in W$  and some  $i$  ( $1 \leq i \leq l$ ). Therefore  $\epsilon(s_\alpha) = \epsilon(s_i) = -1$ .

We shall call an element  $u \in S$  skew (or skew-invariant) if  $u^s = \epsilon(s)u$  for  $s \in W$ .

**COROLLARY.**  $J\pi$  is exactly the set of all skew-invariants in  $S$ .

Let  $S'$  be the set of all skew elements in  $S$ . It is obvious that  $J\pi \subset S'$ . Now suppose  $u \in S'$ . Then  $u^s = -u$  ( $s \in \Sigma$ ) and therefore the polynomial function  $u$  vanishes identically on the hyperplane  $\alpha = 0$  in  $\mathfrak{h}_p$ . But this implies that  $\alpha$  divides  $u$  in  $S$ . This being true for every  $\alpha \in \Sigma$ , we conclude that  $u = \pi v$  for some  $v \in S$ . But since both  $u$  and  $\pi$  are skew, it is clear that  $v \in J$  and therefore  $u \in J\pi$ .

For any  $x \in C(S)$ , let  $\text{sp}_{S/J}x$  denote the relative trace of  $x$  from  $C(S)$  to  $C(J)$ . Let  $\mathfrak{D}_{S/J}$  be the set of those elements  $x \in C(S)$  which have the property that  $\text{sp}_{S/J}xy \in J$  for every  $y$  in  $S$ .

**LEMMA 11.** Let  $x$  be an element in  $C(S)$ . Then  $x\mathfrak{D}_{S/J} \subset S$  if and only if  $x \in S\pi$ .

Let  $y$  be any element in  $S$ . Since  $W$  is the Galois group of  $C(S)/C(J)$ , it is clear that  $\text{sp}_{S/J}(y/\pi) = \sum_{s \in W} (y/\pi)^s = y_0/\pi$  where  $y_0 = \sum_{s \in W} \epsilon(s)y^s$ . But  $y_0$  is obviously a skew-invariant and therefore, from the corollary to Lemma 10, it lies in  $J\pi$ . This proves that  $\pi^{-1} \in \mathfrak{D}_{S/J}$ . Hence if  $x$  is an element in  $C(S)$  such that  $x\mathfrak{D}_{S/J} \subset S$ , it follows that  $x\pi^{-1} \in S$  and therefore  $x \in S\pi$ . The proof of the converse is however more complicated.

Select  $u_i \in S$   $1 \leq i \leq w$  as in the corollary to Lemma 8 and let  $u^i$   $1 \leq i \leq w$  denote the dual base of  $C(S)/C(J)$  so that  $\text{sp}_{S/J}(u_i u^j) = \delta_i^j$  ( $1 \leq i, j \leq w$ ). Then it is easy to verify that  $\mathfrak{D}_{S/J} = \sum_{1 \leq i \leq w} J u^i$ . Hence in order to complete the proof of our lemma, it is sufficient to show that  $\pi u^i \in S$  ( $1 \leq i \leq w$ ). Fix an element  $\alpha$  in  $\Sigma$ . In view of Lemma 9, we may assume that  $\alpha$  is equivalent to  $s\alpha_1$  for some  $s \in W$ . Let  $\beta_1, \dots, \beta_k$  be all the elements among  $(\alpha_1, \dots, \alpha_q)$  which are equivalent to  $t\alpha$  for some  $t \in W$ . Put  $\pi_\alpha = \beta_1 \beta_2 \dots \beta_k$ . Then it is clear that  $\pi_\alpha^s = c(s)\pi_\alpha$  where  $c(s) = \pm 1$  ( $s \in W$ ). By the argument used in the proof of Lemma 10, we deduce easily that  $c(s_1) = -1$  and therefore  $c(s_\beta) = -1$  for  $\beta = \beta_1, \beta_2, \dots, \beta_k$ . Now suppose  $z \in J \cap S\alpha$ . Then  $z = y\alpha$  for some  $y \in S$  and  $z = z^{s_\alpha} = (y\alpha)^{s_\alpha} = -y^{s_\alpha}\alpha$ . This shows that  $y^{s_\alpha} = -y$  and therefore  $\alpha$  divides  $y$  in  $S$ . But then  $\alpha^2$  divides  $z$  (in  $S$ ). However  $z \in J$  and therefore  $(s\alpha)^2$  divides  $z$  for every  $s \in W$ . This shows that  $z \in S\pi_\alpha^2$ . Therefore since  $\pi_\alpha^2 \in J$ , we conclude, from the definition of  $J$ , that  $z\pi_\alpha^{-2} \in S \cap C(J) = J$ . This proves that  $J \cap S\alpha = J\pi_\alpha^2$ .

Let  $S_1$  denote the set of all elements in  $C(S)$  of the form  $a/b$  ( $a, b \in S$ ) where  $b$  and  $\pi_a$  are relatively prime in  $S$ . It is obvious that  $b^s$  and  $\pi_a$  are also relatively prime for every  $s \in W$ . Hence if  $b_0 = \prod_{s \in W} b^s$  and  $b_1 = b_0/b$ , it follows that  $a/b = ab_1/b_0$ . Notice that  $b_0$  lies in  $J$  but not in  $J\pi_a^2$ . Moreover  $S\alpha$  is obviously a prime ideal in  $S$  and therefore  $J \cap S\alpha = J\pi_a^2$  is a prime ideal in  $J$ . Hence if  $J_1$  denotes the quotient ring of  $J$  with respect to  $J\pi_a^2$ ,  $b_0^{-1}$  lies in  $J_1$  and this proves that  $S_1 = SJ_1$ . Finally if  $a/b$  is in  $C(J)$ , it is obvious that  $ab_1 \in J$ . Therefore  $S_1 \cap C(J) = J_1$ .

By a result of Chevalley [3(b), Theorem (A)], we can select  $\omega_1, \dots, \omega_l \in J$  such that  $J = C[\omega_1, \dots, \omega_l]$ . Then  $\omega_1, \dots, \omega_l$  are algebraically independent (over  $C$ ). Obviously  $\pi_a^2$  is also transcendental over  $C$  and therefore we may assume that  $(\pi_a^2, \omega_2, \dots, \omega_l)$  are algebraically independent so that  $C(J)$  is algebraic over  $C(\pi_a^2, \omega_2, \dots, \omega_l)$ . Moreover since  $\pi_a^2 \in J$ , it can be expressed as a polynomial in  $\omega_1, \omega_2, \dots, \omega_l$  with coefficient in  $C$ .  $\omega_1$  must actually appear in this polynomial because otherwise  $\pi_a^2, \omega_2, \dots, \omega_l$  would be algebraically dependent. Therefore  $\pi_a^2$  cannot divide (in  $J$ ) any nonzero element in  $C[\omega_2, \dots, \omega_l]$ . Hence the field  $C_1 = C(\omega_2, \dots, \omega_l)$  is contained in the local ring  $J_1$ .

Now we regard  $C(J)/C_1$  and  $C(S)/C_1$  as fields of algebraic functions of one variable (see Chevalley [3(a)]). It is obvious that  $\pi_a^2$  is a prime element of  $J$  and therefore  $J_1$  is a valuation ring in  $C(J)$ . Let  $p$  be the corresponding place of  $C(J)/C_1$ . For any  $j$  ( $1 \leq j \leq k$ ) and  $x \in S$ , let  $v_j(x)$  denote the highest integer  $m \geq 0$  such that  $\beta_j^m$  divides  $x$  in  $S$ . (If  $x=0$  we put  $v_j(x) = \infty$ .) Then it is seen without difficulty that  $v_j$  can be extended (uniquely) to a valuation of  $C(S)$ . Obviously  $J_1$  is contained in the valuation ring of  $v_j$  and therefore  $v_j$  defines a place  $q_j$  of  $C(S)$  lying above  $p$ . Since  $C(S)$  is normal over  $C(J)$ , every place of  $C(S)$  lying above  $p$  is conjugate to  $q_1$  under  $W$  (see [3(a), p. 54]). But since  $s\beta_1$  ( $s \in W$ ) is equivalent to  $\beta_j$  for some  $j$ , it follows that  $q_1, q_2, \dots, q_k$  are all the distinct places of  $C(S)$  above  $p$ . It is obvious that  $S_1$  is exactly the set of those elements in  $C(S)$  which are integral at  $q_j$  for every  $j$  ( $1 \leq j \leq k$ ). Moreover  $S_1 = J_1 S = \sum_{1 \leq i \leq w} J_1 u_i$ . Since  $v_j(\pi_a^2) = 2$ , it follows that the ramification index of  $q_j$  with respect to  $C(J)$  is 2. Therefore we conclude from Theorem 7 of [3(a), p. 69] that the differential exponent of  $q_j$  is 1. Let  $\mathfrak{D}_p$  be the set of all  $x \in C(S)$  such that  $\text{sp}_{S/J} x S_1 \subset J_1$ . Then obviously  $\mathfrak{D}_p = \sum_{1 \leq i \leq w} J_1 u_i$ . Hence it follows from Lemma 4 of [3(a), p. 73] that  $\min_{1 \leq i \leq w} v_j(u_i) = -1$ . Let  $d$  be a nonzero element in  $S$  of the lowest possible degree such that

$du^i \in S$   $1 \leq i \leq w$ . Then the above result shows that  $\pi_a$  divides  $d$  in  $S$  and  $\pi_a$  and  $d/\pi_a$  are relatively prime in  $S$ . But since  $\alpha$  was an arbitrary element in  $\Sigma$ , this implies that  $d' = d/\pi$  lies in  $S$  and it is relatively prime to  $\pi$ . In order to complete our proof, it only remains to show that  $d' \in C$ .

We shall now introduce some notation which will also be useful later. For any  $H \in \mathfrak{h}_p$ , let  $S_H$  denote the set of all polynomial functions in  $S$  which vanish at  $H$  and put  $J_H = J \cap S_H$ . Obviously  $S_H$  and  $J_H$  are prime ideals in  $S$  and  $J$  respectively and  $J_{sH} = J_H$  ( $s \in W$ ). Now suppose  $\pi(H) \neq 0$ . Then from Lemma 6, the  $w$  elements  $sH$  ( $s \in W$ ) are all distinct. Hence there exist  $w$  distinct prime ideals  $S_{sH}$  in  $S$  lying above  $J_H$ . On the other hand  $J = C + J_H$  and therefore  $S = \sum_{1 \leq i \leq w} Ju_i = \sum_{1 \leq i \leq w} Cu_i + SJ_H$ . This proves that  $\dim S/SJ_H \leq w$ . But in view of what we have said above, the associative algebra  $\bar{S} = S/SJ_H$  has  $w$  distinct non-trivial homomorphisms into  $C$ . Therefore it must be semisimple and of dimension  $w$  and  $\bigcap_{s \in W} S_{sH} = SJ_H$ . Let  $u \rightarrow \bar{u}$  denote the natural homomorphism of  $S$  on  $\bar{S}$  and let  $\text{sp } \bar{u}$  denote the trace of  $\bar{u}$  in the regular representation of  $\bar{S}$ . Then clearly  $\text{sp } \bar{u} = \sum_{s \in W} u(sH)$  ( $u \in S$ ). Moreover since  $\bar{S}$  is semisimple, we can conclude that  $\bar{u} = 0$  if  $\text{sp } \bar{u} = 0$  ( $1 \leq i \leq w$ ).

Now put  $g_{ij} = \text{sp}_{S/J}(u_i u_j)$   $1 \leq i, j \leq w$  and  $g = \det(g_{ij})_{1 \leq i, j \leq w}$ . Then  $g \neq 0$ . Let  $(g^{ij})_{1 \leq i, j \leq w}$  denote the inverse of the matrix  $(g_{ij})_{1 \leq i, j \leq w}$ . It is obvious that  $u^i = \sum_j g^{ij} u_j$  and therefore  $d$  divides  $g$  in  $S$ . We claim  $g(H) \neq 0$ .

For otherwise we could choose complex numbers  $c_1, \dots, c_w$ , not all zero, such that  $\sum_j g_{ij}(H) c_j = 0$  ( $1 \leq i \leq w$ ). Put  $u = \sum_j c_j u_j$ . Then  $\sum_j g_{ij} c_j = \text{sp}_{S/J}(u_i u) = \sum_{s \in W} (u_i u)^s$ . Hence  $\text{sp } \bar{u}_i \bar{u} = \sum_{s \in W} u_i(sH) u(sH) = 0$ . But, as we have seen above, this implies that  $\bar{u} = 0$ . On the other hand  $\bar{u}_1, \dots, \bar{u}_w$  span  $\bar{S}$  and so they must be linearly independent over  $C$ . Therefore  $\bar{u} = \sum_j c_j \bar{u}_j \neq 0$  and we get a contradiction. Hence  $g(H) \neq 0$  whenever  $\pi(H) \neq 0$  ( $H \in \mathfrak{h}_p$ ). This means that every prime factor of  $g$  divides  $\pi$ . But we know that  $\pi$  divides  $d$  and  $d$  divides  $g$ . Therefore  $\pi$ ,  $d$  and  $g$  have exactly the same prime factors and this implies  $d' \in C$ . The proof of Lemma 11 is now complete.

Define  $W'$  and  $J'$  as before. We recall that  $\pi = \alpha_1 \alpha_2 \cdots \alpha_q$ . Let  $\pi'$  denote the product of those elements among  $\alpha_1, \dots, \alpha_q$  which vanish at  $H_0$ . Define  $\text{sp}_{S/J'} x$  ( $x \in C(S)$ ) to be the relative trace from  $C(S)$  to  $C(J')$ .

**COROLLARY.** Select  $u_i$  and  $u'_j$  ( $1 \leq i \leq w, 1 \leq j \leq w'$ ) as in the corollary to Lemma 8 and define  $u^i$  and  $u'^j$  in  $C(S)$  by the relations  $\text{sp}_{S/J'}(u^i u'_k) = \delta_k^i$

$(1 \leq i, k \leq w)$  and  $\text{sp}_{S/J}(u'^j u'_m) = \delta_m^j$  ( $1 \leq j, m \leq w'$ ) respectively. Then  $\pi u^i$  and  $\pi' u'^j$  are in  $S$ . Moreover if  $d$  and  $d'$  are any two elements in  $S$  such that  $du^i$  and  $d'u'^j$  ( $1 \leq i \leq w, 1 \leq j \leq w'$ ) are all in  $S$ , then  $\pi$  divides  $d$  and  $\pi'$  divides  $d'$  in  $S$ .

We have already seen this for  $u^i$ . The proof for  $u'^j$  is entirely similar.

Choose  $v_1, \dots, v_r$  as in Lemma 8 and let  $\text{sp}_{J'/J}x$  ( $x \in C(J')$ ) denote the relative trace of  $x$  from  $C(J')$  to  $C(J)$ .

LEMMA 12. Define  $v^i \in C(J')$   $1 \leq i \leq r$  by the conditions  $\text{sp}_{J'/J}(v^i v_j) = \delta_j^i$  ( $1 \leq i, j \leq r$ ). Then an element  $x \in S$  is divisible by  $\pi/\pi'$  in  $S$  if and only if  $xv^i \in S$   $1 \leq i \leq r$ .

Let  $D$  be a nonzero element in  $S$  of the least possible degree such that  $Dv^i \in S$ . Then the highest common factor (h.c.f.) of  $Dv^1, \dots, Dv^r$  (in  $S$ ) is 1. Define  $u'^j$  as above. Then

$\text{sp}_{S/J}(u'^i v^j u'_k v_m) = \text{sp}_{J'/J} \text{sp}_{S/J'}(u'^i u'_k v^j v_m) = \delta_k^i \text{sp}_{J'/J}(v^j v_m) = \delta_k^i \delta_m^j$   
 $(1 \leq i, k \leq w', 1 \leq j, m \leq r)$ . Since  $S = \sum_{1 \leq k \leq w'} \sum_{1 \leq m \leq r} J v_m u'_k$ , it follows from the corollary to Lemma 11 that  $\pi u'^i v^j \in S$ . Since the h.c.f. of  $Dv^j$  ( $1 \leq j \leq r$ ) is 1, this implies that  $D^{-1} \pi u'^i \in S$ . Therefore we conclude from the same corollary that  $\pi/D\pi'$  is in  $S$ . Conversely  $\pi' u'^i Dv^j \in S$  and therefore  $\pi$  divides  $\pi'D$  again by the same corollary. Hence  $\pi'D/\pi$  lies in  $C$  and the lemma follows.

Put  $J_H' = J' \cap S_H$  for  $H \in \mathfrak{h}_p$ .

LEMMA 13.  $\dim J'/J'J_H = [W:W']$ ,  $\dim S/SJ_H = w$  and  $\dim S/SJ_H' = w'$  for every  $H \in \mathfrak{h}_p$ .

We prove only the first statement since the proofs of the other two are parallel. Suppose  $\sum_{1 \leq i \leq r} c_i v_i \in J'J_H$  ( $c_i \in C$ ). Then since  $J'J_H = \sum_i J_H v_i$ , it follows from the linear independence of  $v_1, \dots, v_r$  over  $C(J)$  (see Lemma 7) that  $c_i \in J_H$ . But obviously this implies that  $c_i = 0$   $1 \leq i \leq r$ . Hence the residue classes of  $v_1, \dots, v_r \bmod J'J_H$  form a base for  $J'/J'J_H$  and therefore  $\dim J'/J'J_H = r = [W:W']$ .

Put  $\sigma^i = \pi u^i$   $1 \leq i \leq w$ . We have seen above that  $\sigma^i \in S$ .

LEMMA 14. For any  $H \in \mathfrak{h}_p$ , let  $e_H = \sum_{1 \leq i \leq w} \sigma^i(H) u_i$ . Then

$$(1) \quad p e_H \equiv p(H) e_H \bmod S J_H \text{ for any } p \in S,$$

$$(2) \quad \sum_{s \in W} \epsilon(s) e_{sH} \equiv \pi(H) \bmod S J_H,$$

(3)  $e_{sH}(tH) = \epsilon(s)\pi(sH)$  or zero according as  $s=t$  or not ( $s, t \in W$ ).

We know that

$$\delta_j^i = \text{sp}_{S/J}(u^i u_j) = \sum_{s \in W} (u^i u_j)^s = \pi^{-1} \sum_{s \in W} \epsilon(s) (\sigma^i u_j)^s.$$

Therefore  $\sum_{s \in W} \epsilon(s) (\sigma^i u_j)^s = \pi \delta_j^i$  ( $1 \leq i, j \leq w$ ). Now suppose  $s_1, s_2, \dots, s_w$  are all the distinct elements of  $W$  and  $\pi(H) \neq 0$ . Then this result implies that

$$\sum_k \epsilon(s_k) \sigma^i(s_k H) u_j(s_k H) = \pi(H) \delta_j^i.$$

Therefore if we regard

$$A = \{\sigma^i(s_k H)\}_{1 \leq i, k \leq w} \text{ and } B = \{u_j(s_k H)/\pi(H)\}_{1 \leq k, j \leq w}$$

as  $w \times w$  matrices, it is clear that they are reciprocals of each other and therefore  $BA$  is also the unit matrix. Hence<sup>9</sup>

$$\sum_j \epsilon(s_i) \sigma^j(s_i H) u_j(s_k H) = \pi(H) \delta_{ik}.$$

This proves that  $e_{s_i H}(s_k H) = \delta_{ik} \epsilon(s_i) \pi(H)$  and therefore  $e_H \in S_{sH}$  if  $s \neq 1$ . Since  $\pi(H) \neq 0$ , we know (see the last part of the proof of Lemma 11) that  $SJ_H = \bigcap_{s \in W} S_{sH}$ . Therefore  $e_H S_H \subset SJ_H$  and hence  $pe_H \equiv p(H)e_H \pmod{SJ_H}$  for  $p \in S$ . Moreover it follows from the relation obtained above that  $\sum_{s \in W} \epsilon(s) e_{sH}(tH) = \pi(H)$  for  $t \in W$ . Therefore  $\sum_{s \in W} \epsilon(s) e_H - \pi(H) \in \bigcap_{t \in W} S_{tH} = SJ_H$ . Thus all the statements of the lemma have been proved under the assumption that  $\pi(H) \neq 0$ .

Now we come to the general case. Select  $x_{ij}^k \in J$  such that  $u_i u_j = \sum_k x_{ij}^k u_k$ .

Then  $e_H u_j = \sum_i \sigma^i(H) u_i u_j = \sum_{i,j} \sigma^i(H) x_{ij}^k u_k \equiv \sum_{i,k} \sigma^i(H) x_{ij}^k(H) u_k \pmod{SJ_H}$ .

But we know from Lemma 13 that  $u_1, \dots, u_w$  are linearly independent mod  $SJ_H$ . Therefore it is clear that the first statement of the lemma holds if and only if  $z_j^k(H) = 0$  ( $1 \leq j, k \leq w$ ) where  $z_j^k = \sum_i \sigma^i x_{ij}^k - \sigma^k u_j$ . But then

it follows from the above proof that  $z_j^k(H) = 0$  whenever  $\pi(H) \neq 0$ . Therefore since  $z_j^k$  are polynomial functions on  $\mathfrak{h}_p$ , we conclude that they are all zero. In the same way the second statement of the lemma is seen to be equivalent to the relations  $\sum_{s \in W} \epsilon(s) \sigma^i(sH) = \pi(H) \delta_1^i$  ( $1 \leq i \leq w$ ). Again

since these equations hold when  $\pi(H) \neq 0$ , they hold in general. The proof of (3) is entirely similar.

**COROLLARY 1.** If  $\pi(H) \neq 0$  then  $e_{sH}$  ( $s \in W$ ) are linearly independent mod  $SJ_H$ .

This is an immediate consequence of the third statement of the above lemma.

**COROLLARY 2.** *Let  $p$  be an element in  $S$  and let  $H \in \mathfrak{h}_p$ . Consider the linear transformation  $L$  in  $S/SJ_H$  corresponding to multiplication by  $p$ . Then  $\det(tI - L) = \prod_{s \in W} (t - p(sH))$  where  $t$  is an indeterminate and  $I$  is the identity mapping of  $S/SJ_H$ .*

Select  $q_{ji} \in J$  such that  $pu_i = \sum_j u_j q_{ji}$  ( $1 \leq i \leq w$ ). Then it is obvious that  $pu_i = \sum_j q_{ji}(H)u_j \bmod SJ_H$ . Put

$$Q(t) = \det(t\delta_{ij} - q_{ij})_{1 \leq i, j \leq w}.$$

Then  $Q(t)$  is a polynomial in  $t$  with coefficients in  $J$ . Obviously it is sufficient to prove that  $Q(t) = \prod_{s \in W} (t - p^s)$ . Let  $q_m$  and  $p_m$  denote the coefficients of  $t^m$  in  $Q(t)$  and  $\prod_{s \in W} (t - p^s)$  respectively. Put  $Q(t:H) = \sum_{m \geq 0} q_m(H)t^m$ . If  $\pi(H) \neq 0$ ,  $e_{sH}$  ( $s \in W$ ) form a base for  $S \bmod SJ_H$  by Corollary 1 above and therefore it follows from the first statement of Lemma 14 that

$$Q(t:H) = \det(tI - L) = \prod_{s \in W} (t - p(sH)).$$

This shows that  $q_m(H) = p_m(H)$  for every  $m$ , if  $\pi(H) \neq 0$ . But since  $q_m$  and  $p_m$  are polynomial functions on  $\mathfrak{h}_p$ , this implies that  $q_m = p_m$ .

Now put  $D = \pi/\pi'$ ,  $\tau^i = Dv^i$   $1 \leq i \leq r$  and select elements  $s_1, \dots, s_r$  in  $W$  such that  $W = \bigcup_{1 \leq i \leq r} W's_i$ . (We recall that  $r = [W:W']$ .)

**LEMMA 15.** *Put  $f_H = \sum_{1 \leq i \leq r} \tau^i(H)v_i$  for  $H \in \mathfrak{h}_p$ . Then*

- (1)  $pf_H \equiv p(H)f_H \bmod SJ_H$  for  $p \in J'$ ,
- (2)  $\sum_{1 \leq i \leq r} \epsilon(s_i)\pi'(s_iH)f_{s_iH} \equiv \pi(H) \bmod SJ_H$ ,
- (3)  $f_{s_iH}(s_jH) = \delta_{ij}D(s_iH)$  ( $1 \leq i, j \leq r$ ).

Finally,  $f_{s_1H}, \dots, f_{s_rH}$  are linearly independent  $\bmod SJ_H$  if  $\pi(H) \neq 0$ .

The proof is similar to that of Lemma 14. Put  $t_k = s_k^{-1}$  ( $1 \leq k \leq r$ ). Since  $C(J')$  is the subfield of  $C(S)$  consisting exactly of those elements which are left fixed by  $W'$ , we conclude from Galois theory that  $t_1, \dots, t_r$  define all the distinct isomorphisms of  $C(J')/C(J)$  into  $C(S)$ . Therefore  $\text{sp}_{J'/J}v = \sum_{1 \leq i \leq r} v^i$  for  $v \in C(J')$ . This shows that

$$\delta_j^i = \text{sp}_{J'/J}(v^i v_j) = \sum_{1 \leq k \leq r} (\tau^i v_j / D) t_k.$$

Multiplying this equation by  $\pi$ , we get

$$\sum_{1 \leq k \leq r} \epsilon(t_k) (\pi' \tau^i v_j)^{t_k} = \pi \delta_j^i \quad (1 \leq i, j \leq r).$$

But this implies that

$$\sum_k \epsilon(s_k) \pi'(s_k H) \tau^i(s_k H) v_j(s_k H) = \pi(H) \delta_j^i.$$

Now first assume that  $\pi(H) \neq 0$ . Then the  $r \times r$  matrices

$$A = \{\epsilon(s_k) \pi'(s_k H) \tau^i(s_k H)\}_{1 \leq i, k \leq r} \text{ and } B = \{v_j(s_k H) / \pi(H)\}_{1 \leq k, j \leq r}$$

are reciprocals of each other and therefore  $BA$  is also the unit matrix. Hence

$$\sum_k \epsilon(s_i) \pi'(s_i H) \tau^k(s_i H) v_k(s_j H) = \pi(H) \delta_{ij}$$

and this proves that  $f_{s_i H}(s_j H) = \delta_{ij} D(s_i H)$   $1 \leq i, j \leq r$ . (By continuity this relation remains valid even when  $\pi(H) = 0$ .) Since  $\pi(H) \neq 0$ , it is clear that  $D(sH) \neq 0$  for any  $s \in W$ . Therefore the linear independence of  $f_{s_1 H}, \dots, f_{s_r H} \bmod SJ_H$  is obvious. Furthermore

$$\sum_{1 \leq i \leq r} \epsilon(s_i) \pi'(s_i H) f_{s_i H}(s_j H) = \epsilon(s_j) \pi'(s_j H) D(s_j H) = \pi(H) \quad (1 \leq i \leq r).$$

On the other hand  $f_{s_i H} \in J'$  and therefore  $f_{s_i H}(sH) = f_{s_i H}(s_j H)$  if  $s \in W' s_j$ . Hence if we put  $g = \sum_{1 \leq i \leq r} \epsilon(s_i) \pi'(s_i H) f_{s_i H} - \pi(H)$ , it follows that  $g(sH) = 0$  for all  $s \in W$ . But this implies that  $g \in \bigcap_{s \in W} S_{sH} = SJ_H$  and so we get the second statement of the lemma. Now suppose  $p \in J'$ . Then  $p(sH) = p(H)$  for  $s \in W'$  and therefore  $p - p(H) \in \bigcap_{s \in W'} S_{sH}$ . On the other hand we have seen above that  $f_H(sH) = 0$  if  $s \notin W'$ . Therefore  $(p - p(H))f_H \in \bigcap_{s \in W} S_{sH} = SJ_H$  and this proves that  $pf_H = p(H)f_H \bmod SJ_H$ .

In order to extend our results to the general case, we use the same method as before. For example, let us prove the first statement. It is enough to show that  $v_i f_H = v_i(H) f_H \bmod SJ_H$   $1 \leq i \leq r$ . Since  $w'v = \sum_{s \in W'} v^s$  for  $v \in J' \cap (SJ_H)$ , it follows that  $J' \cap (SJ_H) = J'J_H$ . Select  $y_{ij}^k \in J$  such that  $v_i v_j = \sum_k y_{ij}^k v_k$ . Then due to Lemma 13, the required relations hold if and only if  $\sum_j \tau^j(H) y_{ij}^k(H) = v_i(H) \tau^k(H)$   $(1 \leq i, k \leq r)$ . The remaining argument is now the same as in the proof of Lemma 14.

**COROLLARY.** Let  $p$  be an element in  $J'$  and let  $H \in \mathfrak{h}_p$ . Consider the linear transformation  $L'$  in  $J' / J_H$  corresponding to multiplication by  $p$ .

Then  $\det(tI' - L') = \prod_{1 \leq i \leq r} (t - p(s_i H))$  where  $t$  is an indeterminate and  $I'$  is the identity mapping of  $J'/J'J_H$ .

This is proved in the same way as Corollary 2 to Lemma 14.

Define  $J_+$  as in the proof of Lemma 8 and let  $d_\pi$  and  $d_i$  denote the degrees of  $\pi$  and  $u_i$  ( $1 \leq i \leq w$ ) respectively. Then the following result is of some interest, although it is not strictly necessary for our purpose. Therefore we state it here without proof.

LEMMA 16. Suppose  $d_w = \max_{1 \leq i \leq w} d_i$ . Then  $d_w = d_\pi$  and  $d_i < d_\pi$  for  $1 \leq i \leq w$ . Moreover  $u_w \equiv c\pi \pmod{SJ_+}$  where  $c$  is a nonzero complex number.

**4. The homomorphism  $\gamma$ .** We now return to the notation of Section 2 and for any  $q \in I_g$  define  $u_q \in \mathfrak{S}_v$  as in Lemma 3. Put  $\rho = \frac{1}{2} \sum_{\alpha \in P_+} \tilde{\alpha}$  where  $\tilde{\alpha}$  denotes the restriction of  $\alpha$  on  $\mathfrak{h}_v$ . Also let  $u \rightarrow 'u$  denote the automorphism of  $\mathfrak{S}_v$  such that  $'H = H - \rho(H)$  ( $H \in \mathfrak{h}_v$ ). We identify  $S(\mathfrak{h}_v)$  with  $\mathfrak{S}_v$  under the isomorphism which preserves every element of  $\mathfrak{h}_v$ . Then  $J = I(\mathfrak{h}_v)$  becomes a subalgebra of  $\mathfrak{S}_v$ . Put  $\gamma(q) = 'u_q$  ( $q \in I_g$ ).

THEOREM 1.  $\gamma$  is a homomorphism of  $I_g$  onto  $I(\mathfrak{h}_v)$  and its kernel is  $I_g \cap \mathfrak{B}\mathfrak{f} = I_g \cap \mathfrak{f}\mathfrak{B}$ .

In order to show that  $\gamma$  is a homomorphism, it is enough to verify that  $u_{q_1 q_2} = u_{q_1} u_{q_2}$  for  $q_1, q_2 \in I_g$ . Now  $q_2 - u_{q_2} \in \mathfrak{f}\mathfrak{B} + \mathfrak{B}\mathfrak{n}$ . Therefore  $q_1(q_2 - u_{q_2}) \in \mathfrak{f}\mathfrak{B} + \mathfrak{B}\mathfrak{n}$  since  $q_1 \mathfrak{f} = \mathfrak{f}q_1$ . But  $q_1 u_{q_2} = (q_1 - u_{q_1})u_{q_2} + u_{q_1} u_{q_2}$  and  $(q_1 - u_{q_1})u_{q_2} \in (\mathfrak{f}\mathfrak{B} + \mathfrak{B}\mathfrak{n})u_{q_2} \subset \mathfrak{f}\mathfrak{B} + \mathfrak{B}\mathfrak{n}$  since  $[\mathfrak{n}, \mathfrak{h}_v] \subset \mathfrak{n}$ . Therefore  $q_1 q_2 - u_{q_1} u_{q_2} \in \mathfrak{f}\mathfrak{B} + \mathfrak{B}\mathfrak{n}$  and this proves that  $u_{q_1 q_2} = u_{q_1} u_{q_2}$ .

Now fix  $q$  in  $I_g$ . Then it follows from Lemma 4 that  $\gamma(q) = 0$  if and only if  $\chi_\nu(u_q) = 0$  for every  $\nu \in \mathfrak{F}_0$ . Define  $\pi$  and  $\phi$  as in Lemma 5 and put  $g(x) = (\phi, \pi(x)\phi)$  for  $x \in G$ . Then if  $\nu$  denotes the restriction on  $\mathfrak{h}_v$  of the highest weight of  $\pi$ , it follows from Lemma 3 that  $g(x; q) = \chi_\nu(u_q)g(x)$ . Since  $g(1) = 1$ , this shows that  $\chi_\nu(u_q) = g(1; q)$ . But since  $g$  is spherical, it is clear that  $g(1; q) = 0$  if  $q \in \mathfrak{f}\mathfrak{B} + \mathfrak{B}\mathfrak{f}$ . Hence  $I_g \cap (\mathfrak{f}\mathfrak{B} + \mathfrak{B}\mathfrak{f})$  is contained in the kernel of  $\gamma$ . On the other hand  $\gamma(q) = 0$  implies that  $g(x; q) = (\phi, \pi(x)\pi(q)\phi) = 0$ . Since this holds for every  $x$  in  $G$  and since  $\pi$  is irreducible, it follows that  $\pi(q)\phi = 0$ . Therefore by applying Lemma 1 of [5(d)] to  $\mathfrak{Y} = \mathfrak{X}\mathfrak{f}$ , we conclude that  $q \in \mathfrak{B}\mathfrak{f}$ . Combining this with the above result, we deduce that  $I_g \cap \mathfrak{B}\mathfrak{f}$  is exactly the kernel of  $\gamma$  and therefore  $I_g \cap \mathfrak{B}\mathfrak{f} \supset I_g \cap \mathfrak{f}\mathfrak{B}$ . Now consider the anti-automorphism  $b \rightarrow b^*$  of  $\mathfrak{B}$  such that  $X^* = -X$  for every  $X \in \mathfrak{g}$ . It is obvious that  $I_g^* = I_g$ . Hence

$(I_g \cap \mathfrak{B})^* = I_g \cap \mathfrak{B} \supset (I_g \cap \mathfrak{f}\mathfrak{B})^* = I_g \cap \mathfrak{B}$ . This proves that  $I_g \cap \mathfrak{B} = I_g \cap \mathfrak{f}\mathfrak{B}$ .

In order to prove that  $\gamma$  maps  $I_g$  into  $J$  we need some additional facts. For any  $h \in A_p$  and  $s \in W$ , put  $h^s = \exp(sH)$  where  $H = \log h$  and let  $dn$  denote the Haar measure on  $N$ .

LEMMA 17. Put

$$F_f(h) = e^{\rho(\log h)} \int_N f(hn) dn \quad (h \in A_p)$$

for any function  $f$  in <sup>17</sup>  $C_c^\infty(G)$  such that  $f(kxk^{-1}) = f(x)$  ( $x \in G, k \in K$ ). Then  $F_f(h^s) = F_f(h)$  for  $s \in W$  and  $h \in A_p$ .

Let  $x \rightarrow x^*$  denote the natural mapping of  $G$  on the factor space  $G^* = G/A_p$ . Put  $D(h) = \prod_{\alpha \in P_+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})$  and  $h^{x^*} = xhx^{-1}$  ( $h \in A_p, x \in G$ ) where  $H = \log h$ . Denote the invariant measure on  $G^*$  by  $dx^*$  and consider the set  $A_p'$  of those elements  $h \in A_p$  where  $D(h) \neq 0$ . Then <sup>18</sup> the integral  $\int_{G^*} g(h^{x^*}) dx^*$  is convergent for  $g \in C_c(G)$  and  $h \in A_p'$ . Moreover the measure  $dn$  can be so normalized that

$$|D(h)| \int_{G^*} g(h^{x^*}) dx^* = e^{\rho(\log h)} \int_{K \times N} g(khnk^{-1}) dk dn \quad (h \in A_p').$$

Let  $s$  be an element in  $W$  and select  $k$  in  $K$  such that  $khk^{-1} = h^s$  for  $h \in A_p$ . Then for any  $x \in G$ , the coset  $(xk)^*$  depends only on  $x^*$  and  $s$  and we denote it by  $x^*s$ . Since  $W$  is a finite group, it follows without difficulty that the measure  $dx^*$  is invariant under these operations of  $W$  on  $G^*$ . Therefore

$$\int_{G^*} g((h^s)^{x^*}) dx^* = \int_{G^*} g(h^{x^*}) dx^*$$

for  $g \in C_c(G)$  and  $h \in A_p'$ . If we apply this to  $f$  and make use of the fact that  $|D(h^s)| = |D(h)|$ , it follows that  $F_f(h^s) = F_f(h)$  for  $h \in A_p'$ . However it is obvious from its definition that  $F_f \in C_c^\infty(A_p)$ . Therefore  $F_f(h^s) = F_f(h)$  for all  $h \in A_p$ .

COROLLARY. For any linear function  $\lambda$  on  $\mathfrak{h}_p$ , put

$$\phi_\lambda(x) = \int_K \exp\{(-1)^{\frac{1}{2}} \lambda(H(xk)) - \rho(H(xk))\} dk \quad (x \in G).$$

Then  $\phi_{s\lambda} = \phi_\lambda$  for  $s \in W$ .

<sup>17</sup> See footnote 1 of [5(k)] for notation.

<sup>18</sup> The facts stated here are quite well known. In any case their proofs offer no difficulty (see [5(f), p. 508]).

Obviously  $\phi_\lambda$  is a spherical function of class  $C^\infty$ . Hence if  $dx$  is the Haar measure on  $G$ , it would be enough to show that for a fixed  $s$ ,

$$\int \phi_\lambda(x)f(x)dx = \int \phi_{s\lambda}(x)f(x)dx$$

for every spherical function  $f$  in  $C_c^\infty(G)$ . Let  $dh$  denote the Haar measure on  $A_p$ . Then it is possible (see [5(c), Lemma 35]) to normalize it in such a way that  $dx = e^{2\rho(\log h)}dkdhdn$  if  $x = khn$  ( $k \in K, h \in A_p, n \in N$ ). Therefore

$$\begin{aligned} \int \phi_\lambda(x)f(x)dx &= \int \exp\{(-1)^{\frac{1}{2}}\lambda(H(x)) - \rho(H(x))\}f(x)dx \\ &= \int \exp\{(-1)^{\frac{1}{2}}\lambda(\log h) + \rho(\log h)\}f(hn)dhdn \\ &= \int \exp\{(-1)^{\frac{1}{2}}\lambda(\log h)\}F_f(h)dh. \end{aligned}$$

Our assertion now follows immediately from the above lemma.

Now suppose  $q \in I_g$ . Then  $\gamma(q)$ , being an element of  $S(\mathfrak{h}_p)$ , can be regarded as a polynomial function on  $\mathfrak{h}_p$ . We denote by  $\gamma(q:H)$  the value of this function at  $H \in \mathfrak{h}_p$ . Also let us recall that  $\mathfrak{h}_p$  has been identified with its dual under the bilinear form  $B$ . Hence the following result is an immediate consequence of Lemma 3 and the definition of  $\gamma(q)$ .

LEMMA 18.  $\phi_\lambda(x;q) = \gamma(q:(-1)^{\frac{1}{2}}\lambda)\phi_\lambda(x)$  for  $q \in I_g$ ,  $x \in G$  and any linear function  $\lambda$  on  $\mathfrak{h}_p$ .

It is obvious from this lemma and the corollary to Lemma 17 that  $\gamma(q:H) = \gamma(q:sH)$  for  $s \in W$  and  $H \in \mathfrak{h}_p$ . Hence  $\gamma(q) = \gamma(q)^*$  and this proves that  $\gamma(q) \in J$ .

We still have to show that  $\gamma$  maps  $I_g$  onto  $J$ . Let  $\lambda$  denote the canonical mapping (see [5(c), p. 192]) of  $S(\mathfrak{g})$  onto  $\mathfrak{B}$  and put  $\mathfrak{P} = \lambda(S(\mathfrak{p}))$  and  $\mathfrak{P}_d = \lambda(S_d(\mathfrak{p}))$  where  $S_d(\mathfrak{p})$  is the set of all homogeneous elements in  $S(\mathfrak{p})$  of degree  $d$ . Since  $\mathfrak{g}$  is the direct sum of  $\mathfrak{k}$  and  $\mathfrak{p}$ , it follows [5(c), Lemma 12] that  $\mathfrak{B} = \mathfrak{K}\mathfrak{P}$  and therefore  $\mathfrak{B} = \mathfrak{P} + \mathfrak{k}\mathfrak{P}$ , the sum being direct. Moreover if  $X \in \mathfrak{k}$  it is obvious that both  $\mathfrak{P}$  and  $\mathfrak{k}\mathfrak{P}$  are invariant under the mapping<sup>19</sup>  $b \rightarrow [X, b]$  ( $b \in \mathfrak{B}$ ) of  $\mathfrak{B}$ . Hence  $I_g = I_g \cap \mathfrak{P} + I_g \cap \mathfrak{k}\mathfrak{P}$  and so it is sufficient to prove the following lemma.

LEMMA 19.  $\gamma$  defines a linear isomorphism of  $I_g \cap \mathfrak{P}$  onto  $J$ .

Since  $I_g \cap \mathfrak{k}\mathfrak{P}$  is the kernel of  $\gamma$  and  $\mathfrak{P} \cap \mathfrak{k}\mathfrak{P} = \{0\}$ , it is clear that  $\gamma$  is univalent on  $I_g \cap \mathfrak{P}$ . Also it follows from Lemma 11 of [5(c)] that

<sup>19</sup>  $[a, b] = ab - ba$  for  $a, b \in \mathfrak{B}$ .

$I_{\mathfrak{g}} \cap \mathfrak{P} = \lambda(I(\mathfrak{p}))$  (in the notation of Lemma 7). Let  $u$  be a homogeneous element in  $J$  of degree  $d$ . We shall prove by induction on  $d$  that there exists an element  $q \in \sum_{e \leq d} I_{\mathfrak{g}} \cap \mathfrak{P}_e$  such that  $\gamma(q) = u$ . This is obvious if  $d = 0$ .

So suppose  $d \geq 1$ . Let  $\bar{p}$  denote the restriction on  $\mathfrak{h}_{\mathfrak{p}}$  of any polynomial function  $p \in S(\mathfrak{g})$ . By Lemma 7, we can select a homogeneous element  $p_0 \in I(\mathfrak{p})$  of degree  $d$  such that  $\bar{p}_0 = u$ . Put  $q_0 = \lambda(p_0)$ . Then  $q_0$  lies in  $I_{\mathfrak{g}} \cap \mathfrak{P}$ . We shall prove that  $u - \gamma(q_0)$  is of degree lower than  $d$  and hence, by induction hypothesis, we can choose  $q_1 \in I_{\mathfrak{g}} \cap \mathfrak{P}$  of degree  $< d$  such that  $u - \gamma(q_0) = \gamma(q_1)$ . Then  $u = \gamma(q)$  where  $q = q_0 + q_1$  and this would prove the lemma.

We know that  $\mathfrak{g}$  is the direct sum of  $\mathfrak{k}$ ,  $\mathfrak{h}_{\mathfrak{p}}$  and  $\mathfrak{n}$ . Hence<sup>11</sup>  $S(\mathfrak{g}) = S(\mathfrak{k})S(\mathfrak{h}_{\mathfrak{p}})S(\mathfrak{n})$ . Now  $\mathfrak{h}_{\mathfrak{p}}$  is orthogonal to  $\mathfrak{k} + \mathfrak{n}$  under the bilinear form  $B(X, Y)$  ( $X, Y \in \mathfrak{g}$ ). Hence if  $p \in S(\mathfrak{g})$ , it is obvious that  $p - \bar{p} \in \mathfrak{k}S(\mathfrak{g}) + S(\mathfrak{g})\mathfrak{n}$  (the products being taken in  $S(\mathfrak{g})$ ). Therefore it is clear that  $p_0 - u \in \mathfrak{k}S_{d-1}(\mathfrak{g}) + S_{d-1}(\mathfrak{g})\mathfrak{n}$ , where  $S_e(\mathfrak{g})$  denotes the space of homogeneous elements in  $S(\mathfrak{g})$  of degree  $e$ . But then

$$q_0 - u \in \lambda(\mathfrak{k}S_{d-1}(\mathfrak{g}) + S_{d-1}(\mathfrak{g})\mathfrak{n}).$$

On the other hand if  $f_1 \in S_{e_1}(\mathfrak{g})$ ,  $f_2 \in S_{e_2}(\mathfrak{g})$ , we know (see [5(a), p. 902]) that

$$\lambda(f_1 f_2) - \lambda(f_1)\lambda(f_2) \in \sum_{e < e_1 + e_2} \lambda(S_e(\mathfrak{g})).$$

Therefore

$$q_0 - u \in \mathfrak{k}\mathfrak{P} + \mathfrak{P}\mathfrak{n} + \sum_{e < d} \lambda(S_e(\mathfrak{g})).$$

Hence it follows from Lemma 3 and the definition of  $\gamma$  that  $\gamma(q_0) - u$  is of degree  $< d$ . The proof of Theorem 1 is now complete.<sup>20</sup>

The following result, which we note here for later use, has been obtained during the above proof.

**LEMMA 20.** *Let  $\lambda$  denote the canonical mapping of  $S(\mathfrak{g})$  onto  $\mathfrak{B}$ . Then if  $p$  is any homogeneous element of degree  $d$  in  $I(\mathfrak{p})$ ,  $\gamma(\lambda(p)) - \bar{p}$  is of degree  $< d$ . (Here  $\bar{p}$  denotes the restriction of  $p$  on  $\mathfrak{h}_{\mathfrak{p}}$ .)*

**5. The mapping  $\mathfrak{D}'$ .** Put  $\Delta(H) = \prod_{\alpha \in P_+} (e^{\alpha(H)} - e^{-\alpha(H)})$  ( $H \in \mathfrak{h}_{\mathfrak{p}}$ ) and  $\Delta(h) = \Delta(\log h)$  ( $h \in A_{\mathfrak{p}}$ ). We call a point  $h$  in  $A_{\mathfrak{p}}$  singular or regular according as  $\Delta(h) = 0$  or not. Let  $A_{\mathfrak{p}}'$  denote the set of regular points in  $A_{\mathfrak{p}}$ . We regard  $\mathfrak{g}$  as a Hilbert space under the positive-definite Hermitian form

<sup>20</sup> Although our definition of  $\gamma$  makes use of an order in the space of real linear functions on  $\mathfrak{h}_{\mathfrak{p}}$ , it can be shown that  $\gamma$  is actually independent of the choice of this order (cf. [5(g), Lemma 20]).

— $B(X, \bar{\theta}(X))$  ( $X \in \mathfrak{g}$ ) where  $\bar{\theta}$  denotes the conjugation of  $\mathfrak{g}$  with respect to its compact real form  $\mathfrak{u} = \mathfrak{k}_0 + (-1)^{\frac{1}{2}}\mathfrak{p}_0$ . Let  $\mathfrak{m}$  be the centralizer of  $\mathfrak{h}_p$  in  $\mathfrak{g}$  and  $\mathfrak{q}$  the orthogonal complement of  $\mathfrak{m}$  in  $\mathfrak{g}$ . Then  $\mathfrak{m} \cap \mathfrak{p} = \mathfrak{h}_p$  (because  $\mathfrak{h}_p$  is maximal abelian in  $\mathfrak{p}$ ) and  $\mathfrak{q} = \mathfrak{q}_\mathfrak{k} + \mathfrak{q}_\mathfrak{p}$  where  $\mathfrak{q}_\mathfrak{k} = \mathfrak{q} \cap \mathfrak{k}$  and  $\mathfrak{q}_\mathfrak{p} = \mathfrak{q} \cap \mathfrak{p}$ .

LEMMA 21. Suppose  $h \in A_p'$ . Then  $\mathfrak{g} = \text{Ad}(h^{-1})\mathfrak{q}_\mathfrak{k} + \mathfrak{h}_p + \mathfrak{k}$  where the sum is direct.

It is known [5(c), Lemma 4] that  $\dim \mathfrak{q}_\mathfrak{k} = \dim \mathfrak{q}_\mathfrak{p}$ . On the other hand  $\dim \mathfrak{g} = \dim \mathfrak{k} + \dim \mathfrak{p}$  and  $\mathfrak{p}$  is the orthogonal sum of  $\mathfrak{h}_p$  and  $\mathfrak{q}_\mathfrak{p}$ . Hence  $\dim \mathfrak{g} = \dim \mathfrak{k} + \dim \mathfrak{h}_p + \dim \mathfrak{q}_\mathfrak{p} = \dim \mathfrak{k} + \dim \mathfrak{h}_p + \dim \mathfrak{q}_\mathfrak{k}$ . Therefore it is sufficient to prove that  $\text{Ad}(h^{-1})\mathfrak{q}_\mathfrak{k} \cap (\mathfrak{h}_p + \mathfrak{k}) = \{0\}$ . Let  $X$  be an element of  $\mathfrak{q}_\mathfrak{k}$  such that  $\text{Ad}(h^{-1})X \in \mathfrak{h}_p + \mathfrak{k}$  and put  $Y = (\text{ad } H)^2 X$  where  $H$  is some element in  $\mathfrak{h}_{p_0}$ . Then  $Y \in \mathfrak{k}$  and  $\text{Ad}(h^{-1})Y \in \mathfrak{k}$ . Therefore  $\text{Ad}(h)Y = \theta(\text{Ad}(h^{-1})Y) = \text{Ad}(h^{-1})Y$  where  $\theta$  is defined as usual (see [5(c), p. 187]). But since  $h \in A_p'$ ,  $\mathfrak{m}$  is the centralizer of  $h^2$  in  $\mathfrak{g}$  and therefore  $Y \in \mathfrak{m}$ . This means that  $(\text{ad } H)^3 X = (\text{ad } H)Y = 0$ . However  $\text{ad } H$  is a self-adjoint operator on  $\mathfrak{g}$  and so we conclude that  $[H, X] = 0$ . This being true for every  $H \in \mathfrak{h}_{p_0}$ , it follows that  $X \in \mathfrak{m}$ . But  $\mathfrak{m} \cap \mathfrak{q} = \{0\}$ , and therefore  $X = 0$ . This proves the lemma.

Let  $\mathfrak{Q}_\mathfrak{k}$  be the image of  $^{11} S(\mathfrak{q}_\mathfrak{k})$  in  $\mathfrak{B}$  under the canonical mapping.

COROLLARY 1. The mapping<sup>12</sup>  $q \mathbf{X} v \mathbf{X} x \rightarrow q^{h^{-1}} v x$  ( $q \in \mathfrak{Q}_\mathfrak{k}, v \in \mathfrak{S}_p, x \in \mathfrak{X}$ ) defines a linear isomorphism of the tensor product  $\mathfrak{Q}_\mathfrak{k} \mathbf{X} \mathfrak{S}_p \mathbf{X} \mathfrak{X}$  onto  $\mathfrak{B}$  if  $h \in A_p'$ .

This is an immediate consequence of the above result and Lemma 12 of [5(c)].

Put  $\mathfrak{m}_\mathfrak{k} = \mathfrak{m} \cap \mathfrak{k}$  and let  $\mathfrak{M}_\mathfrak{k}$  be the subalgebra of  $\mathfrak{B}$  generated by  $(1, \mathfrak{m}_\mathfrak{k})$ . Since  $\mathfrak{k} = \mathfrak{m}_\mathfrak{k} + \mathfrak{q}_\mathfrak{k}$ , it follows from Lemma 12 of [5(c)] that  $\mathfrak{X} = \mathfrak{Q}_\mathfrak{k} \mathfrak{M}_\mathfrak{k}$ . Let  $\mathfrak{Q}_\mathfrak{k}'$  denote the set of those elements in  $\mathfrak{Q}_\mathfrak{k}$  whose homogeneous component<sup>21</sup> of degree zero is zero. Since  $[\mathfrak{m}_\mathfrak{k}, \mathfrak{q}_\mathfrak{k}] \subset \mathfrak{q}_\mathfrak{k}$ , it follows without difficulty that

$$\mathfrak{X}\mathfrak{k} = \mathfrak{k}\mathfrak{X} = \mathfrak{Q}_\mathfrak{k}'\mathfrak{M}_\mathfrak{k} + \mathfrak{Q}_\mathfrak{k}\mathfrak{M}_\mathfrak{k}\mathfrak{m}_\mathfrak{k}.$$

Hence in particular  $\mathfrak{k}\mathfrak{Q}_\mathfrak{k} \subset \mathfrak{Q}_\mathfrak{k}'\mathfrak{M}_\mathfrak{k} + \mathfrak{Q}_\mathfrak{k}\mathfrak{M}_\mathfrak{k}\mathfrak{m}_\mathfrak{k}$ . Moreover  $\mathfrak{B} = (\mathfrak{Q}_\mathfrak{k})^{h^{-1}}\mathfrak{S}_p\mathfrak{X}$  from the above corollary and therefore  $(\text{Ad}(h^{-1})\mathfrak{k})\mathfrak{B} = (\mathfrak{k}\mathfrak{Q}_\mathfrak{k})^{h^{-1}}\mathfrak{S}_p\mathfrak{X}$ . Since  $[\mathfrak{m}, \mathfrak{h}_p] = \{0\}$ , this implies that

<sup>21</sup> Define  $\lambda$  and  $S_d(\mathfrak{g})$  as during the proof of Lemma 19. Then  $\mathfrak{B}$  is the direct sum of  $\lambda(S_d(\mathfrak{g}))$  ( $d \geq 0$ ) and an element  $b \in \mathfrak{B}$  is homogeneous of degree  $d$  if  $b \in \lambda(S_d(\mathfrak{g}))$ .

$$\mathfrak{f}^{h^{-1}}\mathfrak{B} \subset (\Omega_{\mathfrak{f}}')^{h^{-1}}\mathfrak{S}_{\mathfrak{p}}\mathfrak{X} + \mathfrak{B}\mathfrak{f}$$

and therefore

$$\mathfrak{f}^{h^{-1}}\mathfrak{B} + \mathfrak{B}\mathfrak{f} = (\Omega_{\mathfrak{f}}')^{h^{-1}}\mathfrak{S}_{\mathfrak{p}}\mathfrak{X} + (\Omega_{\mathfrak{f}}')^{h^{-1}}\mathfrak{S}_{\mathfrak{p}}\mathfrak{X}\mathfrak{f}.$$

Combining this with Corollary 1 above, we get the following result.

COROLLARY 2.  $\mathfrak{B}$  is the direct sum of  $\mathfrak{S}_{\mathfrak{p}}$  and  $\mathfrak{f}^{h^{-1}}\mathfrak{B} + \mathfrak{B}\mathfrak{f}$  if  $h \in A_{\mathfrak{p}}'$ .

But this obviously implies the lemma below.

LEMMA 22. Let  $h$  be an element in  $A_{\mathfrak{p}}'$ . Then for any  $b \in \mathfrak{B}$  there exists a unique element  $\delta_h'(b) \in \mathfrak{S}_{\mathfrak{p}}$  such that

$$b - \delta_h'(b) \in \mathfrak{f}^{h^{-1}}\mathfrak{B} + \mathfrak{B}\mathfrak{f}.$$

The significance of  $\delta_h'(b)$  is given by the following lemma.<sup>22</sup>

LEMMA 23. Let  $\phi$  be a spherical function on  $G$  of class  $C^\infty$ . Then<sup>11</sup>

$$\phi(h; b) = \phi(h; \delta_h'(b))$$

for  $b \in \mathfrak{B}$  and  $h \in A_{\mathfrak{p}}'$ .

Since  $\phi$  is spherical, it is obvious that  $\phi(h; b') = 0$  for  $b' \in \mathfrak{f}^{h^{-1}}\mathfrak{B} + \mathfrak{B}\mathfrak{f}$ . Therefore our assertion follows from the definition of  $\delta_h'(b)$ .

Now we regard  $A_{\mathfrak{p}}'$  as an open submanifold (see [5(g), p. 110]) of the Lie group  $A_{\mathfrak{p}}$ . Then if  $b \in \mathfrak{B}$ , we intend to show that there exists a differential operator  $\delta'(b)$  on  $A_{\mathfrak{p}}'$  whose local expression (see [5(g), p. 112]) at any point  $h \in A_{\mathfrak{p}}'$  coincides with  $\delta_h'(b)$ .

Define  $\pi$  as in Lemma 11 and let  $\mathfrak{h}_{\mathfrak{p}_0}'$  be the set of those points  $H \in \mathfrak{h}_{\mathfrak{p}_0}$  where  $\pi(H) \neq 0$ . It is obvious that  $\mathfrak{h}_{\mathfrak{p}_0}'$  is mapped onto  $A_{\mathfrak{p}}'$  under the exponential mapping. Let  $\mathfrak{R}$  be the ring of analytic functions on  $A_{\mathfrak{p}}'$  generated<sup>23</sup> (over  $C$ ) by the functions  $g_{\beta}$  ( $\beta \in P_+$ ) defined by

$$g_{\beta}(\exp H) = (e^{2\beta(H)} - 1)^{-1} \quad (H \in \mathfrak{h}_{\mathfrak{p}_0}').$$

Note that  $Hg_{\beta} = -2\beta(H)(g_{\beta} + g_{\beta}^2)$  for  $H \in \mathfrak{h}_{\mathfrak{p}}$  and therefore  $\mathfrak{R}$  is stable under the action of any differential operator in  $\mathfrak{S}_{\mathfrak{p}}$ . Let  $\mathfrak{B}_d$  denote the space of all elements in  $\mathfrak{B}$  of degree  $\leq d$ .

LEMMA 24. Let  $b$  be an element in  $\mathfrak{B}_d$ . Then we can choose  $u \in \mathfrak{S}_{\mathfrak{p}} \cap \mathfrak{B}_d$  and a finite set of elements  $v_i \in \mathfrak{S}_{\mathfrak{p}} \cap \mathfrak{B}_{d-1}$  and  $g_i \in \mathfrak{R}$  ( $1 \leq i \leq r$ ) such that

<sup>22</sup> On account of our identification of  $\mathfrak{S}_{\mathfrak{p}}$  with  $S(\mathfrak{h}_{\mathfrak{p}})$ , elements of  $\mathfrak{S}_{\mathfrak{p}}$  can appear in two different roles, either as differential operators on  $G$  (or  $A_{\mathfrak{p}}$ ) or as polynomial functions on  $\mathfrak{h}_{\mathfrak{p}}$ . They should always be interpreted as differential operators unless some qualification to the contrary is made.

<sup>23</sup> Note that the constant function 1 is not included among the generators of  $\mathfrak{R}$ .

$$b - u - \sum_{1 \leq i \leq r} g_i(h) v_i \in \mathfrak{B}\mathfrak{f} + \mathfrak{f}^{h^{-1}}\mathfrak{B}$$

for all  $h \in A_{\mathfrak{p}}'$ .

We shall use induction on  $d$ . Since  $\mathfrak{g} = \mathfrak{f} + \mathfrak{h}_{\mathfrak{p}} + \theta(\mathfrak{n})$  and the sum is direct, it follows from [5(c), Lemma 12] that<sup>11</sup>

$$\mathfrak{B}_d = \sum_{p+q+r \leq d} \lambda(S_p(\theta(\mathfrak{n}))) \lambda(S_q(\mathfrak{h}_{\mathfrak{p}})) \lambda(S_r(\mathfrak{f}))$$

where  $\lambda$  is the canonical mapping of  $S(\mathfrak{g})$  onto  $\mathfrak{B}$  and  $S_i$  refers to the space of homogeneous elements of degree  $i$ . Hence we can select  $u \in \mathfrak{S}_{\mathfrak{p}} \cap \mathfrak{B}_d$  and  $b_{\beta} \in \mathfrak{B}_{d-1}$  such that

$$b - u - \sum_{\beta \in P_+} \theta(X_{\beta}) b_{\beta} \in \mathfrak{B}\mathfrak{f}.$$

Now let  $X_{\beta} = Y_{\beta} + Z_{\beta}$  ( $Y_{\beta} \in \mathfrak{p}$ ,  $Z_{\beta} \in \mathfrak{f}$ ). Then if  $h \in A_{\mathfrak{p}}'$  and  $H = \log h$ ,

$$\begin{aligned} \text{Ad}(h^{-1})Z_{\beta} &= \frac{1}{2}(\text{Ad}(h^{-1})X_{\beta} + \theta(\text{Ad}(h)X_{\beta})) \\ &= -\frac{1}{2}(e^{\beta(H)} - e^{-\beta(H)})Y_{\beta} + \frac{1}{2}(e^{\beta(H)} + e^{-\beta(H)})Z_{\beta}. \end{aligned}$$

Hence  $Y_{\beta} \equiv (1 + 2g_{\beta}(h))Z_{\beta} \pmod{\mathfrak{f}^{h^{-1}}}$ . Therefore since  $\theta(X_{\beta}) = -Y_{\beta} + Z_{\beta}$ , we get the following result which will be useful later on.

**LEMMA 25.** *For any  $\beta \in P_+$  let  $X_{\beta} = Y_{\beta} + Z_{\beta}$  where  $Y_{\beta} \in \mathfrak{p}$  and  $Z_{\beta} \in \mathfrak{f}$ . Then  $\theta(X_{\beta}) \equiv -2g_{\beta}(h)Z_{\beta} \pmod{\mathfrak{f}^{h^{-1}}}$  for  $h \in A_{\mathfrak{p}}'$ .*

This means that

$$b \equiv u - 2 \sum_{\beta \in P_+} g_{\beta}(h) [Z_{\beta}, b_{\beta}] \pmod{(\mathfrak{B}\mathfrak{f} + \mathfrak{f}^{h^{-1}}\mathfrak{B})}$$

for all  $h \in A_{\mathfrak{p}}'$ . But  $[Z_{\beta}, b_{\beta}] \in \mathfrak{B}_{d-1}$  and so the assertion of Lemma 24 follows immediately by induction hypothesis.

Now it is obvious from the definition of  $\delta'_h(b)$  that  $\delta'_h(b) = u + \sum_{1 \leq i \leq r} g_i(h) v_i$  for  $h \in A_{\mathfrak{p}}'$ , in the notation of Lemma 24. Therefore since  $g_i$  are analytic functions on  $A_{\mathfrak{p}}'$ ,  $\delta'(b) = u + \sum_{1 \leq i \leq r} g_i v_i$  is an analytic differential operator on  $A_{\mathfrak{p}}'$  whose local expression at  $\tilde{h}$  is  $\delta'_h(b)$ .

**6. The relation between  $\gamma$  and  $\delta$ .** Let  $\mu$  be a linear function on  $\mathfrak{h}_{\mathfrak{p}}$ . Then  $e^{\mu}$  is a holomorphic function on the complex Euclidean space  $\mathfrak{h}_{\mathfrak{p}}$ . We agree to denote the function  $h \rightarrow e^{\mu(\log h)}$  on  $A_{\mathfrak{p}}$  also by  $e^{\mu}$ . Now put<sup>24</sup>  $\delta(b) = e^{\rho} \delta'(b) \circ e^{-\rho}$  ( $b \in \mathfrak{B}$ ) where  $\rho$  is defined as in Section 4. The following lemma establishes a connection between  $\delta(q)$  and  $\gamma(q)$  for  $q \in I_{\mathfrak{g}}$ .

<sup>24</sup> Whenever it is necessary to avoid confusion we denote the operator product of two differential operators  $D, D'$  by  $D \circ D'$ . Since a function  $f$  of class  $C^{\infty}$  can also be regarded as a differential operator, one has to distinguish between  $Df$  and  $D \circ f$ .

LEMMA 26. Let  $q$  be an element in  $I_{\mathfrak{g}}$  of degree  $d$ . Then we can select a finite number of elements  $v_i \in \mathfrak{S}_{\mathfrak{p}} \cap \mathfrak{B}_{d-1}$  and  $g_i \in \mathfrak{R}$  ( $1 \leq i \leq r$ ) such that

$$\delta(q) = \gamma(q) + \sum_{1 \leq i \leq r} g_i v_i$$

on  $A_{\mathfrak{p}}'$ .

We have seen in Section 4 that  $I_{\mathfrak{g}} = I_{\mathfrak{g}} \cap \mathfrak{P} + I_{\mathfrak{g}} \cap \mathfrak{B}\mathfrak{f}$ . Moreover it is easy to see that  $\mathfrak{B}$  is the direct sum of  $\mathfrak{P}$  and  $\mathfrak{B}\mathfrak{f}$  and

$$\mathfrak{B}_d = (\mathfrak{B}_d \cap \mathfrak{P}) + (\mathfrak{B}_d \cap \mathfrak{B}\mathfrak{f}).$$

Therefore if  $q_1, q_2$  are the components of  $q$  in  $I_{\mathfrak{g}} \cap \mathfrak{P}$  and  $I_{\mathfrak{g}} \cap \mathfrak{B}\mathfrak{f}$  respectively, they are both of degree  $\leq d$ . Also  $\delta(q_2) = 0$ ,  $\gamma(q_2) = 0$  and every homogeneous component of  $q_1$  lies in  $I_{\mathfrak{g}} \cap \mathfrak{P}$ . Hence it would clearly be sufficient to consider the case when  $q \in I_{\mathfrak{g}} \cap \mathfrak{P}$ .

Let  $b \rightarrow b^*$  denote the anti-automorphism of  $\mathfrak{B}$  given by  $X^* = -X$  ( $X \in \mathfrak{g}$ ). Then if  $p$  is a homogeneous polynomial in  $S(\mathfrak{g})$  of degree  $r$  and  $\lambda$  the canonical mapping of  $S(\mathfrak{g})$  into  $\mathfrak{B}$ , it is clear that  $\lambda(p)^* = (-1)^r \lambda(p)$ . But  $X^* = \theta(X)$  for  $X \in \mathfrak{p}$ . Hence if we extend  $\theta$  to an automorphism of  $\mathfrak{B}$ , it follows that  $\theta(p) = p^*$  for  $p \in \mathfrak{P}$ . Let  $\gamma'(q)$  denote the element in  $\mathfrak{S}_{\mathfrak{p}}$  such that  $q - \gamma'(q) \in \mathfrak{B}\mathfrak{n} + \mathfrak{f}\mathfrak{B}$  (see Lemma 3). Then  $q^* - (\gamma'(q))^* \in \mathfrak{n}\mathfrak{B} + \mathfrak{B}\mathfrak{f}$  and therefore

$$q - \gamma'(q) = \theta(q^*) - \theta(\gamma'(q)^*) \in \theta(\mathfrak{n})\mathfrak{B} + \mathfrak{B}\mathfrak{f}.$$

Therefore we can select  $b_{\beta} \in \mathfrak{B}_{d-1}$  such that

$$q - \gamma'(q) - \sum_{\beta \in P_+} \theta(X_{\beta}) b_{\beta} \in \mathfrak{B}\mathfrak{f}.$$

But then from Lemma 25,

$$q = \gamma'(q) - \sum_{\beta \in P_+} g_{\beta}(h) b_{\beta}' \text{ mod } (\mathfrak{B}\mathfrak{f} + \mathfrak{f}^{h^{-1}}\mathfrak{B})$$

for  $h \in A_{\mathfrak{p}}'$  where  $b_{\beta}' = 2[Z_{\beta}, b_{\beta}] \in \mathfrak{B}_{d-1}$ . Therefore by Lemma 24 we can select a finite number of elements  $v_i' \in \mathfrak{S}_{\mathfrak{p}} \cap \mathfrak{B}_{d-1}$  and  $g_i \in \mathfrak{R}$  ( $1 \leq i \leq r$ ) such that

$$q = \gamma'(q) + \sum_{1 \leq i \leq r} g_i(h) v_i' \text{ mod } (\mathfrak{B}\mathfrak{f} + \mathfrak{f}^{h^{-1}}\mathfrak{B}) \quad (h \in A_{\mathfrak{p}}').$$

But this means that  $\delta'(q) = \gamma'(q) + \sum_i g_i v_i'$  and therefore

$$\delta(q) = \gamma(q) + \sum_{1 \leq i \leq r} g_i v_i$$

where  $v_i = e^{\rho} v_i' \circ e^{-\rho}$ . Since it is obvious that  $v_i \in \mathfrak{S}_{\mathfrak{p}}$ , our assertion is proved.

Now select homogeneous elements  $u_1 = 1, u_2, \dots, u_{10}$  in  $\mathfrak{S}_{\mathfrak{p}}$  in accordance

with the corollary of Lemma 8, so that  $\mathfrak{S}_p = \sum_{1 \leq i \leq w} J u_i$  where  $J = I(\mathfrak{h}_p)$ . We know from Theorem 1 that  $\gamma(I_{\mathfrak{g}}) = J$ .

**THEOREM 2.** *Let  $u$  be an element in  $\mathfrak{S}_p$  and select  $q_i \in I_{\mathfrak{g}}$  ( $1 \leq i \leq w$ ) such that  $u = \sum_{1 \leq i \leq w} \gamma(q_i) u_i$ . Then we can choose a finite number of elements  $g_{ij} \in \mathfrak{R}$  and  $q_{ij} \in I_{\mathfrak{g}}$  ( $1 \leq i \leq w, 1 \leq j \leq r$ ) such that*

$$u = \sum_{1 \leq i \leq w} u_i \circ \delta(q_i) + \sum_{1 \leq i \leq w} \sum_{1 \leq j \leq r} g_{ij} u_i \circ \delta(q_{ij})$$

on  $A_p'$ .

We shall use induction on the degree  $d$  of  $u$ . Obviously  $u$  can be assumed to be homogeneous. Put  $I_p = I_{\mathfrak{g}} \cap \mathfrak{P}$ . If we replace each  $q_i$  by its component in  $I_p$  (with respect to the direct sum  $\mathfrak{B} = \mathfrak{P} + \mathfrak{B}'$ ), neither  $\gamma(q_i)$  nor  $\delta(q_i)$  is affected. Hence we can suppose that  $q_i \in I_p$ . Let  $d_i$  denote the degree of  $u_i$ . Then it follows from our assumptions (see the corollary of Lemma 8) that  $\gamma(q_i)$  is homogeneous of degree  $d - d_i$  if  $d \geq d_i$  and  $\gamma(q_i) = 0$  if  $d < d_i$ . Now fix  $i$  and first suppose  $d \geq d_i$ . Then by Lemma 20,  $q_i$  is of degree  $d - d_i$  and  $\gamma(q_i) u_i = u_i \circ \delta(q_i) - u_i \circ (\delta(q_i) - \gamma(q_i))$  on  $A_p'$ . Since  $\mathfrak{R}$  is stable under the operations of  $\mathfrak{S}_p$ , it follows from Lemma 26 that  $u_i \circ (\delta(q_i) - \gamma(q_i)) = \sum_{1 \leq j \leq r} g_j v_j$  where  $g_j \in \mathfrak{R}$  and  $v_j \in \mathfrak{S}_p \cap \mathfrak{B}_{d-1}$ . On the other hand if  $d < d_i$ ,  $\gamma(q_i) = 0$  and therefore  $q_i = 0$  by Lemma 19. Hence  $\gamma(q_i) u_i = u_i \circ \delta(q_i) = 0$  in this case. This shows that we can select a finite set of elements  $g_j \in \mathfrak{R}$  and  $v_j \in \mathfrak{S}_p \cap \mathfrak{B}_{d-1}$  ( $1 \leq j \leq s$ ) such that

$$u = \sum_{1 \leq i \leq w} u_i \circ \delta(q_i) + \sum_{1 \leq j \leq s} g_j v_j$$

on  $A_p'$ . The statement of the theorem now follows immediately by applying the induction hypothesis to  $v_j$ .

The following consequence of this theorem will be important for our applications.

**COROLLARY.** *Let  $U$  be a nonempty open connected subset of  $A_p'$  and  $\chi$  a homomorphism of  $I_{\mathfrak{g}}$  into  $C$ . Let  $E_{\chi}$  denote the space of all analytic functions  $\phi$  on  $U$  which satisfy the system of differential equations  $\delta(q)\phi = \chi(q)\phi$  ( $q \in I_{\mathfrak{g}}$ ). Then the dimension of  $E_{\chi}$  over  $C$  cannot exceed  $w$ .*

Choose a point  $h_0$  in  $U$ . If  $\dim E_{\chi} > w$ , we can obviously find a function  $\phi \neq 0$  in  $E_{\chi}$  such that  $\phi(h_0; u_i) = 0$  ( $1 \leq i \leq w$ ). But then it follows from the above theorem that  $\phi(h_0; u) = 0$  for every  $u \in \mathfrak{S}_p$ . However since  $U$  is connected and  $\phi$  is analytic, this implies that  $\phi = 0$  and so we get a contradiction. Hence  $\dim E_{\chi} \leq w$ .

**7. The operator  $\delta'(\omega)$ .** The Casimir operator  $\omega$  of  $\mathfrak{g}$  is an element lying in the center of  $\mathfrak{B}$  which is defined as follows. Let  $X_1, \dots, X_n$  be a base for  $\mathfrak{g}$  and put  $g_{ij} = B(X_i, X_j)$   $1 \leq i, j \leq n$ . Then the matrix  $(g_{ij})_{1 \leq i, j \leq n}$  is nonsingular and if  $(g^{ij})_{1 \leq i, j \leq n}$  denotes its inverse,  $\omega = \sum_{1 \leq i, j \leq n} g^{ij} X_i X_j$ . (It is easy to check that  $\omega$  does not depend on the choice of the base used in its definition.) We intend to compute  $\delta'(\omega)$ .

For any linear function  $\mu$  on  $\mathfrak{h}_p$ , let  $H_\mu$  denote the unique element in  $\mathfrak{h}_p$  such that  $B(H, H_\mu) = \mu(H)$  for all  $H \in \mathfrak{h}_p$ . Also put  $\gamma'(q) = e^{-\rho} \gamma(q) \circ e^\rho$  for  $q \in I_{\mathfrak{g}}$ .

**LEMMA 27.**  $\delta'(\omega) = \gamma'(\omega) + 2 \sum_{\alpha \in P_+} (e^{2\tilde{\alpha}} - 1)^{-1} H_{\tilde{\alpha}}$  where  $\tilde{\alpha}$  denotes the restriction of  $\alpha$  on  $\mathfrak{h}_p$ .

For each root  $\alpha$ , select  $X_\alpha$  as before and normalize it in such a way that  $B(X_\alpha, X_{-\alpha}) = 1$  for any  $\alpha \in P$ . Then  $[X_\alpha, X_{-\alpha}] = H_\alpha$  where  $H_\alpha$  is the element in  $\mathfrak{h}$  such that  $B(H, H_\alpha) = \alpha(H)$  for every  $H \in \mathfrak{h}$ . Choose bases  $H_1, \dots, H_l$  and  $H_{l+1}, \dots, H_m$  for  $\mathfrak{h}_p$  and  $\mathfrak{h}_t$  respectively such that  $B(H_i, H_j) = \delta_{ij}$  ( $1 \leq i, j \leq m$ ). Then  $H_1, \dots, H_m$  together with  $X_\alpha, X_{-\alpha}$  ( $\alpha \in P$ ), form a base for  $\mathfrak{g}$  and therefore it is clear that

$$\begin{aligned} \omega &= H_1^2 + \dots + H_m^2 + \sum_{\alpha \in P} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) \\ &= H_1^2 + \dots + H_m^2 + 2 \sum_{\alpha \in P} X_{-\alpha} X_\alpha + \sum_{\alpha \in P} H_\alpha \\ &\equiv H_1^2 + \dots + H_l^2 + 2H_\rho + 2 \sum_{\alpha \in P_+} X_{-\alpha} X_\alpha \pmod{\mathfrak{f}\mathfrak{B} \cap \mathfrak{B}\mathfrak{f}}, \end{aligned}$$

since  $X_\alpha, X_{-\alpha}$  and  $H_\alpha$  lie in  $\mathfrak{f}$  if  $\alpha$  vanishes identically on  $\mathfrak{h}_p$ . This shows that  $\omega \equiv H_1^2 + \dots + H_l^2 + 2H_\rho \pmod{\mathfrak{f}\mathfrak{B} + \mathfrak{B}\mathfrak{n}}$  and hence  $\gamma'(\omega) = H_1^2 + \dots + H_l^2 + 2H_\rho$ . Now for any root  $\alpha$ , which is not identically zero on  $\mathfrak{h}_p$ , let  $X_\alpha = Y_\alpha + Z_\alpha$  ( $Y_\alpha \in \mathfrak{p}, Z_\alpha \in \mathfrak{f}$ ) and put  $g_\alpha(\exp H) = (e^{2\alpha(H)} - 1)^{-1}$  ( $H \in \mathfrak{h}_{p_0'}$ ). We have seen in Section 5 that  $Y_\alpha \equiv (1 + 2g_\alpha(h))Z_\alpha \pmod{\mathfrak{f}^{h-1}}$  for  $h \in A_{p'}$ . Hence

$$\begin{aligned} X_\alpha X_{-\alpha} &= X_\alpha Y_{-\alpha} \equiv 2(1 + g_\alpha(h))Z_\alpha Y_{-\alpha} \\ &\equiv 2(1 + g_\alpha(h))[Z_\alpha, Y_{-\alpha}] \pmod{\mathfrak{f}^{h-1}\mathfrak{B} + \mathfrak{B}\mathfrak{f}}. \end{aligned}$$

If we apply the automorphism  $\theta$  to this congruence, we get

$$\theta(X_\alpha X_{-\alpha}) \equiv -2(1 + g_\alpha(h))[Z_\alpha, Y_{-\alpha}] \pmod{\mathfrak{f}^{h-1}\mathfrak{B} + \mathfrak{B}\mathfrak{f}}.$$

Replacing  $\alpha$  and  $h$  by  $-\alpha$  and  $h^{-1}$  respectively, we find that

$$\theta(X_{-\alpha} X_\alpha) \equiv -2(1 + g_\alpha(h))[Z_{-\alpha}, Y_\alpha] \pmod{\mathfrak{f}^{h-1}\mathfrak{B} + \mathfrak{B}\mathfrak{f}}.$$

This shows that

$$\begin{aligned} \frac{1}{2}\{X_{\alpha}X_{-\alpha} + \theta(X_{-\alpha}X_{\alpha})\} \\ \equiv (1 + g_{\alpha}(h))\{[Z_{\alpha}, Y_{-\alpha}] - [Z_{-\alpha}, Y_{\alpha}]\} \bmod (\mathfrak{f}^{h-1}\mathfrak{B} + \mathfrak{B}\mathfrak{f}). \end{aligned}$$

But  $[X_{\alpha}, X_{-\alpha}] = H_{\alpha}$  and therefore

$$[Z_{\alpha}, Y_{-\alpha}] + [Y_{\alpha}, Z_{-\alpha}] = \frac{1}{2}(H_{\alpha} - \theta(H_{\alpha})) = H_{\alpha}.$$

This proves that

$$\frac{1}{2}\{X_{\alpha}X_{-\alpha} + \theta(X_{-\alpha}X_{\alpha})\} \equiv (1 + g_{\alpha}(h))H_{\alpha} \bmod (\mathfrak{f}^{h-1}\mathfrak{B} + \mathfrak{B}\mathfrak{f})$$

for  $h \in A_{\mathfrak{p}}'$ . On the other hand it is easy to verify that  $\theta(\omega) = \omega$ . Therefore

$$\begin{aligned} \omega &= \frac{1}{2}\{\omega + \theta(\omega)\} \\ &\equiv (H_1^2 + \cdots + H_l^2) + \frac{1}{2} \sum_{\alpha \in P_+} \{X_{\alpha}X_{-\alpha} + X_{-\alpha}X_{\alpha} \\ &\quad + \theta(X_{\alpha}X_{-\alpha} + X_{-\alpha}X_{\alpha})\} \bmod \mathfrak{B}\mathfrak{f}. \end{aligned}$$

But in view of the above result, this implies that

$$\omega \equiv (H_1^2 + \cdots + H_l^2) + \sum_{\alpha \in P_+} (1 + 2g_{\alpha}(h))H_{\alpha} \bmod (\mathfrak{f}^{h-1}\mathfrak{B} + \mathfrak{B}\mathfrak{f})$$

since  $g_{\alpha} - g_{-\alpha} = 1 + 2g_{\alpha}$ . Hence

$$\omega \equiv \gamma'(\omega) + 2 \sum_{\alpha \in P_+} g_{\alpha}(h)H_{\alpha} \bmod (\mathfrak{f}^{h-1}\mathfrak{B} + \mathfrak{B}\mathfrak{f})$$

for  $h \in A_{\mathfrak{p}}'$  and this proves the lemma.

**COROLLARY 1.** Let  $H_1, \dots, H_l$  be any base for  $\mathfrak{h}_{\mathfrak{p}}$  and let  $(g^{ij})_{1 \leq i, j \leq l}$  denote the inverse of the matrix  $g_{ij} = B(H_i, H_j)$  ( $1 \leq i, j \leq l$ ). Then

$$\delta'(\omega) = \Delta^{-1} \sum_{1 \leq i, j \leq l} g^{ij} H_i \circ \Delta H_j.$$

It is easy to check that the right side is actually independent of the choice of the base  $H_1, \dots, H_l$ . Hence it is sufficient to consider the base used in the proof of the above lemma. But then one finds by direct calculation that

$$\begin{aligned} \Delta^{-1} \sum_{1 \leq i, j \leq l} g^{ij} H_i \circ \Delta H_j &= (H_1^2 + \cdots + H_l^2) + \sum_{\alpha \in P_+} \{(e^{\bar{\alpha}} + e^{-\bar{\alpha}})/(e^{\bar{\alpha}} - e^{-\bar{\alpha}})\} H_{\alpha} \\ &= \gamma'(\omega) + 2 \sum_{\alpha \in P_+} (e^{2\bar{\alpha}} - 1)^{-1} H_{\alpha} \end{aligned}$$

and this proves our assertion.

Let  $\bar{\omega}$  denote the restriction on  $\mathfrak{h}_{\mathfrak{p}}$  of the Casimir polynomial of  $\mathfrak{g}$  (see [5(k), p. 98]).

COROLLARY 2.  $\gamma(\omega) = \bar{\omega} - \langle \rho, \rho \rangle$ .

We keep to the notation of the proof of Lemma 27. Since  $\gamma'(\omega) = (H_1^2 + \cdots + H_l^2) + 2H_\rho$  it is clear that

$$\begin{aligned}\gamma(\omega) &= (H_1^2 + \cdots + H_l^2) - 2 \sum_{1 \leq i \leq l} \rho(H_i) H_i + \langle \rho, \rho \rangle + 2(H_\rho - \langle \rho, \rho \rangle) \\ &= (H_1^2 + \cdots + H_l^2) - \langle \rho, \rho \rangle.\end{aligned}$$

On the other hand  $\bar{\omega} = H_1^2 + \cdots + H_l^2$  and so our assertion is obvious.

8. **Some consequences of Lemma 27.** Select  $\alpha_1, \dots, \alpha_l \in \Sigma$  as in Section 3 and let  $L$  denote the set of all linear functions on  $\mathfrak{h}_p$  of the form  $m_1 \alpha_1 + \cdots + m_l \alpha_l$  where  $m_1, \dots, m_l$  are nonnegative integers. Given two linear functions  $\lambda, \lambda'$  on  $\mathfrak{h}_p$ , we write  $\lambda < \lambda'$  (or  $\lambda' > \lambda$ ) if  $\lambda \neq \lambda'$  and  $\lambda' - \lambda \in L$ . It is clear that for any two linear functions  $\lambda_1, \lambda_2$  on  $\mathfrak{h}_p$ , there exist only a finite number of  $\lambda$  such that  $\lambda_1 < \lambda < \lambda_2$ . Define  $S = S(\mathfrak{h}_p)$  as in Section 3 and let  $Q = Q(\mathfrak{h}_p)$  denote the quotient field of  $S$ . Then  $Q$  is the field of rational functions on  $\mathfrak{h}_p$ . Introduce the scalar product  $\langle p, q \rangle$  for  $p, q \in S$  as in [5(k), p. 90] and put  $m(\lambda) = m_1 + \cdots + m_l$  for  $\lambda = m_1 \alpha_1 + \cdots + m_l \alpha_l \in L$ . Now for each  $\lambda \in L$ , we define an element  $\Gamma_\lambda \in Q$  by induction on  $m(\lambda)$  as follows.  $\Gamma_0 = 1$  and <sup>25</sup>

$$\{\langle \lambda, \lambda - 2\rho \rangle - 2\lambda\} \Gamma_\lambda = 2 \sum_{\alpha \in P_+} \sum_{k \geq 1} \Gamma_{\lambda - 2k\alpha} \{\langle \lambda - 2k\alpha, \alpha \rangle - \bar{\alpha}\},$$

where  $\bar{\alpha}$  denotes the restriction of  $\alpha$  on  $\mathfrak{h}_p$  and  $k$  runs over all positive integers such that  $\lambda - 2k\alpha \in L$ . Let  $L'$  denote the set of all elements  $\lambda \neq 0$  in  $L$ . Consider the hyperplane  $\sigma_\lambda'$  ( $\lambda \in L'$ ) consisting of those points  $H \in \mathfrak{h}_p$  where  $2\lambda(H) = \langle \lambda, \lambda - 2\rho \rangle$ . It is clear that any compact subset of  $\mathfrak{h}_p$  meets  $\sigma_\lambda'$  for only a finite number of  $\lambda$  in  $L'$ . Let  $\mathfrak{h}_p''$  denote the complement of  $\bigcup_{\lambda \in L'} \sigma_\lambda'$  in  $\mathfrak{h}_p$ . Then  $\mathfrak{h}_p''$  is an open, connected, dense subset of  $\mathfrak{h}_p$ . It is obvious that the rational function  $\Gamma_\lambda$  ( $\lambda \in L$ ) takes a well-defined value at any point  $H \in \mathfrak{h}_p''$ . We denote this value by  $\Gamma_\lambda(H)$ .

LEMMA 28. *Let  $U$  be a compact subset of  $\mathfrak{h}_p''$ . Then there exists a real number  $c \geq 1$  such that  $|\Gamma_\lambda(H)| \leq c^{m(\lambda)}$  for  $\lambda \in L$  and  $H \in U$ .*

Select positive numbers  $c_1$  and  $c_2$  such that

$$|\langle \lambda, \bar{\alpha} \rangle - \alpha(H)| \leq c_1(m(\lambda) + 1), \quad |\langle \lambda, \lambda - 2\rho \rangle - 2\lambda(H)| \geq c_2 m(\lambda)^2$$

<sup>25</sup> As will become apparent during the proof of Lemma 29, the motivation for this definition comes from Lemma 27.

for every  $\lambda \in L$ ,  $\alpha \in P_+$  and  $H \in U$ . Obviously this is possible. Now if  $\lambda' \ll \lambda$  ( $\lambda' \in L$ ), it is clear that  $m(\lambda') < m(\lambda)$  and therefore

$$|\langle \lambda, \tilde{\alpha} \rangle - \alpha(H)| |\langle \lambda, \lambda - 2\rho \rangle - 2\lambda(H)|^{-1} \leq c_3/m(\lambda) \quad (H \in U)$$

where  $c_3 = 2c_1/c_2$ . Hence it follows that

$$|\Gamma_\lambda(H)| \leq 2c_3(m(\lambda))^{-1} \sum_{\alpha \in P_+} \sum_{k \geq 1} |\Gamma_{\lambda-2k\tilde{\alpha}}(H)| \quad (H \in U, \lambda \in L').$$

Let  $r$  be the number of roots in  $P_+$  and put  $c = \max\{rc_3, 1\}$ . We shall prove by induction on  $m(\lambda)$  that  $|\Gamma_\lambda(H)| \leq c^{m(\lambda)}$  for  $H \in U$  and  $\lambda \in L$ . If  $\lambda = 0$ , this is true. So now suppose  $\lambda \neq 0$ . Since  $\lambda - 2k\tilde{\alpha} \in L$ , it is obvious that  $k \leq \frac{1}{2}m(\lambda)$ . Hence it follows by induction hypothesis that

$$\sum_{k \geq 1} |\Gamma_{\lambda-2k\tilde{\alpha}}(H)| \leq \frac{1}{2}m(\lambda) c^{m(\lambda)-1}$$

and therefore

$$|\Gamma_\lambda(H)| \leq c_3 r c^{m(\lambda)-1} \leq c^{m(\lambda)}$$

For any number  $M \geq 0$ , let  $\mathfrak{h}_p(M)$  denote the set of those points  $H \in \mathfrak{h}_p$  where the real part of  $\alpha_i(H)$  is greater than  $M$  for every  $i$  ( $1 \leq i \leq l$ ).

COROLLARY 1. We can choose  $M \geq 0$  such that the series

$$\sum_{\lambda \in L} |\Gamma_\lambda(H') e^{-\lambda(H)}|$$

converges uniformly for  $H \in \mathfrak{h}_p(M)$  and  $H' \in U$ .

Obviously it is sufficient to choose  $M$  so large that  $c < e^M$ .

COROLLARY 2. Let  $U'$  be an open subset of  $\mathfrak{h}_p''$  whose closure (in  $\mathfrak{h}_p''$ ) is compact. Then if  $M$  is sufficiently large, the function  $\phi'(H':H)$  ( $H' \in U'$ ,  $H \in \mathfrak{h}_p(M)$ ) defined by the series

$$\phi'(H':H) = e^{\langle H', H \rangle} \sum_{\lambda \in L} \Gamma_\lambda(H') e^{-\lambda(H)}$$

is holomorphic on  $U' \times \mathfrak{h}_p(M)$ .

This is an immediate consequence of Corollary 1.

We keep to the notation of Corollary 2 above and denote by  $A_p(M)$  the set of those points  $h \in A_p$  for which  $\log h \in \mathfrak{h}_p(M)$ . Also put  $\Phi'(H':h) = \phi'(H':\log h)$  ( $H' \in U'$ ,  $h \in A_p(M)$ ). Then  $\Phi'$  is an analytic function on  $U' \times A_p(M)$ . The following lemma provides the justification for the above construction.

LEMMA 29. Let  $q$  be an element in  $I_q$ . Then<sup>21</sup>

$$\Phi'(H':h; \delta'(q)) = \gamma'(q; H') \Phi'(H':h) \quad (H' \in U, h \in A_p(M))$$

where  $\gamma'(q; H')$  denotes the value of the polynomial function<sup>22</sup>  $\gamma'(q)$  at  $H'$ .

<sup>21</sup> We recall that  $\gamma'(q) = e^{-\rho\gamma(q)} \circ e\rho$ .

For otherwise suppose our assertion is false. Then from Lemma 26,  $\delta'(q) = \gamma'(q) + \sum_{1 \leq i \leq r} g_i v_i$  where  $g_i \in \mathfrak{R}$  and  $v_i \in \mathfrak{S}_p$ . For any  $v \in S(\mathfrak{h}_p)$ , let  $\partial(v)$  denote the corresponding differential operator on  $\mathfrak{h}_p$  (see [5(k), § 2]). Then if  $\psi$  is a holomorphic function on  $\mathfrak{h}_p(M)$  and  $\Psi(h) = \psi(\log h)$  ( $h \in A_p(M)$ ), it is obvious that  $\Psi(h; v) = \psi(\log h; \partial(v))$  for  $v \in \mathfrak{S}_p$ . Therefore in particular

$$\Psi(h; \sum_{1 \leq i \leq r} g_i v_i) = \sum_{1 \leq i \leq r} g_i(h) \psi(\log h; \partial(v_i)).$$

But  $g_\alpha(\exp H) = (e^{2\alpha(H)} - 1)^{-1} = \sum_{1 \leq k < \infty} e^{-2k\alpha(H)}$  for  $\alpha \in P_+$  and  $H \in \mathfrak{h}_p(M)$ . Hence it is obvious that we can select an element  $v_\lambda \in \mathfrak{S}_p$  for each  $\lambda \in L'$ , such that

$$\sum_{1 \leq i \leq r} g_i(\exp H) \psi(H; \partial(v_i)) = \sum_{\lambda \in L'} e^{-\lambda(H)} \psi(H; \partial(v_\lambda)) \quad (H \in \mathfrak{h}_p(M))$$

for any holomorphic function  $\psi$  on  $\mathfrak{h}_p(M)$ . Hence in particular

$$\Phi'(H'; h; \delta'(q)) - \gamma'(q; H') \Phi'(H'; h) = \phi_1(H'; \log h) \quad (h \in A_p(M))$$

where

$$\phi_1(H'; H) = \phi'(H'; H; \partial(\gamma'(q)) - \gamma'(q; H')) + \sum_{\lambda \in L'} e^{-\lambda(H)} \phi'(H'; H; \partial(v_\lambda))$$

for  $H' \in U'$  and  $H \in \mathfrak{h}_p(M)$ . Now put  $\phi_0(H'; H) = e^{-\langle H', H \rangle} \phi_1(H'; H)$ . Then it follows from our hypothesis that  $\phi_0 \neq 0$ . On the other hand it is obvious that

$$\phi'(H'; H; \partial(v)) = e^{\langle H', H \rangle} \sum_{\lambda \in L} \Gamma_\lambda(H') v(H' - H_\lambda) e^{-\lambda(H)} \quad (H' \in U', H \in \mathfrak{h}_p(M))$$

for  $v \in \mathfrak{S}_p$ . (Here  $v(H' - H_\lambda)$  denotes the value of the polynomial function  $v$  and  $H' - H_\lambda$ .) Hence

$$\begin{aligned} \phi_0(H'; H) &= \sum_{\lambda \in L} \Gamma_\lambda(H') e^{-\lambda(H)} \{ \gamma'(q; H' - H_\lambda) - \gamma'(q; H') \} \\ &\quad + \sum_{\lambda \in L} \sum_{\mu \in L'} \Gamma_\lambda(H') e^{-\lambda(H) - \mu(H)} v_\mu(H' - H_\lambda) \\ &= \sum_{\lambda \in L} e^{-\lambda(H)} c_\lambda(H') \end{aligned}$$

where

$$\begin{aligned} c_\lambda(H') &= \Gamma_\lambda(H') \{ \gamma'(q; H' - H_\lambda) - \gamma'(q; H') \} \\ &\quad + \sum_{0 < \mu < \lambda} \Gamma_\mu(H') v_{\lambda - \mu}(H' - H_\mu). \end{aligned}$$

Obviously  $c_\lambda$  is a rational function on  $\mathfrak{h}_p$  and since  $\phi_0 \neq 0$ , we can choose  $\lambda_0 \in L$  such that  $c_{\lambda_0} \neq 0$ . We shall now show that this is actually impossible.

Let  $\pi$  be an irreducible representation of  $g$  on a finite-dimensional space  $V_0$ . We assume that there exists a unit vector  $\xi$  in  $V_0$  such that  $\pi(\mathfrak{f})\xi = \{0\}$ . As usual, we denote the corresponding representation of  $G_c$  also by  $\pi$  and put  $\Xi(x) = (\xi, \pi(x)\xi)$  ( $x \in G_c$ ). Then  $\Xi$  is a holomorphic function on  $G_c$ . Let  $\Lambda_1' > \Lambda_2' > \dots > \Lambda_r'$  be all the weights of  $\pi$  and  $\Lambda = \Lambda_1 > \Lambda_2 > \dots > \Lambda_r$  all the linear functions on  $\mathfrak{h}_p$  obtained by taking the restrictions of these weights on  $\mathfrak{h}_p$ . Then we know from [5(h), Lemma 2] that  $\Lambda_i \ll \Lambda$  ( $2 \leq i \leq r$ ). Let  $V_i$  be the subspace consisting of those elements  $\eta \in V_0$  for which  $\pi(H)\eta = \Lambda_i(H)\eta$  ( $H \in \mathfrak{h}_p$ ) and let  $E_i$  denote the orthogonal projection of  $V_0$  on  $V_i$ . Then since  $\pi(H)$  is self-adjoint for  $H$  in  $\mathfrak{h}_{p_0}$ , it follows that  $V_1, \dots, V_r$  are mutually orthogonal and therefore

$$\Xi(\exp H) = \sum_{1 \leq i \leq r} |E_i \xi|^2 e^{\Lambda_i(H)} \quad (H \in \mathfrak{h}_p).$$

On the other hand put  $E = \int_K \pi(k) dk$  and let  $\eta$  be a unit vector in  $V_0$  belonging to the highest weight  $\Lambda_1'$ . Then we have seen during the proof of Lemma 5 that  $\xi = cE\eta$  where  $c$  is a nonzero complex number. This shows that  $(\xi, \eta) = (E\xi, \eta) = (\xi, E\eta) = c^{-1} \neq 0$ . But  $\eta \in V_1$  and therefore  $(\xi, \eta) = (\xi, E_1\eta) = (E_1\xi, \eta)$ . Hence  $E_1\xi \neq 0$  and so

$$\Xi(\exp H) = \sum_{1 \leq i \leq r} a_i' e^{\Lambda_i(H)} \quad (H \in \mathfrak{h}_p)$$

where  $a_1' = |E_1\xi|^2 > 0$  and  $a_i' = |E_i\xi|^2 \geq 0$  ( $2 \leq i \leq r$ ).

Let  $\Xi_0$  denote the restriction of  $\Xi$  on  $G$ . Then  $\Xi_0$  is a spherical function and therefore  $\Xi_0(h; b) = \Xi_0(h; \delta'(b))$  ( $b \in \mathfrak{B}, h \in A_p'$ ) by Lemma 23. Moreover if  $b \in I_{\mathfrak{g}}$ , it follows from Lemmas 3 and 5 that  $b\Xi_0 = \gamma'(b; H_{\Lambda})\Xi_0$ . Therefore

$$\Xi_0(h; \delta'(b)) - \gamma'(b; H_{\Lambda})\Xi_0(h) = 0 \quad (b \in I_{\mathfrak{g}}, h \in A_p').$$

Now put  $\Xi_1(H) = (a_1')^{-1} e^{-\Lambda(H)} \Xi(\exp H)$  and  $\Xi_1'(H) = \Xi(\exp H)$  ( $H \in \mathfrak{h}_p$ ). Then  $\Xi_1$  can be written in the form

$$\Xi_1(H) = 1 + \sum_{\lambda \in L'} a_{\lambda} e^{-\lambda(H)} \quad (H \in \mathfrak{h}_p)$$

where  $a_{\lambda}$  are real numbers which are zero for all  $\lambda$  in  $L'$  except a finite number. Then by applying the above relation for  $b = \omega$  and making use of Lemma 27, we find that

$$\begin{aligned} \Xi_1'(H; \partial(\gamma'(\omega))) + 2 \sum_{\alpha \in P'} \sum_{1 \leq k < \infty} e^{-2k\alpha(H)} \Xi_1'(H; \partial(H_{\alpha})) \\ = \gamma'(\omega; H_{\Lambda}) \Xi_1'(H) \end{aligned} \quad (H \in \mathfrak{h}_p(M)).$$

Hence if  $a_0 = 1$ ,

$$\sum_{\lambda \in L} a_{\lambda} \gamma'(\omega: H_{\Lambda} - H_{\lambda}) e^{-\lambda(H)} + 2 \sum_{\alpha \in P_+} \sum_{1 \leq k < \infty} e^{-2k\alpha(H)} \sum_{\lambda \in L} e^{-\lambda(H)} a_{\lambda} \langle \tilde{\alpha}, \Lambda - \lambda \rangle$$

$$= \gamma'(\omega: H_{\Lambda}) \sum_{\lambda \in L} a_{\lambda} e^{-\lambda(H)} \quad (H \in \mathfrak{h}_p(M)).$$

But  $\gamma'(\omega: H) = \langle H, H \rangle + 2\rho(H)$  ( $H \in \mathfrak{h}_p$ ) from Corollary 2 to Lemma 27. Therefore if  $E(M)$  stands for the set of all points  $H \in \mathfrak{h}_p(2M)$  where<sup>27</sup>  $\operatorname{Re} \alpha_i(H) > \frac{1}{2} |\alpha_j(H)|$  ( $1 \leq i, j \leq l$ ), it follows by applying the corollary of Lemma 57 of the Appendix (§ 15) to  $V = E(M)$  that we can equate the coefficients of  $e^{-\lambda}$  on both sides of the above equation and obtain

$$a_{\lambda} \langle \Lambda - \lambda, \Lambda - \lambda + 2\rho \rangle + 2 \sum_{\alpha \in P_+} \sum_{k \geq 1} a_{\lambda - 2k\tilde{\alpha}} \langle \tilde{\alpha}, \Lambda - \lambda + 2k\tilde{\alpha} \rangle$$

$$= a_{\lambda} \langle \Lambda, \Lambda + 2\rho \rangle \quad (\lambda \in L).$$

Here the sum is to be taken over those positive integers  $k$  for which  $\lambda - 2k\tilde{\alpha} \in L$ . This proves that

$$\{\langle \lambda, \lambda - 2\rho \rangle - 2\lambda(H_{\Lambda})\} a_{\lambda}$$

$$= 2 \sum_{\alpha \in P_+} \sum_{k \geq 1} a_{\lambda - 2k\tilde{\alpha}} \{\langle \lambda - 2k\tilde{\alpha}, \tilde{\alpha} \rangle - \tilde{\alpha}(H_{\Lambda})\}.$$

By comparing this with the recurrence relation for  $\Gamma_{\lambda}$ , it is obvious that  $a_{\lambda} = \Gamma_{\lambda}(H_{\Lambda})$  provided  $H_{\Lambda} \notin \sigma_{\mu}'$  for any  $\mu \in L'$  such that  $\lambda - \mu \in L$ . On the other hand it is clear from Lemma 4 that  $\pi$  can be selected in such a way that (1)  $H_{\Lambda} \notin \sigma_{\mu}'$  for  $\mu = \lambda_0$  or  $\mu \ll \lambda_0$  ( $\mu \in L'$ ) and (2) the rational function  $c_{\lambda_0}$  (which is then obviously defined at  $H_{\Lambda}$ ) does not take the value zero at  $H_{\Lambda}$ . However the relation

$$\Xi_0(h; \delta'(q)) - \gamma'(q: H_{\Lambda}) \Xi_0(h) = 0 \quad (h \in A_p')$$

implies that

$$\sum_{\lambda \in L} a_{\lambda} e^{-\lambda(H)} \{\gamma'(q: H_{\Lambda} - H_{\lambda}) - \gamma'(q: H_{\Lambda})\}$$

$$+ \sum_{\lambda \in L} \sum_{\mu \in L'} a_{\lambda} e^{-\lambda(H) - \mu(H)} v_{\mu}(H_{\Lambda} - H_{\lambda}) = 0$$

for  $H \in \mathfrak{h}_{p_0}(M) = \mathfrak{h}_{p_0} \cap \mathfrak{h}_p(M)$ . Again by applying the corollary of Lemma 57 to  $V = \mathfrak{h}_{p_0} \cap E(M)$ , we can equate the coefficient of  $e^{-\lambda}$  in the above expression, to zero. But it is obvious that this coefficient is  $c_{\lambda}(H_{\Lambda})$  for  $\lambda = \lambda_0$  or  $\lambda \ll \lambda_0$  ( $\lambda \in L$ ). Hence  $c_{\lambda_0}(H_{\Lambda}) = 0$  and so we get a contradiction. This completes the proof of Lemma 29.

Let  $\tau$  denote the mapping of  $\mathfrak{h}_p$  into itself given by

$$\tau(H) = -(-1)^{\frac{1}{2}}(H + H_{\rho}) \quad (H \in \mathfrak{h}_p).$$

<sup>27</sup> For any complex number  $c$ ,  $\operatorname{Re} c$  and  $\operatorname{Im} c$  denote the real parts of  $c$  and  $-(-1)^{\frac{1}{2}}c$  respectively.

Then  $\tau^{-1}(H) = (-1)^{\frac{1}{2}}H - H_\rho$ . For any polynomial  $p \in S(\mathfrak{h}_p)$ , let  $p^\tau$  denote the function whose value at  $H \in \mathfrak{h}_p$  is  $p(\tau^{-1}(H))$ . Then the mapping  $p \rightarrow p^\tau$  can be extended to an automorphism of  $Q(\mathfrak{h}_p)$ . It is clear that a rational function  $p \in Q(\mathfrak{h}_p)$  is defined at a point  $H$  if and only if  $p^\tau$  is defined at  $\tau(H)$  and if this is so,  $p(H) = p^\tau(\tau(H))$ . Put  $\sigma_\lambda = \tau(\sigma'_\lambda)$  ( $\lambda \in L'$ ) and  $\mathfrak{h}_p = \tau(\mathfrak{h}_p'')$ . Then  $\sigma_\lambda$  consists of those points  $H \in \mathfrak{h}_p$  where  $\langle \lambda, \lambda \rangle = 2(-1)^{\frac{1}{2}}\lambda(H)$ . This shows that  $\mathfrak{h}_{p_0} \subset \mathfrak{h}_p$ . Put  $\Gamma_\lambda = (\Gamma'_\lambda)^\tau$  ( $\lambda \in L$ ),  $U = \tau(U')$  and

$$\phi(H_0; H) = e^{\rho(H)} \phi'(\tau^{-1}(H_0); H) = e^{(-1)^{\frac{1}{2}}\langle H_0, H \rangle} \sum_{\lambda \in L} \Gamma_\lambda(H_0) e^{-\lambda(H)}$$

for  $H_0 \in U$  and  $H \in \mathfrak{h}_p(M)$  (in the notation of Corollary 2 to Lemma 28). Moreover let

$$\Phi(H; h) = e^{\rho(\log h)} \Phi'(\tau^{-1}(H); h) \quad (H \in U, h \in A_p(M)).$$

Then Lemma 29 can be restated as follows.

LEMMA 30. For any  $q$  in  $I_\theta$ ,

$$\Phi(H; h; \delta(q)) = \gamma(q; (-1)^{\frac{1}{2}}H) \Phi(H; h)$$

if  $H \in U$  and  $h \in A_p(M)$ .

Now put  $\xi_\lambda(H_0; H) = \exp\{(-1)^{\frac{1}{2}}\langle H_0, H \rangle - \lambda(H)\} \Gamma_\lambda(H_0)$  for  $H_0 \in \mathfrak{h}_p$ ,  $H \in \mathfrak{h}_p$  and  $\lambda \in L$ . Then if  $u \in S(\mathfrak{h}_p)$ , it follows from the uniform convergence (see Corollary 1 to Lemma 28) of the series  $\sum_{\lambda \in L} |\xi_\lambda(H_0; H)|$  on any compact subset  $\Omega$  of  $U \times \mathfrak{h}_p(M)$ , that the series<sup>11</sup>  $\sum_{\lambda \in L} \xi_\lambda(H_0; \partial(u); H)$  also converges uniformly on  $\Omega$  to  $\phi(H_0; \partial(u); H)$  (see Lemma 58 of the Appendix). Define  $E(M)$  as above to be the set of all points  $H \in \mathfrak{h}_p(2M)$  such that  $\operatorname{Re} \alpha_i(H) > \frac{1}{2} |\alpha_j(H)|$  ( $1 \leq i, j \leq l$ ). Then if  $M$  is sufficiently large we have the following lemma.

LEMMA 31. Let  $U_1$  be a compact subset of  $U$ . Then for any linear function  $\mu$  on  $\mathfrak{h}_p$  and  $u \in S(\mathfrak{h}_p)$ , the series

$$\sum_{\lambda \in L} |\xi_\lambda(H_0; \partial(u); H) e^{\mu(H)}|$$

converges uniformly for  $H_0 \in U_1$  and  $H \in E(M)$ .

In view of Lemma 58 of the Appendix, it is sufficient to show that the series  $\sum_{\lambda \in L} |\xi_\lambda(H_0; H) e^{\mu(H)}|$  converges uniformly for  $H_0 \in U$  and  $H \in E(M)$ . Since the closure of  $U$  is compact, we can find a real number  $\nu$  such that

$$|\exp\{(-1)^{\frac{1}{2}}\langle H_0, H \rangle + \mu(H)\}| \leq \exp\{\nu \operatorname{Re}(\alpha_1(H) + \dots + \alpha_l(H))\}$$

for all  $H_0 \in U$  and  $H \in E(M)$ . Let  $\beta(H) = \min_{1 \leq i \leq l} \operatorname{Re} \alpha_i(H)$ . Then if  $H \in E(M)$ ,  $\operatorname{Re}(\alpha_1(H) + \cdots + \alpha_l(H)) \leq 2l\beta(H)$  and  $\operatorname{Re} \lambda(H) \geq m(\lambda)\beta(H)$  ( $\lambda \in L$ ). Hence  $|\xi_\lambda(H_0: H)e^{\mu(H)}| \leq |\Gamma_\lambda(H_0)| \exp\{(2vl - m(\lambda))2M\}$ . Therefore if  $m(\lambda) \geq 4vl$ ,

$$|\xi_\lambda(H_0: H)e^{\mu(H)}| \leq |\Gamma_\lambda(H_0)| \exp(-m(\lambda)M)$$

and our assertion follows from Lemma 28.

Now fix an element  $H_0 \in U$ . We now use the notation of Section 3 and define  $W'$  and  $J'$  corresponding to the element  $H_0$ . Let  $V_{H_0}$  denote the subspace consisting of those elements  $v \in S$  which satisfy the condition that  $p(H_0; \partial(v)) = 0$  for every polynomial function  $p \in SJ_{H_0}'$ . Also let  $J_+'$  be the subspace spanned by homogeneous elements in  $J'$  of positive degree.

LEMMA 32.  $S = V_{H_0} + SJ_+'$  and  $\dim V_{H_0} = w'$ .

Put  $V = V_{H_0}$  and let  $p \rightarrow p^\sigma$  ( $p \in S$ ) denote the automorphism of  $S$  defined by  $p^\sigma(H) = p(H + H_0)$  ( $H \in \mathfrak{h}_p$ ). Since  $H_0$  is left fixed by  $W'$ , it is clear that  $(J')^\sigma = J'$ . Moreover  $p$  vanishes at  $H_0$  if and only if  $p^\sigma$  vanishes at zero. Hence  $(J_{H_0}')^\sigma = J_+'$ . Also it is obvious that  $(\partial(H)p)^\sigma = \partial(H)p^\sigma$  ( $H \in \mathfrak{h}_p, p \in S$ ) and therefore  $(\partial(u)p)^\sigma = \partial(u)p^\sigma$  ( $u \in S$ ). This shows that  $V$  is exactly the set of those elements  $v \in S$  which satisfy the condition  $\langle v, p \rangle = 0$  for all  $p \in SJ_+'$ . For any  $p \in S$ , let  $p_*$  denote the polynomial function on  $\mathfrak{h}_p$  given by  $p_*(H) = \operatorname{conj} p(H)$  ( $H \in \mathfrak{h}_{p_0}$ ). Then  $\langle p, p_* \rangle$  is a positive-definite Hermitian form on  $S$  (see [5(k), p. 110]) and since  $J_+'$  is invariant under the mapping  $p \rightarrow p_*$ ,  $V$  is the orthogonal complement of  $SJ_+'$  in  $S$  under this form. Hence  $V \cap SJ_+'' = \{0\}$ . Moreover if  $S_d$  denotes the space of homogeneous elements in  $S$  of degree  $d$ , it is clear that  $V \cap S_d$  is the orthogonal complement of  $S_d \cap SJ_+'$  in  $S_d$ . Therefore since  $\dim S_d$  is finite, it follows that  $S_d = V \cap S_d + S_d \cap SJ_+'$  and so  $S = V + SJ_+'$ . But then  $\dim V = \dim S/SJ_+'' = w'$  from Lemma 13.

COROLLARY. Suppose  $v \in V_{H_0}$  and  $q \in I_0$ . Then

$$\Phi(H_0; \partial(v): h; \delta(q)) = \gamma(q: (-1)^{\frac{1}{2}} H_0) \Phi(H_0; \partial(v): h)$$

for  $h \in A_p(M)$ .

Let  $p$  denote the polynomial function  $H \rightarrow \gamma(q: (-1)^{\frac{1}{2}} H)$  on  $\mathfrak{h}_p$  and let  $(\partial(v) \circ p)_H$  denote the local expression (see [5(k), p. 90]) at  $H \in \mathfrak{h}_p$  of the differential operator  $2\delta(v) \circ p$  on  $\mathfrak{h}_p$ . We know from Lemma 30 that

$$\Phi(H: h; \delta(q)) = p(H) \Phi(H: h) \quad (H \in U, h \in A_p(M))$$

and therefore it is obvious that

$$\Phi(H; \partial(v):h; \delta(q)) = \Phi(H; (\partial(v) \circ p)_H: h)$$

Hence it would be sufficient to prove that  $(\partial(v) \circ p)_{H_0} = p(H_0)\partial(v)$ . So let us suppose that  $D = (\partial(v) \circ p)_{H_0} - p(H_0)\partial(v) \neq 0$ . Then we can choose  $p_1 \in S$  such that  $p_1(H_0; D) \neq 0$ . Put  $p_2 = (p - p(H_0))p_1$ . Then  $p_1(H_0; D) = p_2(H_0; \partial(v))$ . On the other hand  $p \in J$  and therefore  $p - p(H_0) \in J_{H_0}'$  and  $p_2 \in SJ_{H_0}'$ . But since  $v \in V_{H_0}$ , this implies that  $p_2(H_0; \partial(v)) = 0$ . Hence we get a contradiction and so the corollary is proved.

Put  $r = [W:W']$  and select  $s_1 = 1, s_2, \dots, s_r$  in  $W$  such that  $W = \bigcup_{1 \leq i \leq r} s_i W'$ . Then the points  $H_i = s_i H_0$  ( $1 \leq i \leq r$ ) are all distinct. We assume that  $H_i \in \mathfrak{h}_v$ ,  $1 \leq i \leq r$  and  $(-1)^{\frac{1}{2}}(H_i - H_j)$ , regarded as a linear function on  $\mathfrak{h}_v$ , does not lie in  $L$  for any pair of indices  $i \neq j$  ( $1 \leq i, j \leq r$ ). Put  $V_i = V_{H_i}$ ,  $1 \leq i \leq r$  and for any  $v \in V_i$ , let  $\psi_v^{(i)}$  denote the function  $h \rightarrow \Phi(H_i; \partial(v):h)$  on  $A_v(M)$  for  $M$  sufficiently large.

LEMMA 33. *Select nonzero elements  $v_i \in V_i$  ( $1 \leq i \leq r$ ). Then the functions  $\psi_{v_i}^{(i)}$   $1 \leq i \leq r$  are linearly independent over  $C$ .*

It is clear that for fixed  $H' \in \mathfrak{h}_v$ ,  $\lambda \in L$  and  $v \in S$ ,

$$\xi_\lambda(H'; \partial(v):H) \exp\{ -(-1)^{\frac{1}{2}} \langle H', H \rangle + \lambda(H) \}$$

is obviously a polynomial function of  $H$ . We denote by  $p_\lambda^{(i)}$  this polynomial corresponding to  $H' = H_i$  and  $v = v_i$ . Then if  $H \in E(M)$ ,

$$\phi(H; \partial(v_i):H) = \sum_{\lambda \in L} p_\lambda^{(i)}(H) \exp\{ (-1)^{\frac{1}{2}} \langle H_i, H \rangle - \lambda(H) \}.$$

Now suppose  $c_1, \dots, c_r$  are complex numbers such that

$$\sum_i c_i \phi(H_i; \partial(v_i):H) = 0 \quad (H \in E(M)).$$

Then

$$\sum_{1 \leq i \leq r} \sum_{\lambda \in L} c_i p_\lambda^{(i)}(H) \exp\{ (-1)^{\frac{1}{2}} \langle H_i, H \rangle - \lambda(H) \} = 0$$

and it follows from Lemma 31, the corollary to Lemma 57 (§ 15) and our assumption that  $(-1)^{\frac{1}{2}}(H_i - H_j) \notin L$  for  $i \neq j$ , that  $c_i p_\lambda^{(i)} = 0$  for every  $i$  and every  $\lambda \in L$ . On the other hand it is obvious that  $p_0^{(i)}(H) = v_i((-1)^{\frac{1}{2}}H)$  ( $H \in \mathfrak{h}_v$ ). Therefore since  $v_i \neq 0$ , it follows that  $p_0^{(i)} \neq 0$  and so  $c_i = 0$ . This proves the lemma.

It follows from the corollary to Lemma 32 that

$$\delta(q)\psi_v^{(i)} = \gamma(q; (-1)^{\frac{1}{2}}H_0)\psi_v^{(i)} \quad (v \in V_i, q \in I_0, 1 \leq i \leq r).$$

Moreover  $\dim V_i = w'$  by Lemma 29. Hence if  $v_{ij}$   $1 \leq j \leq w'$  is a base for  $V_i$ , we conclude from Lemma 33, that the  $w$  functions  $\psi_{ij} = \psi_{v_{ij}}^{(i)}$  ( $1 \leq i \leq r$ ,

$1 \leq j \leq w'$  are linearly independent and  $\delta(q)\psi_{ij} = \gamma(q: (-1)^{\frac{1}{2}H_0})\psi_{ij}$  ( $q \in I_{\mathfrak{g}}$ ). Therefore in view of Lemmas 18 and 23, we get the following result from the corollary of Theorem 2.

LEMMA 34. *Under the above assumptions there exist unique complex numbers  $c_{ij}$  such that*

$$e^{\rho(\log h)} \int_K \exp\{(-1)^{\frac{1}{2}} \langle H_0, H(hk) \rangle - \rho(H(hk))\} dk \\ = \sum_{1 \leq i \leq r} \sum_{1 \leq j \leq w'} c_{ij} \psi_{ij}(h)$$

for all  $h \in A_p(M)$ .

In particular let us consider the case when  $H_0 = 0$ . Then  $W' = W$  and  $V = V_1$  is the set of all  $v \in S(\mathfrak{h}_p)$  such that  $\langle v, p \rangle = 0$  for  $p \in SJ_+$ . Moreover since  $\mathfrak{h}_{p_0} \subset \mathfrak{h}_p$ , all the required conditions for  $H_0$  hold in this case. Therefore we obtain the following corollary.

COROLLARY. *There exists an element  $v \in V$  and a number  $M \geq 0$  such that*

$$e^{\rho(\log h)} \int_K e^{-\rho(H(hk))} dk = \Phi(0; \partial(v): h)$$

for all  $h \in A_p(M)$ .

**9. An important inequality.** Define  $\bar{\theta}$  as in Section 5 and put  $\|X\|^2 = -B(X, \bar{\theta}(X))$  ( $X \in \mathfrak{g}$ ). As before, we regard  $\mathfrak{g}$  as a Hilbert space under the norm  $\|X\|$ . Let  $\mathfrak{h}_{p_0}^+$  be the set of those points  $H \in \mathfrak{h}_{p_0}$  where  $\alpha(H) > 0$  for every  $\alpha \in \Sigma$ . We put  $A_p^+ = \exp \mathfrak{h}_{p_0}^+$ . Our main object in this section is to prove the following result which will play an important role later.

THEOREM 3. *There exists an integer  $d \geq 0$  and a positive number  $a$  such that*

$$\int_K e^{-\rho(H(hk))} dk \leq a(1 + \|\log h\|)^d e^{-\rho(\log h)}$$

for all  $h \in A_p^+$ .

For the proof we need some auxiliary results. Let  ${}^+\mathfrak{h}_{p_0}$  denote the set of all  $H \in \mathfrak{h}_{p_0}$  such that  $\langle H, H' \rangle \geq 0$  for every  $H'$  in  $\mathfrak{h}_{p_0}^+$ . Also let  $\text{Cl}(A_p^+)$  denote the closure of  $A_p^+$  in  $A_p$ .

LEMMA 35. *Define  $\mathfrak{F}_0$  as in Lemma 2. An element  $H' \in \mathfrak{h}_{p_0}$  lies in  ${}^+\mathfrak{h}_{p_0}$  if and only if  $\lambda(H') \geq 0$  for every  $\lambda \in \mathfrak{F}_0$ . Moreover  $\mathfrak{h}_{p_0}^+ \subset {}^+\mathfrak{h}_{p_0}$  and  $\log h - H(hk) \in {}^+\mathfrak{h}_{p_0}$  for  $h \in \text{Cl}(A_p^+)$  and  $k \in K$ .*

Select  $\alpha_1, \dots, \alpha_l \in \Sigma$  as in Section 3 and choose  $H_i \in \mathfrak{h}_{p_0}^+$  such that  $\alpha_i(H_j) = \delta_{ij}$  ( $1 \leq i, j \leq l$ ). Then it is obvious that  $\mathfrak{h}_{p_0}^+$  consists of all elements of the form  $t_1 H_1 + \dots + t_l H_l$  where  $t_1, \dots, t_l$  are positive numbers. Therefore  $H'$  lies in  $\mathfrak{h}_{p_0}$  if and only if  $\langle H', H_i \rangle \geq 0$  for every  $i$ . For any  $\beta \in P$  define  $H_\beta$  as during the proof of Lemma 27 and, for a fixed  $i$ , consider the linear function  $\mu_i$  on  $\mathfrak{h}$  given by  $\mu_i(H) = B(H_i, H)$  ( $H \in \mathfrak{h}$ ). It is clear that  $\mu_i(H_\beta) = \beta(H_i) \geq 0$  for  $\beta \in P$ . Since  $\beta(H_\beta)$  are positive rational numbers, we can choose a positive integer  $m$  such that  $2m\mu_i(H_\beta)/\beta(H_\beta)$  is an integer for every  $\beta \in P$  and every  $i$  ( $1 \leq i \leq l$ ). Let  $\lambda_i$  denote the restriction of  $2m\mu_i$  on  $\mathfrak{h}_p$ . Then it follows from Theorem 1 of [5(b)] and Lemma 2 that  $\lambda_i \in \mathfrak{F}_0$ . Now suppose  $H' \in \mathfrak{h}_{p_0}$ . Then  $\lambda_i(H') = 2m\langle H', H_i \rangle$  and  $\langle \lambda_i, \alpha_j \rangle = 2m\alpha_j(H_i) = 2m\delta_{ji}$ . Therefore if  $\lambda$  is any linear function on  $\mathfrak{h}_p$ ,  $\lambda = \sum_i c_i \lambda_i$  where  $c_i = \langle \lambda, \alpha_i \rangle / 2m$ . But  $\lambda \geq s_{\alpha_i} \lambda$  if  $\lambda \in \mathfrak{F}_0$  and so  $2mc_i = \langle \lambda, \alpha_i \rangle \geq 0$ . Hence  $\langle H', H_i \rangle \geq 0$  ( $1 \leq i \leq l$ ) if and only if  $\lambda(H') \geq 0$  for every  $\lambda \in \mathfrak{F}_0$ . This proves the first statement of the lemma.

Now let  $\pi$  be an irreducible representation of  $\mathfrak{g}$  on a finite-dimensional space  $V$  and let  $\phi$  be a unit vector in  $V$  belonging to the highest weight  $\Lambda$  of  $\pi$ . Then (see [5(h), Lemma 2]) every other weight of  $\pi$  is of the form  $\Lambda - (m_1 \beta_1 + \dots + m_r \beta_r)$  where  $\beta_1, \dots, \beta_r \in P$  and  $m_1, \dots, m_r$  are positive integers. Hence if  $H \in \mathfrak{h}_{p_0}^+$ , it follows that  $\Lambda(H) \geq \Lambda'(H)$  for every weight  $\Lambda'$  of  $\pi$ . But since  $\pi(H)$  is a self-adjoint operator of trace zero, this implies that  $\Lambda(H) \geq 0$ . Similarly if  $h \in \text{Cl}(A_p^+)$ ,  $e^{\Lambda(\log h)}$  is the largest eigenvalue of the self-adjoint operator  $\pi(h)$ . Therefore

$$|\pi(hk)\phi| \leq e^{\Lambda(\log h)} |\pi(k)\phi| = e^{\Lambda(\log h)}$$

for  $k \in K$ . However it is obvious that  $|\pi(hk)\phi| = e^{\Lambda(H(hk))}$  and so  $\Lambda(\log h - H(hk)) \geq 0$ . This shows in particular that  $\lambda(H) \geq 0$  and  $\lambda(\log h - H(hk)) \geq 0$  for  $\lambda \in \mathfrak{F}_0$ . Therefore by the criterion established above,  $H$  and  $\log h - H(hk)$  lie in  $\mathfrak{h}_{p_0}$ . Thus the lemma is proved.

**COROLLARY 1.** *Let  $H$  and  $H'$  be two elements in  $\mathfrak{h}_{p_0}^+$ . Then  $\langle H', H \rangle > \langle H', sH \rangle$  for  $s \neq 1$  in  $W$ .*

Fix an element  $s \neq 1$  in  $W$  and select  $k \in K$  such that  $\text{Ad}(k)H = sH$ . Then  $\pi(sH) = \pi(k)\pi(H)\pi(k^{-1})$  in the above notation. Therefore since  $\Lambda(H)$  is the greatest eigenvalue of  $\pi(H)$ , it follows that  $\Lambda(sH) \leq \Lambda(H)$ . But in view of Lemma 35, this implies that  $H - sH \in \mathfrak{h}_{p_0}$ . Now  $\lambda_1, \dots, \lambda_l$  are linearly independent and by Lemma 6  $H \neq sH$  since  $H \in \mathfrak{h}_{p_0}^+$ . Therefore  $\lambda_i(H - sH) \neq 0$  for some  $i$ . Moreover we have seen that if  $\lambda$  is any linear function on  $\mathfrak{h}_p$ ,  $\lambda = \sum_i c_i \lambda_i$  where  $c_i = \langle \lambda, \alpha_i \rangle / 2m$ . Therefore  $\langle H', H - sH \rangle$

$= (2m)^{-1} \sum_{1 \leq i \leq l} \alpha_i(H') \lambda_i(H - sH)$ . But from Lemma 35,  $\lambda_i(H - sH)$  are non-negative and  $\alpha_i(H') > 0$  ( $1 \leq i \leq l$ ) since  $H' \in \mathfrak{h}_{p_0}^+$ . Therefore  $\langle H', H - sH \rangle > 0$ .

From [5(j), Lemma 37] we can choose  $s_0 \in W$  such that  $s_0 \mathfrak{h}_{p_0}^+ = -\mathfrak{h}_{p_0}^+$ . Put  $H^* = -s_0 H$  for any  $H \in \mathfrak{h}_p$ . It is obvious that the mapping  $\alpha \rightarrow -s_0 \alpha$  ( $\alpha \in \Sigma$ ) is a permutation on  $\Sigma$  and therefore  $\rho(H^*) = \rho(H)$ .

**COROLLARY 2.**  $(\log h)^* + H(hk) \in {}^+ \mathfrak{h}_{p_0}$  for  $h \in \text{Cl}(A_p^+)$  and  $k \in K$ .

Fix  $h$  and  $k$  and let  $hk = k' \exp H(hk)n$  ( $k' \in K, n \in N$ ). Then  $H(hk) = -H(h^{-1}k')$  (see [5(c), Lemma 36]). Now select  $k_0 \in K$  such that  $\text{Ad}(k_0)H' = s_0 H'$  for every  $H'$  in  $\mathfrak{h}_{p_0}$ . Then if  $h^* = \exp(\log h)^*$ , it is obvious that  $h^* = k_0 h^{-1} k_0^{-1}$  and therefore  $H(h^* k_0 k') = H(h^{-1} k') = -H(hk)$ . But from Lemma 35, this implies that

$$\log h^* + H(hk) = \log h^* - H(h^* k_0 k') \in {}^+ \mathfrak{h}_{p_0}.$$

**COROLLARY 3.**  $\rho(\log h) \geq |\rho(H(hk))|$  for  $h \in \text{Cl}(A_p^+)$  and  $k \in K$ .

It follows from the definition of  $\rho$  that  $s_\alpha \rho < \rho$  and therefore  $\langle \rho, \alpha \rangle > 0$  for any  $\alpha \in \Sigma$ . Hence  $H_\rho \in \mathfrak{h}_{p_0}^+$ . Moreover  $\rho(\log h^*) = \rho(\log h)$  as we have seen above. Hence our assertion follows from the fact that  $\log h - H(hk)$  and  $\log h^* + H(hk)$  are both in  ${}^+ \mathfrak{h}_{p_0}$ .

$$\text{Now put } \psi(h) = e^{\rho(\log h)} \int_K e^{-\rho(H(hk))} dk \quad (h \in A_p).$$

**LEMMA 36.**  $\psi(h) \leq \psi(hh_1)$  for  $h \in A_p$  and  $h_1 \in \text{Cl}(A_p^+)$ .

Fix  $h$  and  $h_1$  and for any  $k \in K$ , let  $k'$  denote the unique element in  $K$  such that  $h_1 k \in k' A_p N$ . Then  $k \rightarrow k'$  is a homeomorphism of  $K$ ,  $H(hh_1 k) = H(hk') + H(h_1 k)$  and  $dk' = e^{-2\rho(H(h_1 k))} dk$  (see [5(c), pp. 240-241]). Hence

$$\begin{aligned} \int_K e^{-\rho(H(hk))} dk &= \int_K e^{-\rho(H(hk'))} dk' = \int_K \exp\{-\rho(H(hk')) - 2\rho(H(h_1 k))\} dk \\ &= \int_K \exp\{-\rho(H(hh_1 k)) - \rho(H(h_1 k))\} dk \leq e^{\rho(\log h_1)} \int_K e^{-\rho(H(hh_1 k))} dk \end{aligned}$$

from Corollary 3 to Lemma 35. This implies that  $\psi(h) \leq \psi(hh_1)$ .

Now we come to the proof of Theorem 3. Select  $H_0 \in \mathfrak{h}_{p_0}^+$  such that  $\alpha_i(H_0) = 1$  ( $1 \leq i \leq l$ ) and put  $F(t) = \psi(\exp tH_0)$  ( $t \in R$ ). Then in the notation of Section 8,  $F(t) = \phi(0; \partial(v): tH_0)$  for  $t > M$ , where  $v$  and  $M$  are defined as in the corollary to Lemma 34. Let  $d$  be the degree of  $v$ . Then it follows from Lemma 31 that

$$F(t) = \sum_{\lambda \in L} p_\lambda(t) e^{-t\lambda(H_0)} \quad (t > M),$$

where  $p_\lambda$  are polynomials of degree  $\leq d$  and the series  $\sum_{\lambda \in L} |p_\lambda(t) e^{-t\lambda(H_0)}|$  converges uniformly for  $t > M$  (if  $M$  is chosen sufficiently large). Hence we can find a finite number of distinct elements  $\lambda_0 = 0, \lambda_1, \dots, \lambda_r \in L$  such that

$$|F(t) - \sum_{0 \leq i \leq r} p_{\lambda_i}(t) e^{-t\lambda_i(H_0)}| \leq 1$$

for  $t > M$ . But since  $\lambda(H_0) = m(\lambda)$  (in the notation of Lemma 28), it is obvious that

$$\lim_{t \rightarrow +\infty} \sum_{1 \leq i \leq r} p_{\lambda_i}(t) e^{-t\lambda_i(H_0)} = 0.$$

Therefore  $F(t) \leq 2 + p_0(t)$  for all sufficiently large positive values of  $t$ . Since the degree of  $p_0$  is  $\leq d$ , this means that we can select a positive constant  $a$  such that  $F(t) \leq a(1+t)^d$  for all  $t \geq 0$ .

Let  $\text{Cl}(\mathfrak{h}_{p_0}^+)$  denote the closure of  $\mathfrak{h}_{p_0}^+$  in  $\mathfrak{h}_{p_0}$  and put  $\beta(H) = \max_{1 \leq i \leq l} \alpha_i(H)$  for  $H \in \mathfrak{h}_{p_0}$ . Then it is obvious that  $\beta(H)H_0 - H \in \text{Cl}(\mathfrak{h}_{p_0}^+)$ . Hence it follows from Lemma 36 that  $\psi(\exp H) \leq F(\beta(H)) \leq a(1 + \beta(H))^d$  for  $H \in \text{Cl}(\mathfrak{h}_{p_0}^+)$ . On the other hand  $\|H\|^2 = \text{sp}(\text{ad } H)^2 \geq \sum_{1 \leq i \leq l} \alpha_i(H)^2 \geq \beta(H)^2$ . Therefore

$$\psi(\exp H) \leq a(1 + \|H\|)^d$$

and this proves the theorem.

*Remark.* Let  $d_\pi$  denote the degree of the polynomial function  $\pi$  of Section 3 and define  $V$  as in the corollary to Lemma 34. Then it follows from Lemma 16 that no nonzero element in  $V$  can have a degree greater than  $d_\pi$ . This shows that  $d \leq d_\pi$ .

**10. The function  $c$ .** For any  $\lambda \in L'$  define  $\sigma_\lambda$  (as in Section 8) to be the hyperplane consisting of those points  $H \in \mathfrak{h}_p$  where  $\langle \lambda, \lambda \rangle = 2(-1)^{\frac{1}{2}\lambda(H)}$  and let  $\tau_\lambda(s, t)$  ( $s, t \in W$ ) denote the set of all  $H \in \mathfrak{h}_p$  for which  $(-1)^{\frac{1}{2}(sH - tH)} = H_\lambda$ . Then if  $\mathfrak{S}$  is the collection of all hyperplanes of the form  $\sigma_\lambda$  or  $\tau_\lambda(s, t)$  ( $s, t \in W; \lambda \in L'$ ), it is obvious that any compact subset of  $\mathfrak{h}_p$  meets only a finite number of hyperplanes in  $\mathfrak{S}$ . Let  $^*\mathfrak{h}_p$  be the complement (in  $\mathfrak{h}_p$ ) of the union of all hyperplanes in  $\mathfrak{S}$ . Then  $^*\mathfrak{h}_p$  is an open, connected and dense subset of  $\mathfrak{h}_p$ . Moreover it follows from its definition that  $^*\mathfrak{h}_p$  is invariant under  $W$ . Define  $\pi$  as in Lemma 11 and let  $^*\mathfrak{h}_p'$  be the set of those  $H \in ^*\mathfrak{h}_p$  where  $\pi(H) \neq 0$ . Since the zeros of  $\pi$  consist of a finite number of hyperplanes,  $^*\mathfrak{h}_p'$  is also connected. We shall now use the notation of Section 8.

**LEMMA 37.** *There exists a holomorphic function  $c$  on  $^*\mathfrak{h}_p'$  with the*

following property. For any compact subset  $U$  of  ${}^*\mathfrak{h}_p$ , we can select a number  $M \geq 0$  such that

$$e^{\rho(\log h)} \int_K \exp\{(-1)^{\frac{1}{2}} \langle H_0, H(hk) \rangle - \rho(H(hk))\} dk \\ = \sum_{s \in W} c(sH_0) \Phi(sH_0: h)$$

for  $H_0 \in U \cap {}^*\mathfrak{h}_p'$  and  $h \in A_p(M)$ . (Here  $\Phi$  has the same meaning as in Lemma 30.)

Put

$$\psi(H_0: h) = e^{\rho(\log h)} \int_K \exp\{(-1)^{\frac{1}{2}} \langle H_0, H(hk) \rangle - \rho(H(hk))\} dk$$

for  $H_0 \in \mathfrak{h}_p$  and  $h \in A_p$ . It is obvious that for fixed  $h \in A_p$  and  $u \in \mathfrak{S}_p$ ,  $\psi(H_0: h; u)$  is a holomorphic function of  $H_0$ . Moreover if  $H_0 \in {}^*\mathfrak{h}_p'$ , the  $w$  elements  $sH_0$  ( $s \in W$ ) are all distinct. Therefore it follows from Lemma 34 that there exist unique complex numbers  $c_s(H_0)$  such that

$$\psi(H_0: h) = \sum_{s \in W} c_s(H_0) \Phi(sH_0: h)$$

for  $h \in A_p(M)$  provided  $M$  is sufficiently large. In view of Corollary 1 of Lemma 28, it is sufficient to show that  $c_s$  ( $s \in W$ ), regarded as functions of  $H_0$ , are holomorphic on  ${}^*\mathfrak{h}_p'$  and  $c_s(H_0) = c_1(sH_0)$  ( $H_0 \in {}^*\mathfrak{h}_p'$ ).

Let  $U_0$  be a nonempty open set in  ${}^*\mathfrak{h}_p$  which is invariant under  $W$ . We assume that the closure of  $U_0$  in  ${}^*\mathfrak{h}_p$  is compact. Select  $M$  so large that for any  $H \in U_0$ ,  $\Phi(H: h)$  is defined for  $h \in A_p(M)$  and choose  $u_1, \dots, u_w \in \mathfrak{S}_p$  as in the corollary to Lemma 8. Also put  $U_0' = U_0 \cap {}^*\mathfrak{h}_p'$ .

**LEMMA 38.** Let  $s_1, s_2, \dots, s_w$  be all the elements of  $W$ . Then  $\det\{\Phi(s_i H_0: h_0; u_j)\}_{1 \leq i, j \leq w} \neq 0$  for any  $H_0 \in U_0'$  and  $h_0 \in A_p(M)$ .

For otherwise we can select complex numbers  $a_s$  ( $s \in W$ ), not all zero, such that  $\sum_{s \in W} a_s \Phi(sH_0: h_0; u_j) = 0$  ( $1 \leq j \leq w$ ). Put  $f(h) = \sum_{s \in W} a_s \Phi(sH_0: h)$  for  $h \in A_p(M)$ . Then  $f$  is an analytic function on  $A_p(M)$  and it follows from Lemma 30 that  $\delta(q)f = \gamma(q: (-1)^{\frac{1}{2}} H_0)f$  for  $q \in I_g$ . Therefore since  $f(h_0; u_j) = 0$  ( $1 \leq j \leq w$ ), we conclude from Theorem 2 that  $f(h_0; u) = 0$  for every  $u \in \mathfrak{S}_p$ . But since  $f$  is analytic and  $A_p(M)$  is connected, this implies that  $f = 0$ . However this contradicts Lemma 33 since  $W' = \{1\}$  in the present case.

Now in order to complete the proof of Lemma 37, fix  $h_0 \in A_p(M)$ . Then by Lemma 38 there exist holomorphic functions  $a_{si}$  on  $U_0'$  ( $s \in W$ ,  $1 \leq i \leq w$ ) such that  $\sum_{1 \leq i \leq w} a_{si}(H) \Phi(s'H: h_0; u_i) = 1$  or 0 according as  $s = s'$

or not  $(s, s' \in W, H \in U_0')$ . Therefore  $\mathbf{c}_s(H) = \sum_{1 \leq i \leq w} a_{si}(H) \psi(H: h_0; u_i)$  ( $s \in W, H \in U_0'$ ) and this proves that  $\mathbf{c}_s$  is holomorphic on  $U_0'$ . On the other hand we know from the corollary to Lemma 17 that  $\psi(sH: h_0) = \psi(H: h_0)$  for  $s \in W$  and  $H \in \mathfrak{h}_p$ . This implies that  $\mathbf{c}_s(H) = \mathbf{c}(sH)$  ( $s \in W, H \in U_0$ ) where  $\mathbf{c} = \mathbf{c}_1$ . Hence Lemma 37 is now proved completely.

**11. A formula for  $\mathbf{c}$ .** We shall now derive an explicit formula for  $\mathbf{c}$  which will be valid on a suitable subdomain of  ${}^* \mathfrak{h}_p'$ . For any root  $\alpha \in P$ , define  $X_\alpha$  and  $X_{-\alpha}$  as in Section 7 and put  $\mathfrak{n}^+ = \sum_{\alpha \in P} CX_\alpha, \mathfrak{n}^- = \sum_{\alpha \in P} CX_{-\alpha}$ . Let  $N_c^+, N_c^-$  and  $A_c$  denote the complex-analytic subgroups of  $G_c$  corresponding to  $\mathfrak{n}^+, \mathfrak{n}^-$  and  $\mathfrak{h}$  respectively and put  $G_c' = N_c^- A_c N_c^+$ . Since  $\mathfrak{g}$  is the direct sum of  $\mathfrak{n}^-, \mathfrak{h}$  and  $\mathfrak{n}^+$ , it follows easily that  $G_c'$  is open in  $G_c$ . We regard  $G_c'$  as an open submanifold of  $G_c$ . Then (see [5(i), Lemma 1]) the mapping  $(n_1, a, n_2) \rightarrow n_1 a n_2$  ( $n_1 \in N_c^-, a \in A_c, n_2 \in N_c^+$ ) is a one-one holomorphic mapping of  $N_c^- \times A_c \times N_c^+$  onto  $G_c'$  which is regular everywhere. For any  $z \in G_c'$ , let  $a(z)$  denote the unique element in  $A_c$  such that  $z \in N_c^- a(z) N_c^+$ . Then  $z \rightarrow a(z)$  is a holomorphic mapping of  $G_c'$  onto  $A_c$ .

**LEMMA 39.** *Let  $n_r', a_r, n_r$  ( $r \geq 1$ ) be three sequences in  $N_c^-, A_c$  and  $N_c^+$  respectively such that  $a_r$  and  $n_r' a_r n_r$  converge in  $G_c$ . Then  $n_r'$  and  $n_r$  are also convergent.*

Obviously it would be enough to prove that  $n_r'$  is convergent. Since  $A_c$  is closed in  $G_c$ ,  $a_r$  converges to an element  $a_0 \in A_c$ . Let  $\pi$  be an irreducible representation of  $\mathfrak{g}$  (and therefore also of  $G_c$ ) on a finite-dimensional vector space  $V$  and  $\xi$  a unit vector belonging to the highest weight  $\Lambda$  of  $\pi$ . For any root  $\alpha$  define  $H_\alpha$  as in Section 7. Then by choosing  $\pi$  suitably, we can assume (see [5(b), Theorem 1]) that  $\Lambda(H_\alpha) > 0$  for every  $\alpha \in P$ . Let  $\chi$  denote the character of  $A_c$  such that  $\pi(a)\xi = \chi(a)\xi$  for  $a \in A_c$ . Then if  $n_r' a_r n_r$  converges to  $z$  in  $G_c$ , it is clear that

$$\pi(n_r')\xi = \chi(a_r^{-1})\pi(n_r' a_r n_r)\xi \rightarrow \chi(a_0^{-1})\pi(z)\xi.$$

Since  $\mathfrak{n}^-$  is a nilpotent Lie algebra, we can select  $Y_r \in \mathfrak{n}^-$  such that  $\exp Y_r = n_r'$ . Then it would be sufficient to prove that  $Y_r$  converges in  $\mathfrak{n}^-$ . Thus it remains to prove the following result.

**LEMMA 40.** *Suppose  $Y$  varies in  $\mathfrak{n}^-$  in such a way that  $\pi(\exp Y)\xi$  converges in  $V$ . Then  $Y$  itself converges in  $\mathfrak{n}^-$ .*

Let  $\beta_1 < \beta_2 < \dots < \beta_r$  be all the roots in  $P$ . Then  $Y = \sum_{1 \leq i \leq r} t_i(Y) X_{-\beta_i}$

$(t_i(Y) \in C)$  and it would be sufficient to prove that  $t_i(Y)$  converges in  $C$  for every  $i$ . Hence suppose that this is false and let  $j$  be the least index such that  $t_j(Y)$  does not converge. Let  $V_j$  be the subspace of  $V$  consisting of all vectors belonging to the weight  $\Lambda - \beta_j$  and let  $E_j$  denote the orthogonal projection of  $V$  on  $V_j$ . Then

$$E_j \pi(\exp Y) \xi = \sum_{m \geq 0} E_j \pi(Y^m) \xi / m!$$

But if  $Y' = \sum_{1 \leq i < j} t_i(Y) X_{-\beta_i}$ , it is obvious that  $E_j \pi(Y^m) \xi = E_j \pi(Y'^m) \xi$  for  $m > 1$ . Moreover  $E_j \pi(Y') \xi = 0$  while  $E_j \pi(Y) \xi = t_j(Y) \pi(X_{-\beta_j}) \xi$ . Hence

$$E_j \pi(\exp Y) \xi = t_j(Y) \pi(X_{-\beta_j}) \xi + E_j \pi(\exp Y') \xi.$$

On the other hand, in view of our hypothesis and the definition of  $j$ , both  $\pi(\exp Y) \xi$  and  $\pi(\exp Y') \xi$  converge in  $V$ . Therefore the same holds for  $t_j(Y) \pi(X_{-\beta_j}) \xi$ . But  $\pi(X_{-\beta_j}) \xi \neq 0$  since  $\Lambda(H_{\beta_j}) > 0$  (see [5(h), Lemma 1]). Hence  $t_j(Y)$  also converges in  $C$ . As this contradicts the definition of  $j$ , our assertion follows.

**COROLLARY 1.** *Let  $z$  be an element in  $G_c$ . Then  $z \in G_c'$  if and only if  $(\xi, \pi(z) \xi) \neq 0$ .*

Put  $f(z) = (\xi, \pi(z) \xi)$  and let  $G_c''$  be the set of all  $z \in G_c$  where  $f(z) \neq 0$ . Since  $f$  is a holomorphic function on  $G_c$  and  $f(1) = 1$ , its set of zeros is a complex subvariety of  $G_c$  of one complex dimension less. Therefore  $G_c''$  is an open connected subset of  $G_c$ . Moreover if  $z \in G_c'$ , it is obvious that  $f(z) = \chi(a(z)) \neq 0$ . Therefore  $G_c' \subset G_c''$ . Hence it would be sufficient to show that  $G_c'$  is closed in  $G_c''$ . Let  $z_r$  ( $r \geq 1$ ) be a sequence in  $G_c'$  which converges to  $z \in G_c''$ . Suppose  $z_r = n_r' a_r n_r$  ( $n_r' \in N_c^-, a_r \in A_c, n_r \in N_c^+$ ). Then  $\chi(a_r) = f(z_r) \rightarrow f(z) \neq 0$  and therefore  $\pi(n_r') \xi = \chi(a_r^{-1}) \pi(z_r) \xi \rightarrow f(z)^{-1} \pi(z) \xi$ . But then by the above lemma,  $n_r'$  converges to an element  $n'$  in  $N_c^-$ . However this implies that  $a_r n_r = n_r'^{-1} z_r \rightarrow n'^{-1} z$ . On the other hand  $A_c N_c^+$  is closed in  $G_c$  and hence  $n'^{-1} z \in A_c N_c^+$ . This proves that  $z \in G_c'$  and therefore  $G_c'$  is closed in  $G_c''$ .

**COROLLARY 2.** *Put  $K' = K \cap G_c'$ . Then  $K'$  is an open dense subset of  $K$  whose complement (in  $K$ ) is of measure zero with respect to the Haar measure of  $K$ .*

Let  $g$  be the restriction on  $K$  of the function  $f$  defined above. Then  $g$  is an analytic function which is not identically zero since  $g(1) = 1$ . Moreover  $K'$  is the set of all points  $k \in K$  where  $g(k) \neq 0$ . From this our statement follows immediately.

Since  $G_c$  is simply connected, we can extend  $\theta$  to an automorphism of  $G_c$ . Put  $\mathfrak{n} = \sum_{\alpha \in P_+} CX_\alpha$  and  $\bar{\mathfrak{n}} = \theta(\mathfrak{n}) = \sum_{\alpha \in P_+} CX_{-\alpha}$  and let  $N_c, \bar{N}_c$  denote the complex analytic subgroups of  $G_c$  corresponding to  $\mathfrak{n}, \bar{\mathfrak{n}}$  respectively. Also let  $\Xi_c$  and  $\bar{\mathfrak{z}}$  denote the centralizers of  $\mathfrak{h}_\mathfrak{p}$  in  $G_c$  and  $\mathfrak{g}$  respectively. Since  $\mathfrak{n}^* + \mathfrak{h}_\mathfrak{p} \subset \mathfrak{n} + \bar{\mathfrak{z}}$ , it follows easily that  $\bar{N}_c \Xi_c N_c \supset N_c A_c N_c^* = G_c'$ .

LEMMA 41. Suppose  $Y \in \bar{\mathfrak{n}}$  and  $\exp(\text{ad } Y)$  maps  $\mathfrak{n}$  into itself. Then  $Y = 0$ .

Otherwise suppose that  $Y \neq 0$ . Let  $\beta_1 < \beta_2 < \dots < \beta_r$  be all the roots in  $P_+$ . Then  $Y = \sum_i c_i X_{-\beta_i}$  where  $c_i \in C$ . Let  $k$  be the least integer such that  $c_k \neq 0$ . Then it is obvious that  $\exp(\text{ad } Y)X_{\beta_k} - X_{\beta_k} \equiv -c_k H_{\beta_k} \pmod{\mathfrak{n}}$ . Therefore since the sum  $\mathfrak{h} + \mathfrak{n}^* + \mathfrak{n}^-$  is direct, it follows that  $\exp(\text{ad } Y)X_{\beta_k}$  does not lie in  $\mathfrak{n}^* + \mathfrak{n}^-$ . But this contradicts our hypothesis that  $\exp(\text{ad } Y)$  maps  $\mathfrak{n}$  into itself. Hence the lemma.

COROLLARY.  $\bar{N}_c \cap (\Xi_c N_c) = \{1\}$  and  $\Xi_c \cap N_c = \{1\}$ .

Suppose  $y \in \bar{N}_c \cap (\Xi_c N_c)$ . Then  $y = \exp Y$  for some  $Y \in \bar{\mathfrak{n}}$  and  $\text{Ad}(y)$  leaves  $\mathfrak{n}$  invariant. Hence  $Y = 0$  and so  $y = 1$ . Now suppose  $x \in \Xi_c \cap N_c$ . Then  $x = \exp X$  for some  $X \in \mathfrak{n}$ . Put  $Y = \theta(X)$ . Since  $x \in \Xi_c$ ,  $\text{Ad}(x)$  leaves  $\bar{\mathfrak{n}}$  invariant and therefore  $\exp(\text{ad } Y)$  leaves  $\mathfrak{n}$  invariant. Hence  $Y = 0$ . This proves that  $x = 1$ .

Define  $N$  as in Section 2 and let  $M$  be the centralizer of  $A_\mathfrak{p}$  in  $K$ . Put  $S = MA_\mathfrak{p}N$ . Obviously  $S$  is a subgroup of  $G$ .

LEMMA 42.  $G \cap (\bar{N}_c \Xi_c N_c) = \bar{N}S$  where  $\bar{N} = \theta(N)$ .

Suppose  $x \in G \cap (\bar{N}_c \Xi_c N_c)$ . Then  $x = \bar{n}\xi n$  ( $\bar{n} \in \bar{N}_c, \xi \in \Xi_c, n \in N_c$ ). Let  $\eta$  denote the conjugation of  $\mathfrak{g}$  with respect to  $\mathfrak{g}_0$ . We extend it to a real automorphism of  $G_c$ . Then  $x = \eta(x) = \eta(\bar{n})\eta(\xi)\eta(n)$ . Since  $\eta$  maps  $\mathfrak{n}$  and  $\bar{\mathfrak{n}}$  into themselves, it follows from the corollary to Lemma 41 that  $\eta(\bar{n}) = \bar{n}$  and  $\eta(n) = n$ . Choose  $X \in \mathfrak{n}$  and  $Y \in \bar{\mathfrak{n}}$  such that  $\exp X = \bar{n}$ ,  $\exp Y = n$ . Since the exponential mapping is univalent on both  $\mathfrak{n}$  and  $\bar{\mathfrak{n}}$ , we conclude that  $X$  and  $Y$  are left fixed by  $\eta$ . Hence  $X \in \mathfrak{n} \cap \mathfrak{g}_0$ ,  $Y \in \bar{\mathfrak{n}} \cap \mathfrak{g}_0$  and therefore  $n \in N$ ,  $\bar{n} \in \bar{N}$  and  $\xi \in G \cap \Xi_c$ . This shows that  $G \cap (\bar{N}_c \Xi_c N_c) = \bar{N}(G \cap \Xi_c)N$ . But since  $MA_\mathfrak{p}$  is the centralizer<sup>13</sup> of  $A_\mathfrak{p}$  in  $G$ , it follows that  $G \cap \Xi_c = MA_\mathfrak{p}$  and therefore  $G \cap (\bar{N}_c \Xi_c N_c) = \bar{N}S$ .

COROLLARY. Put  $K_1 = K \cap (\bar{N}S)$ . Then  $K_1 \supset K'$ .

We have seen above that  $\bar{N}_c \Xi_c N_c \supset G_c'$  and therefore  $K_1 = K \cap (\bar{N}_c \Xi_c N_c) \supset K \cap G_c' = K'$ .

LEMMA 43. Let  $\bar{n} \in \bar{N}$  and  $h \in \text{Cl}(A_{\mathfrak{p}}^+)$ . Then both  $H(\bar{n})$  and  $H(\bar{n}) - H(h\bar{n}h^{-1})$  lie in  ${}^+\mathfrak{h}_{\mathfrak{p}_0}$ .

Let  $\pi$  be an irreducible representation of  $\mathfrak{g}$  on a finite-dimensional space and let  $\phi$  be a unit vector belonging to the highest weight  $\Lambda$  of  $\pi$ . Then  $|\pi(\bar{n})\phi| = e^{\Lambda(H(\bar{n}))}$ . Moreover since  $h \in \text{Cl}(A_{\mathfrak{p}}^+)$ ,  $e^{\Lambda(\log h)}$  is the greatest eigenvalue of the self-adjoint operator  $\pi(h)$  (see the proof of Lemma 35). Hence

$$e^{\Lambda(H(h\bar{n}h^{-1}))} = |\pi(h\bar{n}h^{-1})\phi| \leq e^{\Lambda(\log h)} |\pi(\bar{n}h^{-1})\phi| = |\pi(\bar{n})\phi| = e^{\Lambda(H(\bar{n}))}.$$

This proves that  $\Lambda(H(\bar{n})) - \Lambda(H(h\bar{n}h^{-1})) \geq 0$ . Now choose  $Y \in \bar{n}$  such that  $\bar{n} = \exp Y$ . Then  $\pi(\exp Y)\phi - \phi$  is obviously a sum of vectors belonging to weights lower than  $\Lambda$ . Hence  $\phi$  and  $\pi(\bar{n})\phi - \phi$  are mutually orthogonal and therefore  $|\pi(\bar{n})\phi|^2 = |\phi|^2 + |\pi(\bar{n})\phi - \phi|^2 \geq 1$ . This shows that  $\Lambda(H(\bar{n})) \geq 0$  and the required statements now follow from Lemma 35.

COROLLARY.  $\rho(H(\bar{n})) \geq 0$  for  $\bar{n} \in \bar{N}$ .

Since  $H_{\rho} \in \mathfrak{h}_{\mathfrak{p}_0}^+$ , this is an immediate consequence of the above lemma.

For any  $\bar{n} \in \bar{N}$ , let  $k(\bar{n})$  denote the unique element in  $K$  such that  $\bar{n} \in k(\bar{n})A_{\mathfrak{p}}N$ . We denote by  $C(K)$  the space of continuous functions on  $K$  and by  $dm$  the normalized Haar measure on  $M$ .

LEMMA 44. The Haar measure  $d\bar{n}$  on  $\bar{N}$  can be so normalized that

$$\int_K f(k) dk = \int_{M \times \bar{N}} f(mk(\bar{n})) e^{-2\rho(H(\bar{n}))} dm d\bar{n}$$

for any  $f \in C(K)$ . This normalization is characterized by the condition that

$$\int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1.$$

Put  $S_0 = A_{\mathfrak{p}}N$  and let  $ds_0$  denote the left-invariant measure on  $S_0$ . Similarly let  $ds$  denote the left-invariant measure on  $S = MS_0$ . We may assume that  $ds = dm ds_0$  ( $s = ms_0, m \in M, s_0 \in S_0$ ). Let  $dx$  denote the Haar measure on  $G$  and  $D(s)$  ( $s \in S$ ) the absolute value of the determinant of the restriction of  $\text{Ad}(s)$  on  $\mathfrak{g} + \mathfrak{n}$ . Since  $G = KS_0$ , a simple calculation (see [5(c), Lemma 35]) shows that

$$\int_G F(x) dx = \int_{K \times S_0} F(ks_0) D(s_0) dk ds_0 \quad (F \in C_c(G)),^{17}$$

provided  $dx$  is suitably normalized. Now  $\bar{N}S = K_1S_0$  and therefore we conclude from Corollary 2 to Lemma 40 and the corollary to Lemma 42 that the complement of  $\bar{N}S$  in  $G$  is of measure zero. On the other hand the

mapping  $(\bar{n}, s) \rightarrow \bar{n}s$  of  $\bar{N} \times S$  into  $G$  is univalent from the corollary to Lemma 41. Put  $n_0 = n \cap g_0$  and  $\bar{n}_0 = \bar{n} \cap g_0$ . Then  $g_0 = \theta(n_0) + \bar{n}_0$  and from this it follows easily that the above mapping is everywhere regular. Moreover a straightforward calculation shows that  $dx = D(s)d\bar{n}ds$  ( $x = \bar{n}s$ ). Therefore

$$\int_{K \times S_0} F(ks_0)D(s_0)dkds_0 = \int_{\bar{N} \times S} F(\bar{n}s)D(s)d\bar{n}ds.$$

Now  $\bar{n} = k(\bar{n})s(\bar{n})$  where  $s(\bar{n}) \in S_0$ . Therefore it follows from the left-invariance of  $ds$  that

$$\int_{\bar{N} \times S} F(\bar{n}s)D(s)d\bar{n}ds = \int_{\bar{N} \times S} F(k(\bar{n})s)D(s)D(s(\bar{n}))^{-1}d\bar{n}ds.$$

But  $D(s(\bar{n})) = e^{2\rho(H(\bar{n}))}$ . Hence

$$\int_{K \times S_0} F(ks_0)D(s_0)dkds_0 = \int F(k(\bar{n})ms_0)D(s_0)e^{-2\rho(H(\bar{n}))}dmd\bar{n}ds_0$$

because  $D(m) = 1$ . Now select a function <sup>17</sup>  $g \in C_c(S_0)$  such that

$$\int g(s_0)D(s_0)ds_0 = 1$$

and define  $F$  by  $F(ks_0) = f(k)g(s_0)$  ( $k \in K, s_0 \in S_0$ ) where  $f$  is a given function in  $C(K)$ . Then

$$\int F(ks_0)D(s_0)dkds_0 = \int_K f(k)dk$$

while

$$\int F(k(\bar{n})ms_0)D(s_0)e^{-2\rho(H(\bar{n}))}dmd\bar{n}ds_0 = \int f(k(\bar{n})m)e^{-2\rho(H(\bar{n}))}d\bar{n}dm.$$

But for a fixed  $m \in M$ , the measure  $d\bar{n}$  is invariant under the transformation  $\bar{n} \rightarrow \bar{n}' = m^{-1}\bar{n}m$  and  $k(\bar{n}') = m^{-1}k(\bar{n})m$ ,  $H(\bar{n}') = H(\bar{n})$ . Therefore

$$\int_K f(k)dk = \int f(mk(\bar{n}))e^{-2\rho(H(\bar{n}))}dmd\bar{n}.$$

In particular if we take  $f = 1$ , we get the relation  $\int e^{-2\rho(H(\bar{n}))}d\bar{n} = 1$ .

**COROLLARY 1.** Let  $\nu$  be a linear function on  $\mathfrak{h}_p$  and put  $\nu_+ = \nu + \rho$ ,  $\nu_- = \nu - \rho$ . Then

$$\begin{aligned} \int_K \exp\{\nu(H(hk)) - \rho(H(hk))\}dk \\ = e^{\nu_-(\log h)} \int_{\bar{N}} \exp\{\nu_-(H(h\bar{n}h^{-1})) - \nu_+(H(\bar{n}))\}d\bar{n} \end{aligned}$$

for  $h \in A_p$ .

Let  $\psi(h)$  denote the left side. Then it follows from the above lemma that

$$\psi(h) = \int \exp\{\nu_-(H(hmk(\bar{n}))) - 2\rho(H(\bar{n}))\} dmd\bar{n}.$$

But  $m$  commutes with  $h$  and therefore  $H(hmk(\bar{n})) = H(hk(\bar{n}))$ . Also  $\bar{n} \in k(\bar{n}) \exp H(\bar{n})N$  and therefore  $h\bar{n} \exp(-H(\bar{n})) \in hk(\bar{n})N$ . But this implies that  $H(hk(\bar{n})) = H(h\bar{n}) - H(\bar{n}) = \log h + H(h\bar{n}h^{-1}) - H(\bar{n})$ . Therefore

$$\psi(h) = e^{\nu_-(\log h)} \int \exp\{\nu_-(H(h\bar{n}h^{-1})) - \nu_+(H(\bar{n}))\} d\bar{n}.$$

COROLLARY 2.

$$e^{\rho(\log h)} \int_K e^{-\rho(H(hk))} dk = \int_{\bar{N}} \exp\{-\rho(H(h\bar{n}h^{-1})) - \rho(H(\bar{n}))\} d\bar{n}$$

for  $h \in A_p$ .

This follows by putting  $\nu = 0$  in Corollary 1.

The following lemma is of decisive significance for our purpose.

LEMMA 45. Let  $\epsilon$  be a positive number and  $d$  the integer of Theorem 3. Then

$$\int_{\bar{N}} e^{-\rho(H(\bar{n}))} \{1 + \rho(H(\bar{n}))\}^{-d-\epsilon} d\bar{n} < \infty.$$

Select an element  $H_0 \in \mathfrak{h}_{p_0}^+$  and put  $h_t = \exp tH_0$  ( $t \in R$ ). Then it follows from Theorem 3 and Corollary 2 above that

$$\int_{\bar{N}} \exp\{-\rho(H(h_t\bar{n}h_t^{-1})) - \rho(H(\bar{n}))\} d\bar{n} \leq c(1+t)^d \quad (t \geq 0)$$

for a suitable positive number  $c$ . Put  $\Lambda = \frac{1}{2} \sum_{\alpha \in P} \alpha$  and let  $\sigma$  denote the irreducible finite-dimensional representation of  $\mathfrak{g}$  on a space  $V$  with the highest weight <sup>28</sup>  $\Lambda$ . Choose a unit vector  $\phi_0$  in  $V$  belonging to the weight  $\Lambda$ . Then

$$|\sigma(h\bar{n}h^{-1})\phi_0|^2 = \exp 2\rho(H(h\bar{n}h^{-1}))$$

for  $h \in A_p$  and  $\bar{n} \in \bar{N}$ . Let  $\phi_0, \phi_1, \dots, \phi_p$  be an orthonormal base for  $V$  such that  $\phi_i$  belongs to the weight  $\Lambda_i$  ( $\Lambda_0 = \Lambda$ ). Then it is obvious that  $\pi(\bar{n})\phi_0 - \phi_0$  is orthogonal to  $\phi_0$  and therefore

$$|\sigma(h\bar{n}h^{-1})\phi_0|^2 = 1 + \sum_{1 \leq i \leq p} \exp\{2\Lambda_i(\log h) - 2\Lambda(\log h)\} |(\phi_i, \pi(\bar{n})\phi_0)|^2.$$

<sup>28</sup> It is known (see Weyl [8(c)]) that such a representation exists.

Let  $\beta_1, \dots, \beta_q$  be all the distinct roots in  $P_+$  and select  $X \in \mathfrak{n}_0 = \theta(\mathfrak{n}_0)$  such that  $\bar{n} = \exp X$ . Then  $\pi(\bar{n})\phi_0 - \phi_0 = \sum_{r \geq 1} \sigma(X^r)\phi_0/r!$  and if  $i \neq 0$ ,  $(\phi_i, \pi(\bar{n})\phi_0) = 0$  unless there exist nonnegative integers  $m_1, \dots, m_q$ , not all zero, such that  $\Lambda_i = \Lambda - (m_1\beta_1 + \dots + m_q\beta_q)$  on  $\mathfrak{h}_p$ . Hence if  $a = \min_{1 \leq i \leq q} \beta_i(H_0)$ , it follows that

$$\begin{aligned} |\sigma(h_t \bar{n} h_t^{-1})\phi_0|^2 &\leq 1 + \sum_{1 \leq i \leq p} e^{-2at} |(\phi_i, \pi(\bar{n})\phi_0)|^2 \\ &\leq 1 + e^{-2at} |\pi(\bar{n})\phi_0|^2 = 1 + \exp\{2\rho(H(\bar{n})) - 2at\} \quad (t \geq 0). \end{aligned}$$

This proves that

$$\exp\{2\rho(H(h_t \bar{n} h_t^{-1}))\} \leq 1 + \exp\{2\rho(H(\bar{n})) - 2at\}$$

and therefore

$$\int e^{-\rho(H(\bar{n}))} \{1 + e^{-2at + 2\rho(H(\bar{n}))}\}^{-\frac{1}{2}} d\bar{n} \leq c(1+t)^d \quad (t \geq 0).$$

Now for any integer  $r \geq 0$ , let  $\bar{N}_r$  denote the set of all points  $\bar{n} \in \bar{N}$  where  $\rho(H(\bar{n})) \leq 2^r$ . It is obvious from its definition that  $a$  is positive. Therefore if  $t = 2^r/a$  and  $\bar{n} \in \bar{N}_r$ , it is clear that  $e^{-2at} e^{2\rho(H(\bar{n}))} \leq 1$  and therefore

$$\int_{\bar{N}_r} e^{-\rho(H(\bar{n}))} 2^{-\frac{1}{2}} d\bar{n} \leq c(1 + a^{-1}2^r)^d.$$

Hence

$$\int_{\bar{N}_r} e^{-\rho(H(\bar{n}))} d\bar{n} \leq c_1 2^{rd} \quad (r \geq 0)$$

where  $c_1 = 2^{\frac{1}{2}}c(1 + a^{-1})^d$ . Let  $\bar{N}_{r+1} - \bar{N}_r$  denote the complement of  $\bar{N}_r$  in  $\bar{N}_{r+1}$ . Then if  $\epsilon > 0$ ,

$$\int_{\bar{N}_{r+1} - \bar{N}_r} e^{-\rho(H(\bar{n}))} \{1 + \rho(H(\bar{n}))\}^{-d-\epsilon} d\bar{n} \leq 2^{-r(d+\epsilon)} \int_{\bar{N}_{r+1}} e^{-\rho(H(\bar{n}))} d\bar{n} \leq c_2 2^{-r\epsilon}$$

where  $c_2 = 2^d c_1$ . Therefore

$$\int_{\bar{N} - \bar{N}_0} e^{-\rho(H(\bar{n}))} \{1 + \rho(H(\bar{n}))\}^{-d-\epsilon} d\bar{n} \leq c_2 \sum_{r \geq 0} 2^{-r\epsilon} < \infty.$$

Hence in order to complete the proof, it would be sufficient to verify that  $\bar{N}_0$  is compact. But this is an immediate consequence of Lemma 40.

**COROLLARY.** *Let  $\epsilon$  be a positive number. Then*

$$\int_{\bar{N}} \exp\{-(1 + \epsilon)\rho(H(\bar{n}))\} d\bar{n} < \infty.$$

This is obvious because  $e^{-\epsilon t}(1+t)^{d+1}$  remains bounded for  $t \geq 0$ .

For any  $H \in \mathfrak{h}_p$  we denote by  $\operatorname{Re} H$  and  $\operatorname{Im} H$  the elements  $H_1, H_2$  respectively in  $\mathfrak{h}_{p_0}$  such that  $H = H_1 + (-1)^{\frac{1}{2}} H_2$ .

**THEOREM 4.** Define the function  $\mathbf{c}$  as in Lemma 37 and normalize the Haar measure  $d\bar{n}$  on  $\bar{N}$  by the condition

$$\int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1.$$

Then if  $H'$  is an element in  ${}^*\mathfrak{h}_p'$  such that  $-\operatorname{Im} H' \in \mathfrak{h}_{p_0}^+$ , we have the relation

$$\mathbf{c}(H') = \int_{\bar{N}} \exp\{ -(-1)^{\frac{1}{2}} \langle H', H(\bar{n}) \rangle - \rho(H(\bar{n})) \} d\bar{n}.$$

Put  $\nu(H) = (-1)^{\frac{1}{2}} \langle H', H \rangle$  for  $H \in \mathfrak{h}_p$ . Then it follows from Lemma 37 and Corollary 1 of Lemma 44 that

$$\sum_{s \in W} \mathbf{c}(sH') e^{-\nu(\log h)\Phi(sH':h)} = \int_{\bar{N}} \exp\{ \nu_-(H(h\bar{n}h^{-1})) - \nu_+(H(\bar{n})) \} d\bar{n}$$

for  $h \in A_p(M)$ . Select a point  $H_0 \in \mathfrak{h}_{p_0}^+$  and put  $h_t = \exp tH_0$  ( $t \geq 0$ ). Then it follows from Corollary 1 of Lemma 35 that the real part of  $(-1)^{\frac{1}{2}} \langle sH' - H', H_0 \rangle$  is negative if  $s \neq 1$  ( $s \in W$ ). Therefore it is obvious from Lemma 31 that  $\lim_{t \rightarrow +\infty} e^{-\nu(H_0)\Phi(sH':h_t)} = 1$  or 0 according as  $s = 1$  or not. Hence

$$\mathbf{c}(H') = \lim_{t \rightarrow +\infty} \int_{\bar{N}} \exp\{ \nu_-(H(h_t\bar{n}h_t^{-1})) - \nu_+(H(\bar{n})) \} d\bar{n}.$$

On the other hand we can evidently choose a positive number  $\epsilon < 1$  such that  $-\operatorname{Im} H' - \epsilon H_\rho \in \mathfrak{h}_{p_0}^+$ . Put  $\nu' = \nu - \epsilon\rho$ . Then  $\nu_- = \nu' - (1 - \epsilon)\rho$ ,  $\nu_+ = \nu' + (1 + \epsilon)\rho$  and  $\operatorname{Re} \nu'(H) \geq 0$  for  $H \in {}^*\mathfrak{h}_{p_0}$ . On the other hand

$$\begin{aligned} \nu_-(H(h_t\bar{n}h_t^{-1})) - \nu_+(H(\bar{n})) &= \nu'(H(h_t\bar{n}h_t^{-1}) - H(\bar{n})) - (1 - \epsilon)\rho(H(h_t\bar{n}h_t^{-1})) \\ &\quad - (1 + \epsilon)\rho(H(\bar{n})). \end{aligned}$$

Therefore it follows from Lemma 43 that

$$\operatorname{Re}\{ \nu_-(H(h_t\bar{n}h_t^{-1})) - \nu_+(H(\bar{n})) \} \leq -(1 + \epsilon)\rho(H(\bar{n}))$$

for  $t \geq 0$ . Hence we conclude from the corollary of Lemma 45 that

$$\mathbf{c}(H') = \int_{\bar{N}} \lim_{t \rightarrow +\infty} \exp\{ \nu_-(H(h_t\bar{n}h_t^{-1})) - \nu_+(H(\bar{n})) \} d\bar{n}.$$

Now for a fixed  $\bar{n} \in \bar{N}$ , select  $X \in \bar{n}$  such that  $\exp X = \bar{n}$ . Then  $h_t\bar{n}h_t^{-1}$

$= \exp(\text{Ad}(h_t)X)$  and if  $X = \sum_{\beta \in P_+} a_\beta X_{-\beta}$  ( $a_\beta \in C$ ), it is obvious that  $\text{Ad}(h_t)X = \sum_{\beta} a_\beta e^{-t\beta(H_0)} X_{-\beta}$ . Since  $\beta(H_0) > 0$  for every  $\beta \in P_+$ , it follows that  $h_t \bar{n} h_t^{-1} \rightarrow 1$  as  $t \rightarrow +\infty$  and therefore

$$c(H') = \int_{\bar{N}} \exp\{-v_+(H(\bar{n}))\} d\bar{n}.$$

This proves the theorem.

We recall that  $\mathfrak{h}_{p_0}'$  is the set of those points  $H \in \mathfrak{h}_{p_0}$  where  $\alpha(H) \neq 0$  for every  $\alpha \in P_+$ . We have seen in Section 10 that  $\mathfrak{h}_{p_0}' \subset {}^*\mathfrak{h}_{p_0}'$  and therefore  $c$  is defined on  $\mathfrak{h}_{p_0}'$ .

**COROLLARY.** *Let  $H_0$  be any point in  $\mathfrak{h}_{p_0}'$ . Then*

$$c(H_0) = \lim_{\epsilon \rightarrow 0} \int_{\bar{N}} \exp\{-( -1)^{\frac{1}{2}} \langle H_0, H(\bar{n}) \rangle - (1 + \epsilon) \rho(H(\bar{n}))\} d\bar{n} \quad (\epsilon > 0).$$

Put  $H_\epsilon = H_0 - (-1)^{\frac{1}{2}} \epsilon H_\rho$  where  $\epsilon$  is a positive number. It is obvious that  $H_\epsilon \in {}^*\mathfrak{h}_{p_0}'$  and  $-\text{Im } H_\epsilon \in \mathfrak{h}_{p_0}^+$  if  $\epsilon$  is sufficiently small. Moreover  $c(H_0) = \lim_{\epsilon \rightarrow 0} c(H_\epsilon)$  since  $c$  is holomorphic at  $H_0$ . Hence the corollary follows immediately from Theorem 4.

We shall now give another interpretation of the above corollary. Let  $\mathfrak{D}(\mathfrak{h}_p)$  be the algebra of all polynomial differential operators [5(k), § 2] on  $\mathfrak{h}_p$  and let  $\mathcal{B}(\mathfrak{h}_{p_0})$  denote the space of all functions on  $\mathfrak{h}_{p_0}$  of class  $C^\infty$  such that  $Da$  remains bounded on  $\mathfrak{h}_{p_0}$  for every  $D \in \mathfrak{D}(\mathfrak{h}_{p_0})$ . We topologize  $\mathcal{B}(\mathfrak{h}_{p_0})$  in the usual way and consider a  $\mathcal{B}$ -distribution  $\tau$  on  $\mathfrak{h}_{p_0}$  (see [5(k), § 2]). Let  $dH$  denote the regular Euclidean measure [5(k), p. 91] on  $\mathfrak{h}_{p_0}$  and for any  $a \in \mathcal{B}(\mathfrak{h}_{p_0})$ , put

$$a'(H_0) = \int_{\mathfrak{h}_{p_0}} \exp\{-( -1)^{\frac{1}{2}} \langle H_0, H \rangle\} a(H) dH \quad (H_0 \in \mathfrak{h}_{p_0}).$$

Then the Fourier transform of  $\tau$  is the  $\mathcal{B}$ -distribution  $\bar{\tau}$  given by  $\bar{\tau}(a) = \tau(a')$  ( $a \in \mathcal{B}(\mathfrak{h}_{p_0})$ ).

**THEOREM 5.** *Let  $\bar{\tau}$  denote the Fourier transform of the distribution  $\tau$  defined by*

$$\tau(a) = \int_{\bar{N}} a(H(\bar{n})) e^{-\rho(H(\bar{n}))} d\bar{n} \quad (a \in \mathcal{B}(\mathfrak{h}_{p_0})).$$

*Then  $\bar{\tau}$  coincides on  $\mathfrak{h}_{p_0}'$  with the analytic function  $c$ .*

We know from Lemma 45 that

$$c = \int_{\bar{N}} e^{-\rho(H(\bar{n}))} \{1 + \rho(H(\bar{n}))\}^{-r} d\bar{n} < \infty,$$

if  $r$  is a suitable positive integer. Hence

$$\int_{\bar{N}} |a(H(\bar{n}))| e^{-\rho(H(\bar{n}))} d\bar{n} \leq c \sup_{H \in \mathfrak{h}_{\mathfrak{p}_0}} |(1 + \rho(H))^r a(H)| \quad (a \in \mathcal{B}(\mathfrak{h}_{\mathfrak{p}_0}))$$

and therefore  $\tau$  is a  $\mathcal{B}$ -distribution. Moreover this inequality implies that

$$\tau(a) = \lim_{\epsilon \rightarrow 0} \int_{\bar{N}} a(H(\bar{n})) e^{-(1+\epsilon)\rho(H(\bar{n}))} d\bar{n} \quad (\epsilon > 0).$$

Now fix an element <sup>17</sup>  $a \in C_c^\infty(\mathfrak{h}_{\mathfrak{p}_0}')$ . Then

$$\bar{\tau}(a) = \lim_{\epsilon \rightarrow 0} \int_{\bar{N}} a'(H(\bar{n})) e^{-(1+\epsilon)\rho(H(\bar{n}))} d\bar{n}.$$

On the other hand

$$\begin{aligned} \int_{\bar{N}} a'(H(\bar{n})) e^{-(1+\epsilon)\rho(H(\bar{n}))} d\bar{n} \\ = \int_{\bar{N}} e^{-(1+\epsilon)\rho(H(\bar{n}))} d\bar{n} \int_{\mathfrak{h}_{\mathfrak{p}_0}} \exp\{-(1+\epsilon)\langle H(\bar{n}), H \rangle\} a(H) dH \end{aligned}$$

and it follows from the corollary of Lemma 45 that

$$\int_{\bar{N}} e^{-(1+\epsilon)\rho(H(\bar{n}))} d\bar{n} \int_{\mathfrak{h}_{\mathfrak{p}_0}} |a(H)| dH < \infty.$$

Therefore by Fubini's Theorem

$$\bar{\tau}(a) = \lim_{\epsilon \rightarrow 0} \int_{\mathfrak{h}_{\mathfrak{p}_0}} a(H) dH \int_{\bar{N}} \exp\{-(1+\epsilon)\langle H(\bar{n}), H \rangle - (1+\epsilon)\rho(H(\bar{n}))\} d\bar{n}.$$

Hence if  $H_\epsilon = H - (-1)^{\frac{1}{2}} \epsilon H_\rho$ , we can conclude from Theorem 4 that

$$\bar{\tau}(a) = \lim_{\epsilon \rightarrow 0} \int_{\mathfrak{h}_{\mathfrak{p}_0}} a(H) \mathbf{c}(H_\epsilon) dH.$$

Now let  $\Omega$  denote the carrier of  $a$ . Then  $\Omega$  is a compact subset of  $\mathfrak{h}_{\mathfrak{p}_0}'$  and therefore  $\mathbf{c}$  is holomorphic on some complex neighborhood of  $\Omega$  in  $\mathfrak{h}_{\mathfrak{p}}$ . Hence it is obvious that

$$\bar{\tau}(a) = \int_{\mathfrak{h}_{\mathfrak{p}_0}} a(H) \mathbf{c}(H) dH.$$

This proves that  $\bar{\tau}$  coincides with  $\mathbf{c}$  on  $\mathfrak{h}_{\mathfrak{p}_0}'$ .

**12. Further study of the function  $\mathbf{c}$ .** We shall now investigate the function  $\mathbf{c}$  a little more closely on  $\mathfrak{h}_{\mathfrak{p}_0}'$ . Let  $\mathfrak{F}$  be the space of all linear functions <sup>20</sup> on  $\mathfrak{h}_{\mathfrak{p}}$  and  $\mathfrak{F}_R$  be the subspace (over  $R$ ) consisting of the real <sup>8</sup>

<sup>20</sup> Although  $\mathfrak{F}$  and  $\mathfrak{h}_{\mathfrak{p}}$  have been identified under the bilinear form  $B$ , it is sometimes convenient to distinguish between the two.

functions. For any  $\lambda \in \mathfrak{F}$ , let  $\lambda_R$  and  $\lambda_I$  denote the real and imaginary parts of  $\lambda$  so that  $\lambda = \lambda_R + (-1)^{\frac{1}{2}} \lambda_I$  ( $\lambda_R, \lambda_I \in \mathfrak{F}_R$ ). Then the following lemma establishes an important inequality.

LEMMA 46. *Put*

$$\psi(\lambda; x) = \int_K \exp\{(-1)^{\frac{1}{2}} \lambda(H(xk)) - \rho(H(xk))\} dk$$

for  $\lambda \in \mathfrak{F}$  and  $x \in G$ . Then for any  $b \in \mathfrak{B}$ , we can select an integer  $d \geq 0$  and a positive constant  $a$  such that<sup>30</sup>

$$|\psi(\lambda; x; b)| \leq a(1 + \|\lambda\|)^d \psi((-1)^{\frac{1}{2}} \lambda_I; x)$$

for all  $x \in G$  and  $\lambda \in \mathfrak{F}$ .

Let  $\beta_1, \dots, \beta_r$  be a fundamental system of roots in  $P$ . Select linear functions  $\Lambda_1, \dots, \Lambda_r$  on  $\mathfrak{h}$  such that<sup>9</sup>  $\Lambda_i(H_{\beta_j}) = 2\beta_j(H_{\beta_j})\delta_{ij}$  ( $1 \leq i, j \leq r$ ) where, for any root  $\beta$ ,  $H_{\beta}$  is defined in the same way as in Section 7. Then, by Theorem 1 of [5(b)], there exists an irreducible representation  $\pi_i$  of  $\mathfrak{g}$  on a finite-dimensional space  $V_i$  with the highest weight  $\Lambda_i$ . Select a unit vector  $\xi_i$  in  $V_i$  belonging to  $\Lambda_i$ . Extend  $\lambda$  and  $\rho$  to linear functions on  $\mathfrak{h}$  by defining them to be zero on  $\mathfrak{h}_I$ . Then  $\lambda = \sum_{1 \leq i \leq r} \lambda_i \Lambda_i$  and  $\rho = \sum_{1 \leq i \leq r} \rho_i \Lambda_i$  where  $\lambda_i \in C$  and  $\rho_i \in R$ . First we shall prove the following result.

LEMMA 47. *Suppose  $\lambda \in \mathfrak{F}$  and  $x \in G$ . Then*

$$\psi(\lambda; x) = \psi(-\lambda; x^{-1}) = \int_K \prod_{1 \leq i \leq r} |\pi_i(x^{-1}k)\xi_i|^{-v_i} dk$$

where  $v_i = (-1)^{\frac{1}{2}} \lambda_i + \rho_i$   $1 \leq i \leq r$ .

For any  $k \in K$ , let  $k_x$  denote the unique element in  $K$  such that  $xk \in k_x A_p N$ . Then  $k \rightarrow k_x$  is a topological mapping of  $K$  onto itself and  $dk = \exp\{2\rho(H(xk))\} dk_x$  (see [5(c), p. 241]). Hence

$$\psi(\lambda; x) = \int_K \exp\{(-1)^{\frac{1}{2}} \lambda(H(xk)) + \rho(H(xk))\} dk_x.$$

Replacing  $k$  by  $k_{x^{-1}}$  in this integral and making use of the relation  $H(xk_{x^{-1}}) = -H(x^{-1}k)$  (see [5(c), Lemma 36]), we get

$$\psi(\lambda; x) = \int_K \exp\{-(-1)^{\frac{1}{2}} \lambda(H(x^{-1}k)) - \rho(H(x^{-1}k))\} dk = \psi(-\lambda; x^{-1}).$$

<sup>30</sup> In view of the identification of  $\mathfrak{F}$  and  $\mathfrak{h}_p$ ,  $\|\lambda\|$  is well-defined ( $\lambda \in \mathfrak{F}$ ).

But  $|\pi_i(x^{-1}k)\xi_i| = \exp\{\Lambda_i(H(x^{-1}k))\}$ . Hence

$$\psi(-\lambda: x^{-1}) = \int_K \prod_{1 \leq i \leq r} |\pi_i(x^{-1}k)\xi_i|^{-\nu_i} dk.$$

Now we come to the proof of Lemma 46. Select a base  $X_1, \dots, X_n$  for  $\mathfrak{g}_0$  over  $R$  and for any  $x \in G$  and  $t = (t_1, \dots, t_n) \in R^n$ , put  $x_t = x \exp(t_1 X_1 + \dots + t_n X_n)$ . For any ordered set  $M = (m_1, \dots, m_n)$  of nonnegative integers, we write  $|M| = m_1 + \dots + m_n$ ,  $t^M = t_1^{m_1} \dots t_n^{m_n}$  and denote by  $X(M)$  the coefficient (in  $\mathfrak{B}$ ) of  $t^M$  in  $(|M|!)^{-1} (t_1 X_1 + \dots + t_n X_n)^{|M|}$ . Also put  $|t| = \max_{1 \leq i \leq n} |t_i|$  and  $M + M' = (m_1 + m'_1, \dots, m_n + m'_n)$  if  $M' = (m'_1, \dots, m'_n)$ . Let  $E_j$  be the space of all endomorphisms of  $V_j$  ( $1 \leq j \leq r$ ). For any  $T \in E_j$ , put  $|T| = \sup_{|v| \leq 1} |Tv|$  ( $v \in V_j$ ). Then  $E_j$  is a Banach space under this norm and

$$\pi_j(\exp\{-(t_1 X_1 + \dots + t_n X_n)\}) = \sum_M t^M (-1)^{|M|} \pi_j(X(M)),$$

the series converging absolutely and uniformly in  $E_j$  (see [5(c), § 5]) provided  $|t|$  remains bounded. Define  $\bar{\theta}$  as in Section 5 and let  $b \rightarrow b^*$  ( $b \in \mathfrak{B}$ ) be the anti-automorphism of  $\mathfrak{B}$  over  $R$  which coincides with  $-\bar{\theta}$  on  $\mathfrak{g}$ . Then it is clear that  $\pi_j(b^*)$  is the adjoint of  $\pi_j(b)$  (in the sense of Hilbert space theory). Put

$$b_M = \sum_{M_1 + M_2 = M} (-1)^{|M|} (X(M_1))^* X(M_2)$$

for any  $M$ . Then it is obvious that

$$|\pi_j(x_t^{-1}k)\xi_j|^2 = |\pi_j(x^{-1}k)\xi_j|^2 + \sum_{|M| \geq 1} t^M (\pi_j(x^{-1}k)\xi_j, \pi_j(b_M)\pi_j(x^{-1}k)\xi_j)$$

for all  $k, x$  and  $t$  ( $1 \leq j \leq r$ ). Put

$$\Psi_{M,j}(x) = |\pi_j(x)\xi_j|^{-2} (\pi_j(x)\xi_j, \pi_j(b_M)\pi_j(x)\xi_j) \quad (x \in G).$$

Then  $|\Psi_{M,j}(x)| \leq |\pi_j(b_M)|$ . Hence  $\Psi_{M,j}$  is a bounded analytic function on  $G$  and

$$|\pi_j(x_t^{-1}k)\xi_j|^2 = |\pi_j(x^{-1}k)\xi_j|^2 \{1 + \sum_{|M| \geq 1} t^M \Psi_{M,j}(x^{-1}k)\}.$$

Obviously this series converges uniformly with respect to  $x, k$  and  $t$  provided  $x$  varies within a compact subset of  $G$  and  $|t|$  remains bounded. Therefore by the binomial theorem,

$$\begin{aligned} \exp\{-(1)^{\frac{1}{2}}\lambda(H(x^{-1}k)) - \rho(H(x^{-1}k))\} &= \prod_{1 \leq j \leq r} |\pi_j(x_t^{-1}k)\xi_j|^{-\nu_j} \\ &= \exp\{-(1)^{\frac{1}{2}}\lambda(H(x^{-1}k)) - \rho(H(x^{-1}k))\} \sum_M t^M \Psi_M(x^{-1}k: \lambda) \end{aligned}$$

provides  $|t|$  is sufficiently small. Here  $\Psi_M(x:\lambda)$  is a function on  $G \times \mathfrak{F}$  which can be written as a polynomial in  $v_j$  and  $\Psi_{M',j}$  ( $1 \leq j \leq r$ ,  $|M'| \leq |M|$ ) with constant coefficients. Therefore it is clear that there exists a positive number  $a_M$  and an integer  $d_M \geq 0$  such that

$$|\Psi_M(x:\lambda)| \leq a_M(1 + \|\lambda\|)^{d_M} \quad (x \in G, \lambda \in \mathfrak{F}).$$

Moreover it is obvious that for a fixed  $x$ , the above series converges uniformly with respect to  $k$  and  $t$  provided  $|t|$  is sufficiently small. Therefore by integrating over  $K$ , we get an expansion of  $\psi(-\lambda:x_t^{-1}) = \psi(\lambda:x_t)$  in powers of  $t_1, \dots, t_n$ . This shows that if  $M = (m_1, \dots, m_n)$ ,

$$\begin{aligned} \psi(\lambda:x;X(M)) &= (m_1! \cdots m_n!)^{-1} \{ \partial^{m_1+\dots+m_n} / \partial t_1^{m_1} \cdots \partial t_n^{m_n} \} \psi(\lambda:x_t) \Big|_{t=0} \\ &= \int_K \Psi_M(x^{-1}k:\lambda) \exp \{ -(-1)^{\frac{1}{2}} \lambda(H(x^{-1}k)) - \rho(H(x^{-1}k)) \} dk. \end{aligned}$$

Hence

$$|\psi(\lambda:x;X(M))| \leq a_M(1 + \|\lambda\|)^{d_M} \psi((-1)^{\frac{1}{2}} \lambda_I : x)$$

and the assertion of the lemma follows from the fact that the elements  $X(M)$  form a base for  $\mathfrak{B}$ .

The following result will be needed in another paper.

LEMMA 48. *Let  $\Omega$  be a compact set in  $G$ . Then we can select a positive number  $c$  such that  $\psi(0:xy) \leq c\psi(0:x)$  for all  $x \in G$  and  $y \in \Omega$ .*

It is clear that

$$e^{-\rho(H(y^{-1}x^{-1}k))} = \prod_{1 \leq j \leq r} |\pi_j(y^{-1}x^{-1}k)\xi_j|^{-\rho_j}$$

and

$$|\pi_j(y)|^{-1} |\pi_j(x^{-1}k)\xi_j| \leq |\pi_j(y^{-1}x^{-1}k)\xi_j| \leq |\pi_j(y^{-1})| |\pi_j(x^{-1}k)\xi_j|.$$

Therefore if we put

$$c = \sup_{y \in \Omega} \prod_{1 \leq j \leq r} \{ |\pi_j(y)|^{\rho_j} + |\pi_j(y^{-1})|^{-\rho_j} \},$$

it follows that

$$e^{-\rho(H(y^{-1}x^{-1}k))} \leq ce^{-\rho(H(x^{-1}k))}$$

for  $y \in \Omega$ . In view of Lemma 47 our assertion is now obvious.

Now for any positive number  $\epsilon$ , let  $\mathfrak{F}_\epsilon$  denote the open neighborhood of  $\mathfrak{F}_R$  in  $\mathfrak{F}$  consisting of those points  $\lambda \in \mathfrak{F}$  for which  $^{27} |\operatorname{Im} \lambda_j| < \epsilon/2$  ( $1 \leq j \leq r$ ). Then it follows from Lemmas 2 and 35 and Corollary 2 of Lemma 35 that

$$|\lambda_l(H(hk))| \leq \frac{1}{2}\epsilon \sum_{1 \leq j \leq r} |\Lambda_j(H(hk))|$$

$$\leq \frac{1}{2}\epsilon \sum_{1 \leq j \leq r} \Lambda_j(\log h + \log h^*) = \epsilon \rho(\log h) \quad (h \in \text{Cl}(A_p^+)),$$

since  $\sum_{1 \leq j \leq r} \Lambda_j = \frac{1}{2} \sum_{\beta \in P} \beta$  and  $\rho(\log h^*) = \rho(\log h)$ . Therefore we get the following result.

LEMMA 49. Suppose  $\lambda \in \mathfrak{F}_\epsilon$  and  $h \in \text{Cl}(A_p^+)$ . Then

$$\psi((-1)^{\frac{1}{2}\lambda_l}: h) \leq e^{\epsilon \rho(\log h)} \psi(0: h).$$

Lemmas 46 and 49, together with Theorem 3, give us an estimate for  $\psi(\lambda: h; b)$  which will be of great value in the discussion below.

Select  $u_1, \dots, u_w \in \mathfrak{S}_p$  as in the corollary of Lemma 8 and put

$$\psi_i(\lambda: H) = e^{\rho(H)} \psi(\lambda: \exp H; u_i') \quad (\lambda \in \mathfrak{F}, H \in \mathfrak{h}_{p_0}, 1 \leq i \leq w)$$

where  $u_i' = e^{\rho} u_i \circ e^{-\rho}$ . Fix an element  $H_0 \in \mathfrak{h}_{p_0}^+$  and let  $\Psi(\lambda: t)$  ( $t \in R$ ) denote the one-column matrix  $\psi_i(\lambda: tH_0)$   $1 \leq i \leq w$ . Let us consider  $d\Psi/dt$ . Choose  $q_{ij} \in I_g$  such that  $H_0 u_j = \sum_{1 \leq i \leq w} u_i \gamma(q_{ij})$  ( $1 \leq j \leq w$ ). Then if  $h_t = \exp tH_0$  ( $t \in R$ ), it follows from Theorem 2 and Lemmas 18 and 23 that

$$d\psi_j(\lambda: tH_0)/dt = \sum_{1 \leq i \leq w} \psi_i(\lambda: tH_0) \gamma(q_{ij}: (-1)^{\frac{1}{2}\lambda}) + \theta_j(\lambda: h_t) \quad (t \geq 1)$$

where

$$\theta_j(\lambda: h) = \sum_{1 \leq i \leq w} \sum_{1 \leq k \leq s} g_{jik}(h) \psi_i(\lambda: \log h) \gamma(q_{jk}: (-1)^{\frac{1}{2}\lambda}) \quad (\lambda \in \mathfrak{F}, h \in A_p^+),$$

$g_{jik} \in \mathfrak{R}$ ,  $q_{jk} \in I_g$  and  $s$  is a positive integer. Put  $\beta(H_0) = \min_{\alpha \in P_+} \alpha(H_0)$ . Then

it is obvious that for any  $g \in \mathfrak{R}$ , there exists a positive number  $c$  such that  $|g(h_t)| \leq ce^{-2t\beta(H_0)}$  ( $t \geq 1$ ). Let  $\Theta(\lambda: t)$  denote the one-column matrix  $\theta_i(\lambda: h_t)$   $1 \leq i \leq w$  and put  $\|a\| = (\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} |a_{ij}|^2)^{\frac{1}{2}}$  for any  $m \times n$  matrix

$a = (a_{ij})$  with complex coefficients. Then it follows from Lemmas 46 and 49 and Theorem 3 that

$$\|\Theta(\lambda: t)\| \leq c_1 \exp\{\epsilon t \rho(H_0) - 2t\beta(H_0)\} (1 + \|\lambda\|)^{d_1 t^{d_1}}$$

for  $t \geq 1$  and  $\lambda \in \mathfrak{F}_\epsilon$ . Here  $c_1$  is a positive number and  $d_1$  a nonnegative integer. Let  $\Xi(\lambda)$  denote the  $w \times w$  matrix given by  $\Xi_{ij}(\lambda) = \gamma(q_{ji}: (-1)^{\frac{1}{2}\lambda})$

<sup>31</sup> Put  $\Lambda = \frac{1}{2} \sum_{\beta \in P} \beta$ . Then  $2\Lambda(H\beta_j) = \beta_j(H\beta_j)$  for  $1 \leq j \leq r$  (see Weyl [8(c)]).

Hence  $\Lambda = \Lambda_1 + \dots + \Lambda_r$ .

( $1 \leq i, j \leq w$ ). Then the above equation can be written in matrix notation as follows:

$$d\Psi(\lambda:t)/dt = \Xi(\lambda)\Psi(\lambda:t) + \Theta(\lambda:t) \quad (t \geq 1).$$

Now put  $\Psi_0(\lambda:t) = \exp(-t\Xi(\lambda))\Psi(\lambda:t)$  and  $\Theta_0(\lambda:t) = \exp(-t\Xi(\lambda))\Theta(\lambda:t)$ . Then it is clear that

$$d\Psi_0(\lambda:t)/dt = \Theta_0(\lambda:t) \quad (t \geq 1).$$

We shall now estimate  $\|\Theta_0(\lambda:t)\|$ . It is obvious from the definition of  $\Xi(\lambda)$  that

$$H_0 u_j \equiv \sum_{1 \leq i \leq w} \Xi_{ji}(\lambda) u_i \pmod{SJ_\lambda}.$$

in the notation of Section 3, if we put  $\lambda^* = (-1)^{\frac{1}{2}}\lambda$ . Hence we conclude from Corollary 2 of Lemma 14 that  $\lambda^*(sH_0)$  ( $s \in W$ ) are all the eigenvalues of  $\Xi(\lambda)$ . Put  $\tau(\lambda) = 2w \max_{s \in W} |\lambda_I(sH_0)|$ . Then it follows from Lemma 60 of the Appendix that

$$\|\exp(-t\Xi(\lambda))\| \leq e^{t\tau(\lambda)} \sum_{0 \leq k < w} t^k \|\Xi(\lambda)\|^k \quad (t > 0).$$

Therefore since  $\|\Theta_0(\lambda:t)\| \leq \|\exp(-t\Xi(\lambda))\| \|\Theta(\lambda:t)\|$ , it is obvious that there exists a positive number  $c_2$  and an integer  $d_2 \geq 0$  such that

$$\|\Theta_0(\lambda:t)\| \leq c_2(1 + \|\lambda\|)^{d_2} t^{d_2} \exp\{\tau(\lambda)t + \epsilon t\rho(H_0) - 2t\beta(H_0)\}$$

for  $\lambda \in \mathfrak{F}_\epsilon$  and  $t \geq 1$ . Moreover by choosing  $\epsilon$  sufficiently small, we can obviously arrange that  $\beta(H_0) \geq \tau(\lambda) + \epsilon\rho(H_0)$  for all  $\lambda \in \mathfrak{F}_\epsilon$ . Then in this case

$$\|\Theta_0(\lambda:t)\| \leq c_2(1 + \|\lambda\|)^{d_2} t^{d_2} e^{-t\beta(H_0)} \quad (t \geq 1, \lambda \in \mathfrak{F}_\epsilon).$$

This shows that the integral  $\int_1^\infty \|\Theta_0(\lambda:t)\| dt$  converges uniformly with respect to  $\lambda$  as  $\lambda$  varies in a compact subset of  $\mathfrak{F}_\epsilon$ . Moreover there exists a positive number  $c_3$  such that

$$\int_1^\infty \|\Theta_0(\lambda:t)\| dt \leq c_3(1 + \|\lambda\|)^{d_2} \quad (\lambda \in \mathfrak{F}_\epsilon).$$

But since  $d\Psi_0(\lambda:t)/dt = \Theta_0(\lambda:t)$ , this implies that  $\Psi_0(\lambda:t)$  tends to a limit as  $t \rightarrow +\infty$ . If we denote this limit by  $\Psi_0(\lambda:\infty)$ , we have the relation

$$\Psi_0(\lambda:\infty) = \Psi_0(\lambda:1) + \int_1^\infty \Theta_0(\lambda:t) dt \quad (\lambda \in \mathfrak{F}_\epsilon).$$

In view of the uniform convergence of the integral, it is clear that  $\Psi_0(\lambda: \infty)$  is a holomorphic (matrix-) function of  $\lambda$  on  $\mathfrak{F}_\epsilon$ . Moreover since  $\|\Psi_0(\lambda: 1)\| \leq \|\exp(-\Xi(\lambda))\| \|\Psi(\lambda: 1)\|$ , it follows without difficulty (by making use of Lemmas 46 and 49 and the above estimate for  $\|\exp(-\Xi(\lambda))\|$ ) that  $\|\Psi_0(\lambda: 1)\| \leq c_4(1 + \|\lambda\|)^{d_4}$  for suitable positive numbers  $c_4, d_4$ . Therefore  $\|\Psi_0(\lambda: \infty)\| \leq c_5(1 + \|\lambda\|)^{d_5}$  ( $\lambda \in \mathfrak{F}_\epsilon$ ) for an appropriate choice of  $c_5, d_5$ . Hence if we apply Lemma 58 of the Appendix to  $V = \mathfrak{F}_{\epsilon/2}$ , we get the following result.

LEMMA 50. *If  $\epsilon$  is sufficiently small, there exists an integer  $d \geq 0$  with the following property. For any  $u \in S(\mathfrak{h}_p)$ , the function*

$$(1 + \|\lambda\|)^{-d} \|\Psi_0(\lambda; \theta(u): \infty)\|$$

*remains bounded as  $\lambda$  varies in  $\mathfrak{F}_\epsilon$ .*

Now define  $\pi$  as in Lemma 11 and let  $\mathfrak{F}'_\epsilon$  denote the set of those elements  $\lambda \in \mathfrak{F}_\epsilon$  where  $\pi(\lambda) \neq 0$ . We may assume that  $\epsilon$  is so small that  $\|\lambda\| < \|\Lambda\|/2$  for any  $\lambda \in \mathfrak{F}_\epsilon$  and  $\Lambda \in L'$  (see Section 8 for the definition of  $L'$ ). Then it follows without difficulty that  $\mathfrak{F}_\epsilon \subset {}^* \mathfrak{h}_p$  and therefore  $\mathfrak{F}'_\epsilon \subset {}^* \mathfrak{h}_p'$ . Hence if  $\lambda \in \mathfrak{F}'_\epsilon$ , we conclude from Lemma 37 that

$$\psi_i(\lambda: tH_0) = \sum_{s \in W} c(s\lambda) \Phi(s\lambda: h_t; u_i) \quad (1 \leq i \leq w, t > M)$$

where  $M$  is a suitable positive number. Define  $e_{sH} \in S(\mathfrak{h}_p)$  ( $s \in W, H \in \mathfrak{h}_p$ ) as in Lemma 14 and put  $e_s = e_{s\Lambda}$ . Moreover define the  $w \times w$  matrix  $E(s)$  as follows:

$$e_s u_i = \sum_{1 \leq j \leq w} E_{ij}(s) u_j \bmod SJ_\lambda. \quad (E_{ij}(s) \in C).$$

Then it follows from Lemma 14 that  $\sum_{s \in W} \epsilon(s) E(s) = \pi(\lambda^*) I$  where  $I$  is the unit matrix. Moreover  $e_s H_0 = H_0 e_s = \lambda^* (s^{-1} H_0) e_s \bmod SJ_\lambda$  by the same lemma and therefore

$$E(s) \Xi(\lambda) = \Xi(\lambda) E(s) = \lambda^* (s^{-1} H_0) E(s).$$

Hence

$$\begin{aligned} \pi(\lambda^*) \Psi_0(\lambda: t) &= \sum_{s \in W} \epsilon(s) E(s) \exp(-t \Xi(\lambda)) \Psi(\lambda: t) \\ &= \sum_{s \in W} \epsilon(s) \exp\{-\lambda^* (s^{-1} H_0) t\} E(s) \Psi(\lambda: t). \end{aligned}$$

Now put  $\xi(\mu: h) = \exp\{\mu^* (\log h)\}$  ( $\mu \in \mathfrak{F}, h \in A_p$ ) where  $\mu^* = (-1)^4 \mu$ . Then it is obvious that  $\xi(\mu: h; u) = u(\mu^*) \xi(\mu: h)$  for  $u \in \mathfrak{S}_p$  and therefore  $\xi(\mu: h; u) = 0$  for  $u \in SJ_{\mu^*}$ . Also

$$\sum_{1 \leq j \leq w} E_{ij}(s) u_j \equiv e_s u_i \equiv u_i(s\lambda^*) e_s \pmod{SJ_\lambda}. \quad (1 \leq i \leq w)$$

from Lemma 14. Therefore

$$\sum_{1 \leq j \leq w} E_{ij}(s) \xi(s'\lambda; h; u_j) = u_i(s\lambda^*) e_s(s'\lambda^*) \xi(s'\lambda; h) \quad (s, s' \in W).$$

Hence it follows from the third statement of Lemma 14 that

$$\sum_j E_{ij}(s) \xi(s'\lambda; h; u_j) = \begin{cases} 0 & \text{if } s \neq s' \\ u_i(s\lambda^*) \epsilon(s) \pi(\lambda^*) \xi(s\lambda; h) & \text{if } s = s'. \end{cases}$$

Let  $\psi_{0j}(\lambda: \infty)$  ( $1 \leq j \leq w$ ) denote the coefficients of the one-column matrix  $\Psi_0(\lambda: \infty)$ . Then

$$\pi(\lambda^*) \psi_{0i}(\lambda: \infty) = \lim_{t \rightarrow +\infty} \sum_{s, s' \in W} \sum_j \epsilon(s) \xi(s\lambda; h_t^{-1}) E_{ij}(s) c(s'\lambda) \Phi(s'\lambda; h_t; u_j).$$

But since  $\beta(H_0) \geq \tau(\lambda) + \epsilon \rho(H_0)$ , it follows that  $|\lambda_t(sH_0)| - \beta(H_0) \leq -\epsilon \rho(H_0)$  for  $s \in W$ . Therefore it is obvious that

$$\lim_{t \rightarrow +\infty} |\Phi(s'\lambda; h_t; u_j) - \xi(s'\lambda; h_t; u_j)| = 0 \quad (s' \in W)$$

and hence

$$\begin{aligned} \pi(\lambda^*) \psi_{0i}(\lambda: \infty) &= \lim_{t \rightarrow +\infty} \sum_{s, s' \in W} \sum_j \epsilon(s) \xi(s\lambda; h_t^{-1}) E_{ij}(s) c(s'\lambda) \xi(s'\lambda; h_t; u_j) \\ &= \pi(\lambda^*) \lim_{t \rightarrow +\infty} \sum_{s \in W} \xi(s\lambda; h_t^{-1}) c(s\lambda) u_i(s\lambda^*) \xi(s\lambda; h_t) \\ &= \pi(\lambda^*) \sum_{s \in W} u_i(s\lambda^*) c(s\lambda) \end{aligned}$$

in view of what we have said above. Thus we have obtained the following result.

LEMMA 51.  $\psi_{0i}(\lambda: \infty) = \sum_{s \in W} u_i(s\lambda^*) c(s\lambda)$  for any  $\lambda \in \mathfrak{F}'_\epsilon$ .

Define  $\sigma^i$  ( $1 \leq i \leq w$ ) as in Lemma 14.

COROLLARY.  $\sum_{1 \leq i \leq w} \sigma^i(\lambda^*) \psi_{0i}(\lambda: \infty) = \pi(\lambda^*) c(\lambda)$  for  $\lambda \in \mathfrak{F}'_\epsilon$ .

This follows immediately from the above lemma and the third statement of Lemma 14.

We have seen that  $\Psi_0(\lambda: \infty)$  is a holomorphic function of  $\lambda$  on  $\mathfrak{F}_\epsilon$ . Therefore in view of the above corollary and Lemma 50, we have the following result.

LEMMA 52. Suppose  $\epsilon$  is sufficiently small. Then there exists a holomorphic function  $\mathbf{b}$  on  $\mathfrak{F}_\epsilon$  such that  $\mathbf{b}(\lambda) = \pi(\lambda) c(\lambda)$  for  $\lambda \in \mathfrak{F}'_\epsilon$ . Moreover

we can select an integer  $d \geq 0$  with the property that for any  $u \in S(\mathfrak{h}_p)$ ,  $(1 + \|\lambda\|)^{-d} |\mathbf{b}(\lambda; \partial(u))|$  remains bounded as  $\lambda$  varies in  $\mathfrak{S}_\epsilon$ .

Define the  $\mathcal{B}$ -distributions  $\tau$  and  $\bar{\tau}$  on  $\mathfrak{h}_{p_0}$  as in Theorem 5.

COROLLARY. The distribution  $\pi\bar{\tau}$  coincides on  $\mathfrak{h}_{p_0}$  with  $\mathbf{b}$ .

We use the notation of the proof of Theorem 5. It is sufficient to show that

$$\bar{\tau}(\pi a) = \int_{\mathfrak{h}_{p_0}} \mathbf{b}(H) a(H) dH$$

for any  $a \in C_c^\infty(\mathfrak{h}_{p_0})$ . Put  $H_\epsilon = H - (-1)^{\frac{1}{2}} \epsilon H_\rho$  ( $H \in \mathfrak{h}_{p_0}$ ) where  $\epsilon$  is a positive number. Then it is obvious that  $\pi(H_\epsilon) \neq 0$ . Hence if  $\epsilon$  is sufficiently small,  $H_\epsilon \in {}^*\mathfrak{h}_p'$  and therefore  $\mathbf{c}(H_\epsilon)$  is well-defined for all  $H \in \mathfrak{h}_{p_0}$ . Moreover if we use the method of proof of Theorem 5, it follows without difficulty that

$$\bar{\tau}(\pi a) = \lim_{\epsilon \rightarrow 0} \int_{\mathfrak{h}_{p_0}} a(H) \pi(H) \mathbf{c}(H_\epsilon) dH.$$

On the other hand it is obvious that  $|\alpha(H)| \leq |\alpha(H_\epsilon)|$  for  $\alpha \in \Sigma$  and  $H \in \mathfrak{h}_{p_0}$  and therefore  $|\pi(H) \mathbf{c}(H_\epsilon)| \leq |\mathbf{b}(H_\epsilon)|$ . Let  $\Omega$  be the carrier of  $a$ . Then since  $\Omega$  is compact and since  $\mathbf{b}$  is holomorphic on a complex neighborhood of  $\mathfrak{h}_{p_0}$  in  $\mathfrak{h}_p$ , it is obvious that  $|\mathbf{b}(H_\epsilon)|$  remains uniformly bounded for all sufficiently small values of  $\epsilon$  as  $H$  varies in  $\Omega$ . Hence we conclude that

$$\bar{\tau}(\pi a) = \int_{\mathfrak{h}_{p_0}} \lim_{\epsilon \rightarrow 0} \{a(H) \pi(H) \mathbf{c}(H_\epsilon)\} dH.$$

But it is obvious that  $\lim_{\epsilon \rightarrow 0} \pi(H) \mathbf{c}(H_\epsilon) = \mathbf{b}(H)$  if  $\pi(H) \neq 0$ . Therefore

$$\bar{\tau}(\pi a) = \int_{\mathfrak{h}_{p_0}} a(H) \mathbf{b}(H) dH$$

and this proves the corollary.

**13. The case  $l = 1$ .** We assume in this section that <sup>32</sup>  $\dim \mathfrak{h}_p = 1$ . Then <sup>19</sup> we can select  $\alpha \in \Sigma$  such that  $2\alpha$  is the only other possible element in  $\Sigma$ . Let  $p$  and  $q$  denote the number of roots in  $P_+$  which coincide on  $\mathfrak{h}_p$  with  $\alpha$  and  $2\alpha$  respectively. Then  $\rho = \frac{1}{2}(p + 2q)\alpha$ . Let  $H$  be the element in  $\mathfrak{h}_p$  such that  $\alpha(H) = 1$ . Then  $\langle H, H \rangle = 2 \sum_{\beta \in P_+} \beta(H)^2 = 2(p + 4q)$ . This

<sup>32</sup> A special case of this type has been discussed by Bargmann [1]. We follow the same method.

implies that  $H_\alpha = (2p + 8q)^{-1}H$  and  $H_\rho = \frac{1}{2}(p + 2q)H_\alpha$ . Hence it follows from Lemma 27 that

$$2(p + 4q)\delta'(\omega) = H^2 + (p + 2q)H + \{2p(e^{2\alpha} - 1)^{-1} + 4q(e^{4\alpha} - 1)^{-1}\}H.$$

Put  $t(h) = \alpha(\log h)$  for  $h \in A_p$ . Then  $t$  can be regarded as the coordinate function on the one-dimensional Lie group  $A_p$  and so it is clear that

$$\begin{aligned} 2(p + 4q)\delta'(\omega) &= d^2/dt^2 + \{p \coth t + 2q \coth 2t\}d/dt \\ &= d^2/dt^2 + \{(p + q) \coth t + q \tanh t\}d/dt, \end{aligned}$$

since  $\coth 2t = \frac{1}{2}(\coth t + \tanh t)$ . Now put  $z = -(\sinh t)^2$ . Then the above relation becomes

$$\frac{1}{2}(p + 4q)\delta'(\omega) = z(z - 1)d^2/dz^2 + \frac{1}{2}\{(p + 2q + 2)z - (p + q + 1)\}d/dz.$$

On the other hand if  $\lambda$  is any linear function on  $\mathfrak{h}_p$ , we know from Corollary 2 of Lemma 27 that

$$\gamma(\omega: (-1)^{\frac{1}{2}}\lambda) = -\langle \lambda, \lambda \rangle - \langle \rho, \rho \rangle = -\frac{1}{2}(p + 4q)^{-1}\{\lambda(H)^2 + \rho(H)^2\}.$$

Therefore if we put  $\phi_\lambda(h) = \psi(\lambda: h)$  ( $h \in A_p$ ), it follows from Lemma 18 that

$$\begin{aligned} &z(z - 1)d^2\phi_\lambda/dz^2 \\ &+ \frac{1}{2}\{(p + 2q + 2)z - (p + q + 1)\}d\phi_\lambda/dz + \frac{1}{4}\{\lambda(H)^2 + (\frac{1}{2}p + q)^2\}\phi_\lambda = 0. \end{aligned}$$

Now let

$$a = \{p + 2q + 2\lambda^*(H)\}/4, \quad b = \{p + 2q - 2\lambda^*(H)\}/4, \quad c = (p + q + 1)/2$$

where  $\lambda^* = (-1)^{\frac{1}{2}}\lambda$ . Then the above equation becomes

$$z(z - 1)d^2\phi_\lambda/dz^2 + \{(a + b + 1)z - c\}d\phi_\lambda/dz + ab\phi_\lambda = 0.$$

Now  $z$  can be regarded as a coordinate function on  $A_p^+$ . Therefore  $\phi_\lambda = g(z)$  where  $g$  is analytic function on the interval  $-\infty < z < 0$ . Moreover  $W = \{1, s\}$  where  $sH = -H$ . Therefore  $\phi_\lambda(h) = \phi_\lambda(h^s) = \phi_\lambda(h^{-1})$  ( $h \in A_p$ ) and so  $\phi_\lambda$  can be expanded in powers of  $t^2$  in the neighborhood of the point  $h = 1$ . But this implies that  $g$  is actually analytic also at  $z = 0$ . Moreover  $g(0) = \phi_\lambda(1) = 1$ . Therefore since  $c > 0$ , it follows by a direct substitution of the power series for  $g$  around the origin, in the above differential equation, that  $g(z)$  is the hypergeometric function  $F(a, b, c, z)$ . But it is known (see Bargmann [1, p. 627]) that

$$F(a, b, c, z) = |z|^{-a} \Gamma(c) \Gamma(b-a) \{\Gamma(b) \Gamma(c-a)\}^{-1} F(a, 1+a-c, 1+a-b, z^{-1}) \\ + |z|^{-b} \Gamma(c) \Gamma(a-b) \{\Gamma(b) \Gamma(c-b)\}^{-1} F(b, 1+b-c, 1+b-a, z^{-1}) \\ (z < 0)$$

provided  $a-b=\lambda^*(H)$  is not an integer. (Here  $\Gamma$  denotes the classical Gamma function.) Therefore if  $\lambda$  is real, it is clear that

$$\lim_{t' \rightarrow +\infty} |e^{t'\rho(H)} \phi_\lambda(\exp t'H) - c(\lambda) e^{t'\lambda^*(H)} - c(-\lambda) e^{-t'\lambda^*(H)}| = 0$$

where

$$c(\lambda) = \Gamma(\tfrac{1}{2}(p+q+1)) \Gamma(\lambda^*(H)) \\ \times \{\Gamma(4^{-1}(p+2q+2\lambda^*(H))) \Gamma(4^{-1}(p+2+2\lambda^*(H)))\}^{-1}.$$

Moreover in the present case  $\pi = \alpha$  and therefore

$$\pi(\lambda) = \lambda(H_\alpha) = (2p+8q)^{-1} \lambda(H).$$

Hence

$$(-1)^{\frac{1}{2}} b(\lambda) = (2p+8q)^{-1} \Gamma((p+q+1)/2) \Gamma(1+\lambda^*(H)) \\ \times \{\Gamma((p+2q+2\lambda^*(H))/4) \Gamma((p+2+2\lambda^*(H))/4)\}^{-1},$$

and by analytic continuation this formula holds for all  $\lambda \in \mathfrak{F}_\epsilon$  if  $\epsilon$  is sufficiently small. Moreover if  $\lambda \in \mathfrak{F}_R$ , it is obvious that  $|b(-\lambda)| = |b(\lambda)|$ . Hence we have obtained the following result.

**LEMMA 53.** *If  $\epsilon$  is sufficiently small,  $b$  is never zero on  $\mathfrak{F}_\epsilon$ . Moreover  $|b(s'\lambda)| = |b(\lambda)|$  for  $s' \in W$  and  $\lambda \in \mathfrak{F}_R$ .*

We shall see in another paper that a similar result holds when  $l > 1$ .

**14. The complex case.**<sup>33</sup> In this section we assume that  $G$  is a complex group. Then it follows easily (see [5(f), p. 513]) that no root  $\alpha \in P$  can vanish identically either on  $\mathfrak{h}_p$  or on  $\mathfrak{h}_r$ . Moreover the restrictions on  $\mathfrak{h}_p$  of two distinct roots  $\alpha, \beta \in P$ , are linearly independent unless  $\beta = -\theta\alpha$ . Hence  $P = P_+$  and we can select a subset  $Q$  of  $P$  such that exactly one of the two roots  $\alpha, -\theta\alpha$  lies in  $Q$  for any  $\alpha \in P$ . Put  $d(H) = \prod_{\alpha \in Q} (e^{\alpha(H)} - e^{-\alpha(H)})$  ( $H \in \mathfrak{h}_p$ ). It is obvious that  $\pi(H) = \prod_{\alpha \in Q} \alpha(H)$  and therefore  $d(sH) = \epsilon(s)d(H)$

<sup>33</sup> After this paper had been written, two recent notes of F. A. Beresin (Doklady Akad. Nauk. SSSR N. S., vol. 107 (1956), pp. 9-12, vol. 110 (1956), pp. 897-900) came to my attention. The results of this section are also contained in these notes.

( $s \in W$ ). Moreover  $\rho(H) = \frac{1}{2} \sum_{\alpha \in P} \alpha(H) = \sum_{\alpha \in Q} \alpha(H)$ . Hence it follows by elementary considerations of divisibility (see Weyl [8(b)]) that

$$d(H) = \sum_{s \in W} \epsilon(s) e^{\rho(s^{-1}H)} \quad (H \in \mathfrak{h}_p).$$

Now put  $D(h) = d(\log h)$  for  $h \in A_p$  and let us use the notation of Section 7. Then it is obvious<sup>22</sup> that  $\omega D = \langle \rho, \rho \rangle D$ . If we take this fact and Lemma 27 into account, a simple calculation gives the following result.

LEMMA 54.  $\delta'(\omega) = D^{-1}\gamma(\omega) \circ D$  in the present case.

Fix  $H' \in \mathfrak{h}_p$  and consider the function

$$F(H':h) = D(h)^{-1} \exp \{ \rho(\log h) + \langle H', \log h \rangle \}$$

( $h \in A_p^+$ ) on  $A_p^+$ . In view of the above lemma,  $F$  is an eigenfunction of the operator  $\delta'(\omega)$  and the corresponding eigenvalue is  $\gamma'(\omega: H')$ . Moreover  $D(h)^{-1} e^{\rho(\log h)} = \prod_{\alpha \in Q} \{1 - e^{-2\alpha(\log h)}\}^{-1}$ . Therefore it follows easily from the reasoning of Section 8 that  $\Phi' = F$  in the notation of Lemma 29. Now put

$$\phi_\lambda(x) = \int_K \exp \{ (-1)^{\frac{1}{2}} \lambda(H(xk)) - \rho(H(xk)) \} dk \quad (x \in G, \lambda \in \mathfrak{F})$$

and fix  $\lambda \in \mathfrak{F}_{K'} = \mathfrak{h}_{p_0}'$ . Then it follows from Lemma 37 that<sup>24</sup>

$$D(h)\phi_\lambda(h) = \sum_{s \in W} c(s\lambda) \exp \{ s\lambda^*(\log h) \} \quad (h \in A_p(M))$$

where  $\lambda^* = (-1)^{\frac{1}{2}} \lambda$ . However since both sides are analytic functions on  $A_p$  this equation must hold for all  $h \in A_p$ . On the other hand  $\phi_\lambda(h^s) = \phi_\lambda(h)$  while  $D(h^s) = \epsilon(s)D(h)$ . Therefore since the points  $s\lambda$  ( $s \in W$ ) are all distinct, it follows (see [5(b), Lemma 41]) that  $c(s\lambda) = \epsilon(s)c(\lambda)$ . Hence

$$D(h)\phi_\lambda(h) = c(\lambda) \sum_{s \in W} \epsilon(s) \exp \{ s\lambda^*(\log h) \} \quad (h \in A_p).$$

Now  $\pi$ , being an element in  $\mathfrak{F}_p$ , can be considered as a differential operator on  $A_p$ . In view of the expression for  $D$  obtained earlier, it is clear that  $D(1; \pi) = w\pi(\rho)$  where  $w$  is the order of  $W$ . Hence if we apply the operator  $\pi$  to the above equation and put  $h=1$ , we get

$$w\pi(\rho) = wc(\lambda)\pi(\lambda^*)$$

since  $\phi_\lambda(1) = 1$ . Hence

$$\begin{aligned} \pi(\lambda)D(h)\phi_\lambda(h) \\ = (-1)^{-q/2}\pi(\rho) \sum_{s \in W} \epsilon(s) \exp \{ (-1)^{\frac{1}{2}} s\lambda(\log h) \} \end{aligned} \quad (h \in A_p)$$

<sup>24</sup>  $s\mu(H)$  denotes the value of  $s\mu$  at  $H$  ( $s \in W, \mu \in \mathfrak{F}, H \in \mathfrak{h}_p$ ).

where  $q$  is the number of roots in  $Q$ . However both sides are holomorphic functions of  $\lambda$  on  $\mathfrak{F}$  and so the above equation must hold for all  $\lambda \in \mathfrak{F}$ . Thus we have obtained another proof of an earlier result (see [5(e), Theorem 7]). Also  $\mathbf{b}(\lambda) = (-1)^{-q/2} \pi(\rho)$ . This shows that  $\mathbf{b}$  is a constant in this case.

It is interesting to note that from the above results one can deduce the following extension of Lemma 54, which we state here without proof.<sup>35</sup>

LEMMA 55.  $\delta'(q) = D^{-1} \gamma(q) \circ D$  for every  $q \in I_{\mathfrak{g}}$ .

**15. Appendix.** LEMMA 56. Let  $t$  be a real variable and  $k_1, \dots, k_r$  distinct real numbers. Then

$$\limsup_{t \rightarrow +\infty} \left| \sum_{1 \leq i \leq r} c_i \exp \{ (-1)^{\frac{1}{2}} k_i t \} \right| \geq \left\{ \sum_{1 \leq i \leq r} |c_i|^2 \right\}^{\frac{1}{2}}$$

for any complex numbers  $c_1, \dots, c_r$ .

Put  $f(t) = \sum_i c_i \exp \{ (-1)^{\frac{1}{2}} k_i t \}$ . Then it follows by direct computation that

$$\lim_{T \rightarrow +\infty} T^{-1} \int_0^T |f(t)|^2 dt = \sum_i |c_i|^2.$$

On the other hand if  $a = \limsup_{t \rightarrow +\infty} |f(t)|$ , it is obvious that

$$\lim_{T \rightarrow +\infty} T^{-1} \int_0^T |f(t)|^2 dt \leq a^2.$$

Therefore  $a^2 \geq \sum_i |c_i|^2$ .

COROLLARY. Let  $k_1, \dots, k_r$  be nonzero complex numbers and  $p_0, p_1, \dots, p_r$  polynomials in  $t$  with complex coefficients. Suppose

$$\limsup_{t \rightarrow +\infty} |p_0(t) + p_1(t) e^{k_1 t} + \dots + p_r(t) e^{k_r t}| \leq a$$

for some real number  $a$ . Then  $p_0$  is a constant and  $|p_0| \leq a$ .

Let  $l_i$  denote the real part of  $k_i$ . If  $l_i < 0$ , it is obvious that  $\lim_{t \rightarrow +\infty} |p_i(t) e^{k_i t}| = 0$ . Therefore if we suppose that  $l_i \geq 0$  for  $1 \leq i \leq s$  while  $l_i < 0$  for  $i > s$ , it follows that

$$\limsup_{t \rightarrow +\infty} |p_0(t) + \sum_{1 \leq i \leq s} p_i(t) e^{k_i t}| \leq a.$$

<sup>35</sup> Cf. Theorem 2 of [5(g)]

Hence without loss of generality we may assume that  $s=r$  and  $k_1, \dots, k_r$  are distinct and  $p_i \neq 0$  ( $0 \leq i \leq r$ ). We shall now first show that  $l_1 = l_2 = \dots = l_r = 0$ . For otherwise suppose  $l = \max(l_1, \dots, l_r) > 0$ . We may assume that  $l_1 = l_2 = \dots = l_j = l$  while  $l_i < l$  for  $i > j$ . Let  $n$  be the highest among the degrees of  $p_1, \dots, p_j$  and put  $c'_i = \lim_{t \rightarrow +\infty} p_i(t)/t^n$ . Then  $c'_1, \dots, c'_j$  cannot all be zero and it is clear that

$$\begin{aligned} 0 &= \limsup_{t \rightarrow +\infty} t^{-n} e^{-lt} |p_0(t) + p_1(t)e^{k_1 t} + \dots + p_r(t)e^{k_r t}| \\ &= \limsup_{t \rightarrow +\infty} |c'_1 e^{(k_1-l)t} + c'_2 e^{(k_2-l)t} + \dots + c'_j e^{(k_j-l)t}|. \end{aligned}$$

But since  $k_i - l$  ( $1 \leq i \leq j$ ) are distinct pure imaginary numbers, we get a contradiction with Lemma 56. This proves that  $l = 0$ .

Now let  $m$  be the maximum of the degrees of  $p_0, p_1, \dots, p_r$ . We claim  $m = 0$ . For otherwise since  $l_i = 0$   $1 \leq i \leq r$ , it follows that

$$\lim_{t \rightarrow +\infty} t^{-m} |p_0(t) + \sum_{1 \leq i \leq r} p_i(t)e^{k_i t}| = 0.$$

This implies that

$$\lim_{t \rightarrow +\infty} |c_0 + \sum_{1 \leq i \leq r} c_i e^{k_i t}| = 0$$

where  $c_i = \lim_{t \rightarrow +\infty} p_i(t)/t^m$  ( $0 \leq i \leq r$ ). But in view of the definition of  $m, c_0, \dots, c_r$  cannot all be zero and so again we get a contradiction with the above lemma. This shows that  $m = 0$  and hence  $p_0, \dots, p_r$  are all constants. Therefore again by the above lemma,  $|p_0| \leq a$ .

Let  $E$  be a vector space over  $R$  of finite dimension. We shall say that a subset  $F$  of  $E$  is full if  $tH \in F$  whenever  $H \in F$  and  $t \geq 1$  ( $t \in R$ ).

LEMMA 57. Let  $k_i \neq 0$  ( $1 \leq i \leq r$ ) be a finite set of linear functions<sup>36</sup> and  $p_0, \dots, p_r$  polynomial functions on  $E$ . Suppose  $V$  is a nonempty, open and full subset of  $E$  and  $a$  a real number such that

$$|p_0(H) + p_1(H)e^{k_1(H)} + \dots + p_r(H)e^{k_r(H)}| \leq a$$

for all  $H \in V$ . Then  $p_0$  is a constant and  $|p_0| \leq a$ .

Let  $m$  be the degree of  $p_0$ . Since  $V$  is open, we can select  $H_0 \in V$  in such a way that  $k_i(H_0) \neq 0$  ( $1 \leq i \leq r$ ) and  $q_0(t) = p_0(tH_0)$  is a polynomial of degree  $m$  in the real variable  $t$ . Put  $q_i(t) = p_i(tH_0)$  and  $k'_i = k_i(H_0)$   $1 \leq i \leq r$ . Then since  $tH_0 \in V$  for  $t \geq 1$ , it follows that

$$|q_0(t) + q_1(t)e^{k'_1 t} + \dots + q_r(t)e^{k'_r t}| \leq a \quad (t \geq 1).$$

<sup>36</sup> We permit linear or polynomial functions on  $E$  to take complex values.

Hence we conclude from the corollary of Lemma 56 that  $q_0$  is a constant and  $|q_0| \leq a$ . This proves that  $m=0$  and therefore  $|p_0| = |q_0| \leq a$ .

COROLLARY. Let  $k_i$  ( $i \geq 1$ ) be a sequence of distinct linear functions and  $p_i$  ( $i \geq 1$ ) a sequence of polynomial functions on  $E$  and  $V$  a nonempty, open and full subset of  $E$ . Suppose the following two conditions hold.

(1) For each linear function  $k$  on  $E$  the series

$$\sum_{1 \leq i < \infty} |p_i(H) \exp(k(H) + k_i(H))|$$

converges uniformly for  $H \in V$ .

(2)  $\sum_{1 \leq i < \infty} p_i(H) e^{k_i(H)} = 0$  for  $H \in V$ .

Then  $p_i = 0$  for every  $i$ .

For otherwise select an index  $j$  such that  $p_j \neq 0$ . For a given  $\epsilon > 0$ , choose an integer  $N \geq j$  such that

$$\sum_{i > N} |p_i(H) \exp(k_i(H) - k_j(H))| \leq \epsilon$$

for all  $H \in V$ . This is possible by condition (1). Moreover it is obvious from condition (2) that

$$\left| \sum_{1 \leq i \leq N} p_i(H) \exp(k_i(H) - k_j(H)) \right| \leq \epsilon \quad (H \in V).$$

On the other hand  $k_i - k_j \neq 0$  for  $i \neq j$  and so we conclude from the above lemma that  $p_j$  is a constant and  $|p_j| \leq \epsilon$ . But  $\epsilon$  being arbitrary, this implies that  $p_j = 0$ . As this contradicts the choice of  $j$ , our assertion is proved.

Let  $(z_1, \dots, z_l)$  denote the Cartesian coordinates of a point  $z$  in the complex Euclidean space  $C^l$  of dimension  $l$ . We regard  $C^l$  as a vector space over  $C$  in the usual way and put  $|z| = \max_{1 \leq i \leq l} |z_i|$ . The distance between two sets  $U$  and  $V$  in  $C^l$  is defined to be  $\inf |z - \xi|$  ( $z \in U, \xi \in V$ ).

LEMMA 58. Let  $U$  be an open set in  $C^l$  and  $V$  a subset of  $U$  and let  $\epsilon$  denote the distance between  $V$  and the complement of  $U$  in  $C^l$ . Then if  $\xi$  is any holomorphic function on  $U$ ,

$$\begin{aligned} \sup_{z \in V} |(\partial^{k_1+\dots+k_l} / \partial z_1^{k_1} \dots \partial z_l^{k_l}) \xi(z)| \\ \leq k_1! k_2! \dots k_l! (2\pi/\epsilon)^{k_1+\dots+k_l} \sup_{z \in U} |\xi(z)| \end{aligned}$$

for any integers  $k_1, \dots, k_l \geq 0$ .

We may obviously assume that  $\epsilon > 0$ . Let  $\epsilon'$  be any positive number less than  $\epsilon$ . Then if  $z \in V$  and  $|z - \xi| \leq \epsilon'$ , it follows that  $\xi \in U$ . Hence

$$\begin{aligned} & (\partial^{k_1+\dots+k_l}/\partial z_1^{k_1} \dots \partial z_l^{k_l}) \xi(z) \\ &= k_1! \dots k_l! \{2\pi(-1)^{\frac{1}{2}}\}^{-1} \oint_1 \dots \oint_l \xi(\xi) \{(\xi_1 - z_1)^{k_1+1} \dots (\xi_l - z_l)^{k_l+1}\}^{-1} \\ & \quad \times d\xi_1 \dots d\xi_l \end{aligned}$$

where  $\oint_i$  denotes complex integration with respect to  $\xi_i$  on the circle  $|z_i - \xi_i| = \epsilon'$ . Therefore it is obvious that

$$|(\partial^{k_1+\dots+k_l}/\partial z_1^{k_1} \dots \partial z_l^{k_l}) \xi(z)| \leq k_1! \dots k_l! \sup_{\xi \in U} |\xi(\xi)| \epsilon'^{-k} (2\pi)^{k-1}$$

where  $k = k_1 + \dots + k_l$ . This being true for every  $\epsilon' < \epsilon$ , our assertion now follows immediately.

The following lemma and its corollary will be needed in another paper.

LEMMA 59. Suppose  $\epsilon$  is positive in Lemma 57 and  $\eta = z_1 \xi$ . Then if  $\sup_{z \in U} (1 + |z|)^{-d} |\eta(z)| < \infty$  for some integer  $d \geq 0$ , we can conclude that  $\sup_{z \in V} (1 + |z|)^{-d} |\xi(z)| < \infty$ .

Select a positive number  $\epsilon' < \min(1/6, \epsilon/3)$  and let  $V'$  be the set of those points  $z \in V$  where  $|z_1| < \epsilon'$ . Obviously it would be enough to verify that  $\sup_{z \in V'} (1 + |z|)^{-d} |\xi(z)| < \infty$ . Now fix a point  $z^0 = (z_1^0, z_2^0, \dots, z_l^0)$  in  $V'$ . Then

$$\xi(z^0) = (2\pi(-1)^{\frac{1}{2}})^{-1} \oint z_1^{-1} \eta(z_1, z_2^0, \dots, z_l^0) (z_1 - z_1^0)^{-1} dz_1$$

where  $\oint$  denotes complex integration over the circle  $|z_1| = 2\epsilon'$  in the  $z_1$ -plane. Now let  $z' = (z_1, z_2^0, \dots, z_l^0)$  where  $|z_1| = 2\epsilon'$ . Then it is clear that

$$1 + |z^0| \geq 1 + |z'| - 3\epsilon' \geq \frac{1}{2}(1 + |z'|).$$

Hence it follows that

$$(1 + |z^0|)^{-d} |\xi(z^0)| \leq (2^d/\epsilon') \sup_{z \in U} (1 + |z|)^{-d} |\eta(z)|.$$

This proves the lemma.

COROLLARY. Let  $p$  be a polynomial function on  $C^1$  which can be written as a product of linear factors. Then the above lemma also holds if  $\eta = p\xi$ .

This follows by an easy induction on the degree of  $p$ .

For any  $m \times m$  matrix  $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq m}$  (with complex coefficients) put  $\|\Lambda\| = (\sum_{i,j} |\lambda_{ij}|^2)^{\frac{1}{2}}$ .

LEMMA 60. Let  $\Lambda$  be an  $m \times m$  matrix and let  $\lambda_1, \dots, \lambda_m$  denote all the eigenvalues of  $\Lambda$ . Put  $\nu = \max_{1 \leq i \leq m} |\operatorname{Re} \lambda_i|$ . Then

$$\|e^\Lambda\| \leq m^{\frac{1}{2}} e^{2m\nu} \sum_{0 \leq k < m} \|\Lambda\|^k.$$

It is well known that we can find a unitary matrix  $U$  such that  $\Lambda' = U\Lambda U^{-1}$  has zeros everywhere below the diagonal. Since  $\Lambda$  and  $\Lambda'$  have the same eigenvalues and  $\|\Lambda'\| = \|\Lambda\|$ ,  $\|e^{\Lambda'}\| = \|e^\Lambda\|$ , we may replace  $\Lambda$  by  $\Lambda'$  in our problem. Therefore we can assume that  $\Lambda = A + N$  where  $A$  is a diagonal and  $N$  a supertriangular<sup>37</sup> matrix. Now consider  $e^{t\Lambda} = e^{t(A+N)}$  ( $t \in R$ ) and put  $n(t) = e^{-tA} e^{t(A+N)}$ . Then it is clear that

$$\begin{aligned} dn(t)/dt &= e^{-tA} (-A) e^{t(A+N)} + e^{-tA} (A + N) e^{t(A+N)} \\ &= e^{-tA} N e^{t(A+N)} = e^{-tA} N e^{tA} n(t). \end{aligned}$$

This differential equation can be solved by the method of iteration and one finds that

$$n(t) = 1 + \sum_{k \geq 1} \int_{0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_k \leq t} N(\tau_k) N(\tau_{k-1}) \cdots N(\tau_1) d\tau_1 \cdots d\tau_k$$

for  $t \geq 0$ . Here 1 stands for the unit matrix and  $N(\tau) = e^{-\tau A} N e^{\tau A}$  ( $\tau \in R$ ). But since  $N(\tau)$  is also supertriangular, the product  $N(\tau_k) \cdots N(\tau_1)$  is zero if  $k \geq m$ . Moreover  $A = A_1 + (-1)^{\frac{1}{2}} A_2$  where  $A_1, A_2$  are both diagonal and real. Then since  $e^{(-1)^{\frac{1}{2}} \tau A_2}$  is unitary, it follows that

$$\|N(\tau)\| = \|e^{-\tau A_1} N e^{\tau A_1}\| \quad (\tau \in R).$$

But the diagonal elements of  $A$  are exactly the eigenvalues of  $\Lambda$ . Therefore if  $|\tau| \leq 1$ , no coefficient of  $e^{\tau A_1}$  can exceed  $e^\nu$  in absolute value. Hence  $\|N(\tau)\| \leq e^{2\nu} \|N\|$  for  $|\tau| \leq 1$ . This shows that

$$\int_{0 \leq \tau_1 \leq \dots \leq \tau_k \leq t} \|N(\tau_k) \cdots N(\tau_1)\| d\tau_1 \cdots d\tau_k \leq \|N\|^k e^{2k\nu}$$

for  $0 \leq t \leq 1$  and  $k \geq 1$ . Therefore

$$\|n(1)\| \leq \|1\| + \sum_{1 \leq k < m} \|N\|^k e^{2k\nu}.$$

<sup>37</sup> This means that all the coefficients of  $N$  on or below the diagonal are zero.

But  $e^\Lambda = e^\Lambda n(1)$  and so  $\|e^\Lambda\| \leq e^\nu \|n(1)\|$ . Moreover  $\|\Lambda\|^2 = \|A\|^2 + \|N\|^2 \geq \|N\|^2$ . Therefore

$$\begin{aligned} \|e^\Lambda\| &\leq e^\nu \|n(1)\| \leq m^{\frac{1}{2}} e^{2m\nu} \sum_{0 \leq k < m} \|N\|^k \\ &\leq m^{\frac{1}{2}} e^{2m\nu} \sum_{0 \leq k < m} \|\Lambda\|^k \end{aligned}$$

since  $\|1\| = m^{\frac{1}{2}}$ .

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## INTERSECTION MULTIPLICITIES OF MAXIMAL CONNECTED BUNCHES.\*

By J. P. MURRE.

**Introduction.** It is possible to use the connectedness of the total transform of a simple point for a birational transformation as a starting point for the theory of intersection multiplicities. As explained in [VI] page 249, it is then possible to attach an integer, not only to a proper component of intersection, but also to every maximal connected\*\* bunch in the intersection of two varieties of complementary dimensions on a complete ambient variety (subject to some restrictions on this ambient variety). In this paper such an integer is defined, provided the ambient variety is projective and every point of the bunch under consideration is simple on the ambient variety.

The technique used to derive from the connectedness theorem ([II], [VIII] and [IX]) the theory of intersection multiplicities is essentially the method of Severi [IV] and van der Waerden [V].†). The elimination of all the arbitrary elements entering into this method is based upon a certain commutativity argument (see § 2) which is, essentially, also contained in [V] (cf. page 639). It is assumed that the multiplicity for proper components is already defined (we proceed by induction on the excess of the bunch, see § 2).

The terminology and notations are from [VI].

**1. Preparations.** Let  $A^a$  be a variety in projective  $N$ -space  $P^N$ , defined over a field  $k$ . Let  $t_{ij}$ ,  $j=0, 1, \dots, N$ , be  $(N+1)^2$  independent transcendentals over  $k$ . Consider the projective transformation of  $P^N$ , the matrix of which is  $T=(t_{ij})$ . Since the  $t_{ij}$  are independent transcendentals over  $k$ , we shall say that this projective transformation is *generic over  $k$* . Let  $(x)$  be a generic point of  $A$  over  $k(t)$ , consider the locus  $A^T$  of the point  $(y)=T^{-1}(x)$  over  $k(t)$  (this locus exists since  $k(t, y)/k(t)=k(t, x)/k(t)$

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\*\* For the meaning of the term "connected" we refer to [II], No. 1 or we take the usual definition in topology for the Zariski topology.

† Mr. J. de Boer has kindly informed me that W. L. Chow has also defined and studied the intersection multiplicity for maximal connected bunches by methods similar to those of Severi and van der Waerden.

is regular). It is easily checked that if  $F_i(X) = 0$ ,  $i = 1, 2, \dots$  is a system of equations for  $A$  over  $k$ , then  $F_i(TX) = 0$ ,  $i = 1, 2, \dots$  is a system of equations for  $A^T$  over  $k(t)$ ; hence if a point  $Q \in A^T$  then  $TQ \in A$ , and conversely. Clearly  $A^T$  is the transform of  $A$  by  $T^{-1}$  (hence, in particular,  $\deg A^T = \deg A = g$ ). In [V], § 2, it is shown that there exists a (singular) matrix  $T'$  such that  $A^T$  specializes (as a cycle) to  $\sum_{i=1}^g L_i$  over the specialization  $T \rightarrow T'$  with reference to  $k$ , where every  $L_i$  is an  $a$ -dimensional linear space generic over  $k$ ,  $L_i \neq L_j$  for  $i \neq j$ , and  $\bigcap_i L_i$  is an  $(a-1)$ -dimensional linear space generic over  $k$ . From this, one obtains by the theory of specializations of cycles (cf. [III], page 104, § 7), the following lemma:

LEMMA 1. *Let  $A^a$  and  $B^b$  be varieties in projective  $N$ -space ( $N \leq a + b$ ), defined over a field  $k$ . Let  $T$  be a projective transformation generic over  $k$ . Then*

- i.  $A^T \cdot B$  is defined and  $A^T \cdot B = \sum_i C_i$  with  $C_i \neq C_j$  for  $i \neq j$  [i.e.,  $i(A^T \cdot B, C_i; P^N) = 1$ ].
- ii. *A generic point of  $C_i$  over a field of definition for  $C_i$  is generic on  $B$  over  $k$ .*

If  $A = \sum_i n_i A_i$  is a projective cycle, rational over  $k$ , and if  $T$  is a projective transformation, generic over  $k$ , then we define  $A^T = \sum_i n_i A_i^T$ .

LEMMA 2. *Let  $A^a$  and  $B^b$  be  $k$ -prime rational cycles in projective  $N$ -space ( $N \leq a + b$ ), with reduced expressions  $A = \sum_i A_i$  and  $B = \sum_i B_i$  (i.e., the coefficients are 1). Let  $T$  be a projective transformation generic over  $k$ . Then we have  $A^T \cdot B = \sum_i C_i$ ,  $C_i \neq C_j$  for  $i \neq j$ , and  $\sum_i C_i$  is  $k(t)$ -prime rational. Furthermore, a generic point of  $C_i$  over a field of definition for  $C_i$  is generic on  $B$  over  $k$ .*

*Proof.* It follows by Lemma 1i. that  $A^T \cdot B$  is defined. Consider first  $A_i^T \cdot B_j = \sum_h C_{ijh}$ . Assume now that  $C_{i_0 j_0 h_0} = C_{i_1 j_1 h_1}$ . It follows from Lemma 1ii. that  $j_0 = j_1$ . Let  $U$  be the inverse transformation of  $T$ ; clearly  $U$  is generic over  $k$ . From our preceding remarks, it follows that  $A_i \cdot B_j^U = \sum_h C_{ijh}^U$ . Again by Lemma 1ii., it follows that  $i_0 = i_1$ ; then by Lemma 1i.,  $h_0 = h_1$ . Therefore  $A^T \cdot B = \sum_i C_i$ ,  $C_i \neq C_j$  for  $i \neq j$ . Clearly  $\sum_i C_i$  is  $k(t)$ -rational; hence we only need show that the  $C_i$  are conjugate to each other over  $k(t)$ . Consider (cf. [V], page 627) the cycle  $W$  in  $P^N \times P^N \times P^{N^2+2N}$ , which is the locus of  $(P, Q, S)$  over  $k$ , where  $P$  and  $Q$  are independent generic points of  $A$  and  $B$  over  $k$  and where  $P = SQ$  ( $S$  is interpreted as an  $(N+1)$ -square

matrix, hence as a projective transformation in  $P^N$ ). Let  $X_i$ , resp.  $X'_i$ , ( $i=0, \dots, N$ ) be corresponding projective coordinates in the first and second projective space  $P^N$  and let  $Y_{ij}$  ( $i, j=0, \dots, N$ ) be projective coordinates in  $P^{N^2+2N}$ . The following equations are among the equations for  $W$

$$(1) \quad X_i \left( \sum_{j=0}^N Y_{0j} X'_j \right) - X_0 \left( \sum_{j=0}^N Y_{ij} X'_j \right) = 0, \quad i=1, \dots, N.$$

If  $P = (p_0, \dots, p_N)$  and  $Q = (q_0, \dots, q_N)$ , then we can assume that  $p_0 \neq 0$  and  $q_0 \neq 0$ . It follows easily that the  $s_{ij}$  with  $j \neq 0$  and  $s_{00}$  can be taken as independent transcendentals over  $k(P, Q)$ , the  $s_{i0}$  ( $i \neq 0$ ) being then uniquely determined by (1). It follows in particular that  $\dim W = a + b + N^2 + N$ . Let  $R_i$  be a generic point of  $C_i$  over the algebraic closure of  $k(t)$ ; it can be assumed that the  $X_0$ -coordinate of the  $R_i$  and the  $TR_i$  are different from zero for all  $i$ .  $TR_i$  being in  $A$ , it follows that  $(P, Q) \rightarrow (TR_i, R_i)$  over  $k$ , and then by the equations (1),  $(P, Q, S) \rightarrow 2(TR_i, R_i, T)$  over  $k$  (as follows from the preceding remarks about the  $s_{ij}$ ). Since  $R_i$  has the dimension  $a + b - N$  over  $k(t)$ , it follows for dimension reasons that the specialization is generic. Hence  $R_i \rightarrow R_j$  over  $k(t)$ , and conversely. This completes the proof since the assertion on the generic points follows from Lemma 1ii.

Again, let  $A^a$  be a variety in projective  $N$ -space, defined over  $k$ . Let  $L^t$  be a linear variety such that  $L^t \cap A = \emptyset$ . We will denote the *projecting cone* through  $A$  with  $L^t$  as center of projection by  $\Gamma_{(A, L)}$ . The letter  $\Gamma$  will be reserved for varieties of this type. If  $k'$  is a field of definition for  $A$  and  $L$ , then  $\Gamma_{(A, L)}$  is the locus over  $k'$  of a point  $T$  which is on the line  $T_1 T_2$  generic over the field  $k'(T_1, T_2)$ , where  $T_1$  and  $T_2$  are independent generic points of  $A$  and  $L$  over  $k'$ . In [I], page 154,  $\Gamma_{(A, L)}$  is called the *cross-join* of  $A$  and  $L$ . In the following, we assume  $L$  generic over  $k$  and  $t < N - a$ . We will use the following two properties, the proofs of which are omitted (see [I], page 154, Lemma 1): **a.**  $\Gamma_{(A, L)}$  is a variety of dimension  $a + t + 1$ ; **b.** if  $S$  is a simple point on  $A$  such that  $L$  is still generic over  $k(S)$ , then  $S$  is simple on  $\Gamma_{(A, L)}$  and the tangent space to  $\Gamma_{(A, L)}$  at  $S$  is spanned by the tangent space to  $A$  at  $S$  and  $L$ . Furthermore, we note that (as follows from the definition of  $\Gamma_{(A, L)}$ ) if  $T''$  is a point on  $\Gamma_{(A, L)}$  and if  $T'' \notin A$ , then there is a point  $T'_1 \in A$  such that the line  $T'' T'_1$  meets  $L$ .

The following lemma is essentially well-known (cf. [I], page 170, Lemmas IV and V).

LEMMA 3. *Let  $A^a$  be a simple subvariety of a variety  $V^n$  in projective  $N$ -space, let  $A$  and  $V$  be defined over a field  $k$ . Let  $L^{N-n-1}$  be a linear space generic over  $k$ . Then we have (if  $\Gamma = \Gamma_{(A, L)}$ ):*

- i.  $\Gamma \cdot V = A + \sum \eta_\lambda A_\lambda$ , and every  $A_\lambda$  is simple on  $V$ .
- ii. if  $B^b$  is any simple subvariety defined over  $k$ , then every component  $C$  of  $A_\lambda \cap B$ , not contained in  $A \cap B$ , is proper, and a generic point of  $C$  over a field of definition for  $C$  is generic on  $B$  over  $k$ .
- iii. if  $C$  is any simple subvariety of  $V$ , algebraic over  $k$ , then there is a point of  $C$  not in any  $A_\lambda$  (this implies that, if  $D$  is a component of  $A_\lambda \cap B$ , contained in a component  $C$  of  $A \cap B$  simple on  $V$ , then  $D$  is strictly contained in  $C$ ).

*Proof.* Let  $K$  be the compositum of the smallest field of definition for  $L$  and  $k$ . Let  $A_\lambda$  be a component of  $\Gamma \cap V$  of dimension  $a'$ , we shall show  $a' \leq a$ . Assume  $a' > a$ . Let  $M^{N-a-1}$  be a linear space generic over  $K$ , let  $K_1$  be the algebraic closure of the smallest field of definition for  $M$  containing  $k$ . By [VII], Lemma 1,  $M \cdot V = V_1^{n-a-1}$ , where  $V_1$  is a variety. Furthermore,  $M \cap A = \phi$  for dimension reasons (hence  $V_1 \cap A = \phi$ ). Let  $R_1$  and  $R_2$  be independent generic points of  $V_1$  and  $A$  over  $K_1$ , let  $R$  be a point on the line  $R_1 R_2$  generic over the field  $K_1(R_1, R_2)$ . Let  $X$  be the locus of  $R$  over  $K_1$  ( $X$  is the cross-join of  $A$  and  $V_1$ , see [I], page 154). Clearly  $\dim X \leq n$ . Since  $\dim A_\lambda > a$ , it follows that there is a point  $R'_1 \in A_\lambda \cap M$ , hence  $R'_1 \in V_1$  and  $R'_1 \notin A$ . Now  $R'_1 \in \Gamma$ ,  $R'_1 \notin A$ , hence there is a point  $R'_2 \in A$  such that the line  $R'_1 R'_2$  meets the linear variety  $L^{N-n-1}$ . However,  $R'_1 R'_2$  is on  $X$ , hence  $X \cap L \neq \phi$ .  $X$  is a variety of dimension not greater than  $n$ , defined over a field  $K_1$ , and  $L^{N-n-1}$  is generic over  $K_1$ ; hence  $X \cap L = \phi$ . Contradiction. Hence  $\dim A_\lambda \leq a$ . By [VI], Chap. VI, Cor. 1 of Th. 1 it follows that  $\dim A_\lambda \geq a$ . Hence  $\dim A_\lambda = a$ . It follows that  $A$  is a component of  $\Gamma \cap V$ ;  $i(\Gamma \cdot V, A; P^N) = 1$  follows by applying [VI], Chap. VI, Th. 6 to  $\Gamma$  and  $V$  and to the tangent spaces to  $\Gamma$  and  $V$  at a generic point of  $A$  over  $K$ ; those tangent spaces have the tangent space to  $A$  at this point as intersection (by **b.**), and therefore they are transversal. This completes part of the proof of **i.**

Next, let  $B^b$  be any simple subvariety of  $V$ , defined over  $k$ . Let  $S$  be a generic point over the algebraic closure of  $K$  of a component  $C$  of  $A_\lambda \cap B$ . Let  $S \notin A$ . There is a point  $S' \in A$  such that  $S \in \Gamma_{(S', L)}$ . Let  $H$  be a hyperplane, containing  $L$  and not going through  $S$  and  $S'$ , let  $k_1$  be the smallest field of definition for  $H$  containing  $k$ . The coordinates in  $P^N$  being  $H, X, \dots, X_N$ ;  $L$  is defined by the equations  $H = 0, \sum_{j=1}^N v_{ij} X_j = 0$  ( $i = 1, \dots, n$ ),  $v_{ij}$  independent transcendentals over  $k_1$ . Introduce the affine space  $S_H^N = P^N - H$ . Consider in  $S_H^N \times S_H^N$  the variety  $A' \times B'$ , where  $A'$  and  $B'$

are the affine varieties corresponding to  $A$  and  $B$  respectively, and the linear variety  $\Lambda^{2N-n}$  defined by the equations  $\sum_{j=1}^N v_{ij}(X_j - x'_j) = 0$ ,  $i = 1, \dots, n$ , where  $S' = (1, x'_1, \dots, x'_N)$ .<sup>1</sup> Let  $S = (1, x_1, \dots, x_N)$ . Then since  $S \neq S'$ , there is a  $j$  such that  $x'_j \neq x_j$ ; let  $x'_1 \neq x_1$ . Let  $k_2$  be the field obtained by adjoining to  $k_1$  all the  $v_{ij}$  except the  $v_{1j}$  (for all  $j$ ). Then  $K = k_1(v) = k_2(v)$  is contained in  $k_2(x, x')$ ; in fact, we have the following inclusions:  $k \subset k_1 \subset k_2 \subset K \subset k_2(x, x')$ . Since  $S' \times S \in A' \times B'$  and  $\dim(A' \times B') = a + b$ , we have  $\dim_{k_2} k_2(x, x') \leq a + b$ ; furthermore, by [VI], Chap. V, Prop. 1,  $\dim_K K(x, x') \geq a + b - n$ . Combining, we obtain

$$a + b \geq \dim_{k_2} k_2(x, x') = \dim_{k_2} K + \dim_K K(x, x') \geq n + (a + b - n).$$

Therefore  $\dim_{k_2} k_2(x, x') = a + b$ ; hence  $\dim_{k_2} k_2(x) = b$ , i. e.,  $S = (x)$  is generic on  $B$  over  $k_2$  (so a fortiori over  $k$ ). Furthermore,  $\dim_K K(x, x') = a + b - n$ ; hence  $\dim_K K(x)$  (which by [VI], Chap. VI, Cor. 1 of Th. 1 is at least  $a + b - n$ , since  $S$  is simple on  $V$ ) is  $a + b - n$ , i. e.,  $C$  is a proper component of  $A_\lambda \cap B$  on  $V$ . This completes the proof of **ii.**, and if we take for  $B$  the variety  $V$  itself, then also the remaining part of **i.** is proved.

Next let  $C$  be any simple subvariety of  $V$ , algebraic over  $k$ . Let  $Q$  be a point of  $C$ , simple on  $V$  and algebraic over  $k$ . We will show  $Q \notin A_\lambda$ . Let  $Q \in A_\lambda$ . Let  $Q_1$  be a generic point of  $A_\lambda$  over  $K$ ,  $Q_1 \notin A$ ; hence there is a point  $P \in A$  such that the line  $PQ_1$  meets  $L$ , i. e.,  $P \in \Gamma_{(Q_1, L)} \cap V$ . Since  $L$  is generic over  $k(Q)$ , it follows that (if we apply **i.** to  $V$ ,  $A = Q_1$  and  $L$ )  $\Gamma_{(Q_1, L)} \cdot V = Q_1 + P + \sum_{i=2}^h Q_i$  with  $Q_i \neq Q_1$  for  $i \neq 1$ . Since  $\Gamma_{(Q_1, L)} = \Gamma_{(P, L)}$ , it follows that  $Q_1$  is algebraic over  $K(P)$ ; hence  $L$  is generic over  $k(P)$ , i. e.  $\Gamma_{(P, L)} \cdot V = P + \sum_{i=1}^h Q_i$  with  $Q_i \neq P$  (again by **i.**). Let  $(Q_1, P) \rightarrow (Q, P')$  over  $K$ . Then we still have that  $Q \in \Gamma_{(P', L)} \cap V$  (by [III], page 53g). Assume first that  $Q \neq P'$ . Again let  $H$  be a hyperplane, containing  $L$ , but not going through  $Q$  and  $P'$ . Write  $Q = (1, y_1, \dots, y_N)$  and  $P' = (1, y'_1, \dots, y'_N)$ . Since we can assume that  $A$  is different from  $V$  (if  $A = V$  then there is no  $A_\lambda$ ), we have  $\dim_{k_1} k_1(P') < n$  (the notations are as above). However, since  $Q$  is algebraic over  $k \subset k_1$ , it follows that the  $n$  equations (for  $\Gamma_{(P', L)}$ )  $\sum_{j=1}^N v_{ij}(X_j - y'_j) = 0$ ,  $i = 1, \dots, n$ , cannot be fulfilled by  $Q$  since the  $v_{ij}$  are independent transcendentals over  $k_1$ . Next assume  $Q = P'$ . The tangent

<sup>1</sup>  $\Gamma_{(S', L)}$  is then defined by the equations  $\sum_{j=1}^N v_{ij}(X_j - x'_j H) = 0$ ,  $i = 1, \dots, n$ , hence  $S' \times S \in A$ .

spaces to  $V$  at  $Q = P'$  and  $\Gamma_{(Q,L)}$  are transversal (their intersection is the point  $Q$ , for otherwise the tangent space to  $V$  at  $Q$  which is of dimension  $n$  and algebraic over  $k$  would meet the linear space  $L^{N-n-1}$  which is generic over  $k$ ). From this, it follows by [VI], Chap. VI, Th. 6 that  $Q = P'$  is a proper component of multiplicity 1 of  $\Gamma_{(P',L)} \cap V$ . On the other hand, consider a specialization  $(P, \sum_{i=1}^h Q_i) \rightarrow (P', Q + \sum_{i=2}^h Q'_i)$  over  $K$ . Consider the variety  $V$  and the linear varieties  $\Gamma_{(P,L)}$  and  $\Gamma_{(P',L)}$ . It follows easily by [VI], Chap. V, Cor. 1 of Th. 3 that the proper component  $P' = Q$  of  $\Gamma_{(P',L)} \cap V$  has a multiplicity greater than 1. Contradiction. Hence  $Q \notin A_\lambda$ .

The following is the main result of this section.

**PROPOSITION 1.\*\*** *Let  $A^a$  be a simple subvariety of a projective variety  $V^a$ . Let  $k$  be a field of definition for  $A$  and  $V$ . Then there exists a variety  $U$ , a variety  $W \subset U \times V$ , for  $U$  and  $W$  an algebraically closed field of definition  $K$  containing  $k$ , and a point  $Q'$  simple on  $U$  and algebraic over  $K$ , such that the following is true: if  $B^b$  ( $a + b \geq n$ ) is any simple variety on  $V$ , defined over a field  $k' \supset k$  which is free with respect to  $K$  over  $k$ , then*

- i. *for a point  $Q$  which is generic on  $U$  over  $K'$  (the composition of  $K$  and  $k'$ ) we have that  $W \cdot (Q \times V) = Q \times W(Q)$ , where  $W(Q) = A^*$  is a  $K(Q)$ -prime rational cycle with reduced expression  $A^* = \sum_{\mu} A^*_{\mu}$  (i.e., with coefficients 1), which is such that  $A^* \cdot B = \sum_i C_i$  with  $C_i \neq C_j$  for  $i \neq j$  and  $\sum_i C_i$  is  $K'(Q)$ -prime rational. (This remains true if  $B$  is a  $k'$ -prime rational cycle with reduced expression  $B = \sum_i B_i$ ). Furthermore, a generic point of  $C_i$  over a field of definition for  $C_i$  is generic on  $B$  over  $k'$ .*
- ii.  $W \cdot (Q' \times V) = Q' \times (A + \sum_{\lambda} \eta_{\lambda} A_{\lambda})$ , where  $A_{\lambda}$  is such that
  - a. *every component  $C$  of  $A_{\lambda} \cap B$ , which is not contained in  $A \cap B$ , is a proper component of  $A_{\lambda} \cap B$  on  $V$ , and a generic point of  $C$  over a field of definition for  $C$  is generic on  $B$  over  $k$ .*
  - b. *if  $D$  is a simple subvariety of  $V$  defined over a field which contains  $k$  and which is free with respect to  $K$  over  $k$ , then there is always a point of  $D$  outside all the  $A_{\lambda}$ .*

Furthermore, this system  $U$ ,  $W$ ,  $K$ , and  $Q'$  can always be selected in such a way that  $K$  is free with respect to an arbitrary given extension field of  $k$ .

(Notes: 1. It follows in particular from **iiib.** that every component of  $A_{\lambda} \cap B$ ,

\*\* Cf. also Lemma 2, page 456, of W. L. Chow, "On equivalence classes of cycles in an algebraic variety," *Annals of Mathematics*, vol. 64 (1956), pp. 450-479.

which is contained in a simple component of  $A \cap B$ , is contained strictly in that component.

2. If  $A \cdot B$  is defined, then all the  $A_\lambda \cdot B$  are defined.
3. In the following, we often denote such a system  $U, W, K$  and  $Q'$  by  $[U, W, K, Q'; A, V, k].$

*Proof.* Let  $V^n$  be in  $P^N$ , let  $L^{N-n-1}$  be a linear space generic over  $k$ . For  $K$ , we take the algebraic closure of the smallest field of definition for  $L$  containing  $k$ . For  $U$ , we take  $P^{N^2+2N}$ ; every point of  $U$  can be interpreted as an  $(N+1)$ -square matrix to which belongs a projective transformation of  $P^N$ , and, conversely, to every  $(N+1)$ -square matrix belongs a point of  $U$ . Let  $Q'$  be the point belonging to the identity matrix. For  $W$  we take the locus over  $K$  of the point  $(T, P)$ , here  $T$  is a generic point on  $U$  over  $K$  and  $P$  is a generic point over  $K(T)$  of the  $K(T)$ -prime rational cycle  $V \cdot \Gamma^T$  (see Lemma 2), where  $\Gamma = \Gamma_{(A, L)}$ .

We have to show that  $U, W, K$  and  $Q'$  fulfill the requirements. Consider  $W(Q)$  for  $Q$  generic over  $K$  on  $U$ ; the  $V$ -cycle  $W(Q)$  considered as a  $P^N$ -cycle is equal, by construction, to  $\Gamma^Q \cdot V$ . By Lemma 2, applied to  $\Gamma$  and  $V$ , the cycle  $W(Q)$  is  $K(Q)$ -prime rational and has the required expression. If  $B^b$  is any simple subvariety of  $V$ , defined over a field  $k'$  (as described in the proposition, and if  $Q$  is generic over  $K'$ ), then  $W(Q) \cap B$  is, as a point-set, the same as  $\Gamma^Q \cap B$ ; hence, by Lemma 1i. and 1ii., every component of  $W(Q) \cap B$  is proper and simple on  $V$ . Let  $C_i$  be such a component, then  $C_i$  is also a component of  $\Gamma^Q \cap B$ . Now if  $W(Q) = A^* = \sum_\mu A^*_\mu$ , then by [VI], Chap. VI, Th. 9, and since  $i(\Gamma^Q \cdot V, A^*_\mu; P^N) = 1$  by Lemma 1i., we have

$$\begin{aligned} i(W(Q) \cdot B, C_i; V) &= \sum_\mu i(A^*_\mu \cdot B, C_i; V) \\ &= \sum_\mu i(A^*_\mu \cdot B, C_i; V) i(\Gamma^Q \cdot V, A^*_\mu; P^N) = i(\Gamma^Q \cdot B, C_i; P^N) = 1. \end{aligned}$$

Therefore the  $V$ -cycle  $A^* \cdot B = \sum_i C_i$ , if considered as a cycle in  $P^N$ , is equal to  $\Gamma^Q \cdot B$ ; from this, all the remaining assertions of 1. follow by Lemma 2. The same reasoning holds if  $B$  is a  $k'$ -prime rational cycle and equal to  $\sum_i B_i$ .

Consider the locus  $W_1$  of  $(T, R)$  over  $K$ , where, as before  $T$  is generic on  $U$  over  $K$ , and where  $R$  is generic over  $K(T)$  on  $\Gamma^T$ . (Note that  $W_1 \cdot (Q' \times P^N) = Q' \times \Gamma$ , as follows by considering a system of equations for  $W_1$ .) Next we show that  $W$  is a proper component of multiplicity 1 of  $W_1 \cap (U \times V)$  and that every other component of that intersection has on  $U$  a projection smaller than  $U$ . Clearly  $W \subset W_1 \cap (U \times V)$ . Let  $Y$  be a component of  $W_1 \cap (U \times V)$  with projection  $U$  on  $U$ . Consider  $Y \cap (T \times P^N)$  for  $T$  generic over  $K$  on  $U$ ; by assumption, this is not empty. Let  $T \times R'$

be a generic point over the algebraic closure of  $K(T)$  of a component  $T \times Z$  of  $Y \cap (T \times P^N)$ ; by [VI], Chap. VI, Th. 11,  $T \times R'$  is generic on  $Y$  over  $K$ . We have

$$\begin{aligned} T \times Z \subset Y \cap (T \times P^N) &\subset W_1 \cap (U \times V) \cap (T \times P^N) \\ &= (T \times \Gamma^T) \cap (U \times V) \subset T \times (V \cap \Gamma^T). \end{aligned}$$

Hence  $T \times R'$  is on  $W$  and  $Y = W$  for dimension reasons. Since

$$i(\Gamma^T \cdot V, C_i; P^N) = 1$$

by Lemma 1i., it follows by [VI], Chap. VI, Th. 7 that

$$i[(T \times \Gamma^T) \cdot (U \times V), T \times C_i; U \times P^N] \cdot i[W_1 \cdot (T \times P^N), T \times \Gamma^T; U \times P^N] = 1$$

(the second factor is 1 by [VI], Chap. VI, Th. 11). Therefore, by the associativity of the  $i$ -symbol ([VI], Chap. VI, Th. 5),

$$i[W_1 \cdot (U \times V), W; U \times P^N] \cdot i[W \cdot (T \times P^N), T \times C_i; U \times P^N] = 1.$$

Hence  $i[W_1 \cdot (U \times V), W; U \times P^N] = 1$ .

Consider next the point  $Q'$ . If  $Y$  is any component of  $W_1 \cap (U \times V)$ , different from  $W$ , then  $Y\{Q'\}$  either is empty or consists entirely of singular points on  $V$ . For, by [VI], Chap. VI, Cor. 1 of Th. 1,  $\dim Y \geq N^2 + 2N + a$ ; since the projection of  $Y$  on  $U$  is, as we have seen, smaller than  $U$ , it follows (again by the same corollary<sup>2</sup>) that if there is a simple point in  $Y\{Q'\}$ , then  $Y\{Q'\}$  must have a component of dimension greater than  $a$ . However,  $Y\{Q'\} \subset V \cap \Gamma$ , for

$$\begin{aligned} Q' \times Y\{Q'\} &= Y \cap (Q' \times V) \subset W_1 \cap (U \times V) \cap (Q' \times P^N) \\ &= (Q' \times \Gamma) \cap (U \times V); \end{aligned}$$

hence  $Y\{Q'\}$  has only components of dimension  $\leq a$  by Lemma 3i. Since every component of  $W_1(Q') \cap V = \Gamma \cap V$  is simple on  $V$  by Lemma 3i., it follows that the components of  $W_1(Q') \cap V$  are the same as the components of  $W\{Q'\}$ . Hence, for dimension reasons,  $W(Q')$  is defined, and if  $V \cap W_1(Q') = A + \sum_{\lambda} \eta'_{\lambda} A_{\lambda}$  (see Lemma 3i.), then  $W(Q') = \eta A + \sum_{\lambda} \eta_{\lambda} A_{\lambda}$ .

<sup>2</sup> Apparently, there arises a difficulty if we want to apply [VI], Chap. VI, Cor. 1 of Th. 1 in the above mentioned situation. For if  $Y'$  is the projection on  $U$  of  $Y$ , then we have to consider the intersection  $Y \cap (Q' \times V)$  on  $Y' \times V$ ; however, it could be that  $Q'$  is singular on  $Y'$ . Therefore we apply the corollary to the following situation: Let  $X$  be a curve on  $U$ , going through  $Q'$  and such that a generic point of  $X$  over a field of definition for  $X$  is generic on  $U$  over  $K$ . Let the simple point in  $Y\{Q'\}$  be denoted by  $S$ . Apply the above mentioned corollary to  $Y \cap (X \times V)$ ; it follows that there is a component  $Z$ , containing  $Q' \times S$ , which is at least of dimension  $a + 1$ . Since  $\text{pr}_V Z \subset X$  and  $\text{pr}_V Z \neq X$  (for  $X$  contains a generic point of  $U$  over  $K$  and  $\text{pr}_V Z \subset \text{pr}_V Y \neq U$ ), it follows that  $Z \subset Y\{Q'\}$ .

Using the fact that  $i[W_1 \cdot (U \times V), W; U \times P^N] = 1$ , the associativity of the  $i$ -symbol ([VI], Chap. VI, Th. 5) and some trivialities, we obtain

$$\begin{aligned} \eta &= i[W \cdot (Q' \times V), Q' \times A; U \times V] = i[W \cdot (Q' \times P^N), Q' \times A; U \times P^N] \\ &= i[W \cdot (Q' \times P^N), Q' \times A; U \times P^N] \cdot i[W_1 \cdot (U \times V), W; U \times P^N] \\ &= i[W_1 \cdot (Q' \times P^N), Q' \times \Gamma; U \times P^N] i[(Q' \times \Gamma) \cdot (U \times V), \\ &\quad Q' \times A; U \times P^N] = i(\Gamma \cdot V, A; P^N) = 1. \end{aligned}$$

All the remaining assertions follow from Lemma 3ii. and iii. This completes the proof.

**2. Definition of the  $i$ -symbol and some of its properties.** First we introduce some concepts and notations which are frequently used in the following. If  $\sum_i n_i R_i$  is a 0-dim. cycle on a variety  $V$ , and if  $\mathcal{B}$  is a bunch on  $V$ , then we understand by "the number of points of  $\sum_i n_i R_i$  which are in  $\mathcal{B}$ " the number of points  $R_i$  which are in  $\{\mathcal{B}\}$ , each counted with the integer  $n_i$ .<sup>3</sup> If  $\mathcal{B}$  and  $\mathcal{D}$  are bunches on  $V$ , such that every component  $C$  of  $\mathcal{B}$  is strictly contained in a component  $D$  of  $\mathcal{D}$ , then we shall say that  $\mathcal{B}$  is strictly contained component-wise in  $\mathcal{D}$ , and we write  $\mathcal{B} < \mathcal{D}$  or  $\mathcal{D} > \mathcal{B}$ .

Let  $A^a$  and  $B^b$  be subvarieties of a projective variety  $V^n$ . Let  $\mathcal{B}$  be a maximal connected bunch of  $A \cap B$ , and assume that every point of  $\{\mathcal{B}\}$  is simple on  $V$ . If  $C$  is a component of  $\mathcal{B}$ , i.e., a component of  $A \cap B$ , then we define the *excess* of  $C$  as a component of the intersection of  $A$  and  $B$  on  $V$  as the integer  $\dim C - a - b + n$ ; the maximum of these integers over all the components of  $\mathcal{B}$  we call the *excess of  $\mathcal{B}$*  as a maximal connected bunch of  $A \cap B$  on  $V$ .

From now on, we assume  $a + b = n$ . We want to define an integer  $i(A \cap B, \mathcal{B}; V)$ , called the *intersection multiplicity of  $A$  and  $B$  on  $V$  at  $\mathcal{B}$* . By induction on the excess of  $\mathcal{B}$ , we assume that such an integer is defined and that the symbol is commutative (i.e.,  $i(A \cap B, \mathcal{B}; V) = i(B \cap A, \mathcal{B}; V)$ ) if the excess of  $\mathcal{B}$  is smaller than some integer  $d$ . (For excess  $\mathcal{B} = 0$  there is such a multiplicity theory in [VI], Chap. VI.) Now let  $\mathcal{B}$  be a maximal connected bunch of  $A \cap B$ , with excess equal to  $d$ .

Let  $k$  be a field of definition for  $A$ ,  $B$  and  $V$ . Let (with respect to the varieties  $A$  and  $V$  and the field  $k$ ) the varieties  $U$  and  $W$ , the field  $K$  and the point  $P'$  have the properties mentioned in Proposition 1. For convenience of notation, we shall denote this system by  $\alpha = [U, W, K, P'; A, V, k]$ . Let  $P$  be a generic point on  $U$  over  $K$ ; writing, as in Proposition 1,  $W(P) = A^*$ , we have, by our assumptions on the system  $\alpha$ ,  $A^* \cdot B = \sum_i R_i$  with  $R_i \neq R_j$  for

<sup>3</sup>  $\{\mathcal{B}\}$  denotes the point-set attached to  $\mathcal{B}$ . (Cf. [VI], page 84).

$i \neq j$  and  $\sum_i R_i$  prime rational over  $K(P)$  (here  $k' = k$ , hence  $K' = K$ ). Let  $\mathcal{D}_\lambda$  be those maximal connected bunches of  $A_\lambda \cap B$  which are in  $\{\mathcal{E}\}$ ; by our assumptions on the system  $\alpha$ , these are the only bunches of  $A_\lambda \cap B$  which have points in common with  $\{\mathcal{E}\}$ , and we have  $\mathcal{D}_\lambda \prec \mathcal{E}$  for all couples  $\lambda, i$  (see Prop. 1iia. and iib.). By the induction assumption,  $i(A_\lambda \cap B, \mathcal{D}_\lambda; V)$  is defined. Let  $\sum_i R'_i$  be a specialization of  $\sum_i R_i$  over the specialization  $P \rightarrow P'$  with reference to  $K$ .<sup>3a</sup> (Note that, by the specialization theorem, [III], page 104,  $A^*$  specializes uniquely to  $A + \sum_\lambda \eta_\lambda A_\lambda$  over the specialization  $P \rightarrow P'$  with reference to  $K$ ). Let  $\sigma$  be the number of points of  $\sum_i R'_i$  which are in  $\{\mathcal{E}\}$ . Our first purpose is to show that  $\sigma$  is independent of the particular specialization  $\sum_i R'_i$  of  $\sum_i R_i$  over the specialization  $P \rightarrow P'$  with reference to  $K$ . Consider the locus  $G$  over  $K$  of the point  $(P, R_i)$ . Since  $\sum_i R_i$  is  $K(P)$ -prime rational, this is independent of  $i$ ;  $G$  is a subvariety of  $U \times V$ . Consider the total transform of the point  $P'$  (which, by assumption, is simple on  $U$ ) by  $G$ . Since  $R_i \in A^* \cap B$ , it follows that the maximal connected bunches of  $G\{P'\}$  are contained in the maximal connected bunches of  $A \cap B$  and  $A_\lambda \cap B$  (by [III], page 53g since  $A^* \rightarrow A + \sum_\lambda \eta_\lambda A_\lambda$  and  $B \rightarrow B$  if  $P \rightarrow P'$  with reference to  $K$ ); by our previous remarks, those bunches are either entirely inside or entirely outside  $\{\mathcal{E}\}$ . Apply [II], Th. 3 to those maximal connected bunches of  $G\{P'\}$  which are contained in  $\{\mathcal{E}\}$ ; it follows that  $\sigma$  is independent of the particular specialization  $\sum_i R'_i$  of  $\sum_i R_i$  over the specialization  $P \rightarrow P'$  with reference to  $K$ .

Define

$$(1) \quad i_a(A \cap B, \mathcal{E}; V) = \sigma - \sum_\lambda \eta_\lambda \sum_i i(A_\lambda \cap B, \mathcal{D}_\lambda; V).^4$$

First of all we shall show that if  $\{\mathcal{E}\}$  is reduced to a point  $C$  (and hence  $i_a(A \cap B, \mathcal{E}; V) = \sigma$ ), then  $i_a(A \cap B, \mathcal{E}; V) = i(A \cdot B, C; V)$ . Consider again the variety  $G$  introduced above. By [VI], Chap. VI, Th. 12, it follows that  $\sigma = i[G \cdot (P' \times V), P' \times C; U \times V]$ . Clearly  $G$  is a component of  $(U \times B) \cap W$ , and it is the only component with projection  $U$  on  $U$ . For dimension reasons, it is a proper component on  $U \times V$ ; moreover, it is a proper component of multiplicity 1 as follows from the computation (if  $R_i \in A^*_\mu$ )

$$\begin{aligned} 1 &= i(A^*_\mu \cdot B, R_i; V) = i[(P \times A^*_\mu) \cdot (U \times B), P \times R_i; U \times V] \\ &= i[(P \times V) \cdot W, P \times A^*_\mu; U \times V] i[(P \times A^*_\mu) \cdot (U \times B), P \times R_i; U \times V] \\ &= i[W \cdot (U \times B), G; U \times V] i[(P \times V) \cdot G, P \times R_i; U \times V] \end{aligned}$$

<sup>3a</sup> Specializations of (positive) cycles on  $V$  have to be understood as follows: take a specialization of the cycle in projective space and omit in the summation all components which are singular on  $V$ . (Cf. W. L. Chow, loc. cit. in \*\*.)

<sup>4</sup> This integer can be negative; see footnote 11 in [V], page 639.

by (except for some trivialities) the associativity of the  $i$ -symbol, [VI], Chap. VI, Th. 5. Moreover, there is no component  $G'$  of  $(U \times B) \cap W$  which contains  $P' \times C$ . For if  $G'$  should be such a component, then the projection of  $G'$  on  $U$  is smaller than  $U$ ; hence, by [VI], Chap. VI, Cor. 1 of Th. 1, every component  $D$  of  $G'\{P'\}$  containing  $P' \times C$  is of dimension greater than zero.<sup>5</sup> However,  $D \subset A \cap B$ , for

$$\begin{aligned} G' \cap (P' \times V) &\subset (U \times B) \cap W \cap (P' \times V) \\ &\subset [(U \times B) \cap (P' \times A)] \cup \left[ \bigcup_{\lambda} (U \times B) \cap (P' \times A_{\lambda}) \right] \\ &\subset P' \times (A \cap B) \cup \left[ \bigcup_{\lambda} A_{\lambda} \cap B \right]. \end{aligned}$$

Hence, since for all  $\lambda$ ,  $C \notin A_{\lambda}$  (Prop. 1iib.), it follows that  $D \subset A \cap B$ ; but then  $C$  is not a component of  $A \cap B$  since it is contained in the component  $D$ . Therefore, we have that

$$\begin{aligned} \sigma &= i[G \cdot (P' \times V), P' \times C; U \times V] \\ &= i[G \cdot (P' \times V), P' \times C; U \times V] i[W \cdot (U \times B), (U \times B), G; U \times V] \\ &= i[(P' \times V) \cdot W, P' \times A; U \times V] i[(P' \times A) \cdot (U \times B), P' \times C; U \times V] \end{aligned}$$

(by [VI], Chap. VI, Th. 5). Since the first factor of the last product is 1 (see Prop. 1ii), it follows that  $\sigma = i[(P' \times A) \cdot (U \times B), P' \times C; U \times V] = i(A \cdot B, C; V)$  by [VI], Chap. VI, Th. 7.

Next we want to show that  $i_{\alpha}(A \cap B, \mathcal{B}; V)$  is, in fact, independent of the system  $\alpha = [U, W, K, P'; A, V, k]$ . Consider a system  $\beta = [X, Y, K_1, Q'; B, V, k]$  having the properties of Proposition 1, and let  $K_1$  be free with respect to  $K(P)$  over  $k$ ; let the compositum of  $K$  and  $K_1$  be  $K_2$ . We have  $Y \cdot (Q' \times V) = Q' \times (B + \sum_{\mu} \pi_{\mu} B_{\mu})$ . Let those maximal connected buches of  $A \cap B_{\mu}$  which are in  $\{\mathcal{B}\}$  be denoted by  $\mathcal{E}_{\mu j}$ , and let those maximal connected bunches of  $A_{\lambda} \cap B_{\lambda}$  which are in  $\{\mathcal{B}\}$  be denoted by  $\mathcal{F}_{\lambda \mu h}$ . (By the assumptions on the system  $\beta$ , and since  $A$  and  $A_{\lambda}$  are defined over a field  $K$  which is free with respect to  $K_1$  over  $k$ , it follows by Prop. 1iib. that  $\mathcal{E}_{\mu j} \prec \mathcal{B}$  and for a fixed  $\lambda, \mu$  and  $h$  there is an index  $i$  such that  $\mathcal{F}_{\lambda \mu h} \prec \mathcal{D}_{\lambda i}$ ;<sup>6</sup> similarly, by the assumptions on the system  $\alpha$ , and since the  $B_{\mu}$  are defined over  $K_1$ , which is

<sup>5</sup> We have to use here the same kind of argument as in footnote 2.

<sup>6</sup> The variety  $A_{\lambda}$  is defined over the field  $K$  which is free with respect to  $K_1$  over  $k$ , and the system  $\beta$  has the properties of Proposition 1. Therefore, it follows by Prop. 1iia. and iib. that a component  $G$  of  $A_{\lambda} \cap B_{\mu}$  which has points in common with  $\{\mathcal{B}\}$  either is contained strictly in a component of  $A_{\lambda} \cap B$  (which then must have points in common with  $\{\mathcal{B}\}$ , i. e., which must be in a  $\mathcal{D}_{\lambda i}$ ) or is proper. In the latter case, this component  $G$  is a point, generic on  $A_{\lambda}$  over  $K$ ; however this is impossible since  $G \in \{\mathcal{B}\}$ , and hence  $A_{\lambda}$  should be contained in  $A$ .

free with respect to  $K$  over  $k$ , it follows that for a fixed  $\lambda, \mu$  and  $h$  there is an index  $j$  such that  $\mathcal{F}_{\lambda\mu h} \prec \mathcal{E}_{\mu j}$ . Furthermore, we note that these bunches are the only maximal connected bunches of the intersections under consideration which have points in common with  $\{\mathcal{C}\}$ . By our induction assumption on the commutativity of the  $i$ -symbol, we have

$$i(A_\lambda \cap B_\mu, \mathcal{F}_{\lambda\mu h}; V) = i(B_\mu \cap A_\lambda, \mathcal{F}_{\lambda\mu h}; V).$$

Let  $Q$  be a generic point of  $X$  over  $K_2(P)$ . By the properties of the system  $\beta$ , we have that the following cycles are defined:  $A^* \cdot B^* = \sum_i S_i$ ,  $A \cdot B^* = \sum_i T_i$  and  $A_\lambda \cdot B^* = \sum_i T_{\lambda i}$ ; furthermore,  $S_i \neq S_j$ ,  $T_i \neq T_j$ ,  $T_{\lambda i} \neq T_{\lambda j}$  and  $\sum_i S_i$ ,  $\sum_i T_i$  and  $\sum_i T_{\lambda i}$  are prime rational over  $K_2(P, Q)$ ,  $K_2(Q)$  and  $K_2(Q)$  respectively. Similarly, by the properties of the system  $\alpha$ , we have [except  $A^* \cdot B = \sum_i R_i$ ,  $R_i \neq R_j$  and  $\sum_i R_i$  prime rational over  $K_2(P)$ ] also that  $A^* \cdot B_\mu = \sum_i R_{\mu i}$  with  $R_{\mu i} \neq R_{\mu j}$  and  $\sum_i R_{\mu i}$  prime rational over  $K_2(P)$ .

Next consider a specialization  $(Q, \sum_i T_i) \rightarrow (Q', \sum_i T'_i)$  over  $K_1$  (hence, by [VI], Chap. II, Th. 4, a fortiori over  $K_2$  since  $K_2$  is the compositum of  $K$  and  $K_1$  and  $K$  is free with respect to  $(Q)$  over  $K_1$ ). Let  $\tau$  be the number of points of  $\sum_i T'_i$  which are in  $\{\mathcal{C}\}$ ; then by definition

$$(2) \quad i_\beta(B \cap A, \mathcal{C}; V) = \tau - \sum_\mu \pi_\mu \sum_j i(B_\mu \cap A, \mathcal{E}_{\mu j}; V).$$

We intend to show that  $i_\alpha(A \cap B, \mathcal{C}; V) = i_\beta(B \cap A, \mathcal{C}; V)$ .

Let  $E$  be the locus of  $(P, Q, S_i)$  over  $K_2$ ; since  $S_i \in A^* \cap B^*$  and since  $A^*$  and  $B^*$  specialize to  $A + \sum_\lambda \eta_\lambda A_\lambda$  and  $B + \sum_\mu \pi_\mu B_\mu$  if  $P \rightarrow P'$  and  $Q \rightarrow Q'$  over  $K_2$ , it follows (by [III], page 53g) that the maximal connected bunches of  $E\{P', Q'\}$  are contained in the maximal connected bunches of  $A \cap B$ ,  $A \cap B_\mu$ ,  $A_\lambda \cap B$  and  $A_\lambda \cap B_\mu$ . By what is shown above about the bunches of those intersections, it follows that the maximal connected bunches of  $E\{P', Q'\}$  are either entirely inside or entirely outside  $\{\mathcal{C}\}$ . Apply [II], Th. 3 to  $E \subset U \times X \times V$ , to the simple point  $(P', Q')$  of  $U \times X$  and to those maximal connected bunches of  $E\{P', Q'\}$  which are in  $\{\mathcal{C}\}$ . It follows that the number of points of every specialized cycle of  $\sum_i S_i$ , over the specialization  $(P, Q) \rightarrow (P', Q')$  with reference to  $K_2$ , which are in  $\{\mathcal{C}\}$  is always the same. Consider, in particular, the specialization

$$(P, Q, \sum_i S_i) \rightarrow (P, Q', A^* \cdot [B + \sum_\mu \pi_\mu B_\mu]) \\ \rightarrow (P', Q', \sum_i S'_i = \sum_i R'_i + \sum_\mu \pi_\mu \sum_i R'_{\mu i})$$

over  $K_2$  [it is easily seen by [VI], Chap. II, Th. 4 that  $(P, A^* \cdot B) \rightarrow (P', \sum_i R'_i)$  not only over  $K$  but also over  $K_2$ ]; let  $n'$ ,  $\sigma$  and  $\sigma_\mu$  be, respectively, the number of points of  $\sum_i S'_i$ , of  $\sum_i R'_i$  and of  $\sum_i R'_{\mu i}$  which are in  $\{\mathcal{C}\}$  [ $\sigma$  has been

introduced above]. Clearly  $n' = \sigma + \sum_{\mu} \pi_{\mu} \sigma_{\mu}$ . On the other hand, consider the specialization

$$(P, Q, \sum_i S_i) \rightarrow (P', Q, [A + \sum_{\lambda} \eta_{\lambda} A_{\lambda}] \cdot B^*) \\ \rightarrow (P', Q', \sum_i S''_i = \sum_i T'_i + \sum_{\lambda} \eta_{\lambda} \sum_i T'_{\lambda i})$$

over  $K_2$ ; let  $n''$ ,  $\tau$  and  $\tau_{\lambda}$  be the number of points of  $\sum_i S''_i$ , of  $\sum_i T'_i$  and of  $\sum_i T'_{\lambda i}$  which are in  $\{\mathcal{C}\}$ . Clearly  $n'' = \tau + \sum_{\lambda} \eta_{\lambda} \tau_{\lambda}$ . As we have just seen,  $n' = n''$ . Hence

$$(3) \quad \sigma + \sum_{\mu} \pi_{\mu} \sigma_{\mu} = n' = n'' = \tau + \sum_{\lambda} \eta_{\lambda} \tau_{\lambda}.$$

By the definition of the  $i$ -symbol, we have (if we make use of the above mentioned relations between the  $\mathcal{D}_{\lambda i}$ , the  $\mathcal{E}_{\mu j}$  and the  $\mathcal{F}_{\lambda \mu h}$ ) after a summation:

$$(4) \quad \sum_{\mu} \pi_{\mu} \sum_j i(A \cap B_{\mu}, \mathcal{E}_{\mu j}; V) \\ = \sum_{\mu} \pi_{\mu} \sigma_{\mu} - \sum_{\mu} \pi_{\mu} \sum_{\lambda, h} \eta_{\lambda} i(A_{\lambda} \cap B_{\mu}, \mathcal{F}_{\lambda \mu h}; V)$$

and

$$(5) \quad \sum_{\lambda} \eta_{\lambda} \sum_i i(B \cap A_{\lambda}, \mathcal{D}_{\lambda i}; V) \\ = \sum_{\lambda} \eta_{\lambda} \tau_{\lambda} - \sum_{\lambda} \eta_{\lambda} \sum_{\mu, h} \pi_{\mu} i(B_{\mu} \cap A_{\lambda}, \mathcal{F}_{\lambda \mu h}; V).$$

(The equations (4) and (5) are essentially nothing else but the definitions of  $i(A \cap B_{\mu}, \mathcal{E}_{\mu j}; V)$  and  $i(B \cap A_{\lambda}, \mathcal{D}_{\lambda i}; V)$  respectively.) In view of the commutativity of the  $i$ -symbols (induction assumption), the last sums in (4) and (5) are equal. From this, one obtains from (4), (5) and (3) that

$$\sigma + \sum_{\mu} \pi_{\mu} \sum_j i(A \cap B_{\mu}, \mathcal{E}_{\mu j}; V) = \tau + \sum_{\lambda} \eta_{\lambda} \sum_i i(B \cap A_{\lambda}, \mathcal{D}_{\lambda i}; V).$$

From this, again after having used the commutativity of the  $i$ -symbol for the bunches  $\mathcal{D}_{\lambda i}$  and  $\mathcal{E}_{\mu j}$  which have an excess smaller than excess  $\mathcal{C}$ , it follows from the definitions (1) and (2) that  $i_{\alpha}(A \cap B, \mathcal{C}; V) = i_{\beta}(B \cap A, \mathcal{C}; V)$ .

From the fact that  $i_{\beta}$  is independent of the system  $\alpha$ , it follows first that  $i_{\alpha}(A \cap B, \mathcal{C}; V)$  is independent of  $\alpha$ ; therefore we can write  $i(A \cap B, \mathcal{C}; V)$ , and next we can conclude from our result  $i_{\alpha}(A \cap B, \mathcal{C}; V) = i_{\beta}(B \cap A, \mathcal{C}; V)$  the commutativity of the  $i$ -symbol.

**THEOREM 1.** *Let  $A^a$  and  $B^b$  be subvarieties of a projective variety  $V^n$  ( $n = a + b$ ). Let  $\mathcal{C}$  be a maximal connected bunch of  $A \cap B$  on  $V$ , such that every point of  $\{\mathcal{C}\}$  is simple on  $V$ . Then  $i(A \cap B, \mathcal{C}; V) = i(B \cap A, \mathcal{C}; V)$ .*

Another immediate consequence of the definition is the birational invariance of the  $i$ -symbol for birational transformations which are biregular at every point of  $\{\mathcal{C}\}$ .

**THEOREM 2.** *Let  $A^a$  and  $B^b$  be subvarieties of a projective variety  $V^n$  ( $n = a + b$ ). Let  $\mathcal{C}$  be a maximal connected bunch of  $A \cap B$ , such that*

every point of  $\{\mathcal{E}\}$  is simple on  $V$ . Let  $T$  be a birational correspondence between  $V$  and a projective variety  $V'$ , biregular at every point of  $\{\mathcal{E}\}$ . Let  $A'$ ,  $B'$  and  $\mathcal{E}'$  correspond with  $A$ ,  $B$  and  $\mathcal{E}$  by  $T$ . Then  $\mathcal{E}'$  is a maximal connected bunch of  $A' \cap B'$  and  $i(A \cap B, \mathcal{E}; V) = i(A' \cap B', \mathcal{E}'; V')$ .

*Proof.* By induction on the excess of  $\mathcal{E}$ . Let the notations be as above. We observe that, from the fact that  $T$  is biregular at every point of  $\{\mathcal{E}\}$ , it follows that  $T$  is biregular at every point  $R_i$  of  $A^* \cdot B$  and at every  $A_\lambda$  (a generic point of  $A_\lambda$  over  $K$  is, namely, generic on  $V$  over  $k$ , as follows from Prop. 1.11a.). From this, one concludes by standard reasoning that every integer in (1) is equal to the corresponding integer with primes; hence  $i(A \cap B, \mathcal{E}; V) = i(A' \cap B', \mathcal{E}'; V')$ .

### 3. Some further properties of the $i$ -symbol.

**THEOREM 3.** Let  $B^b$  be a simple subvariety of a projective variety  $V^n$ . Let  $\mathcal{E}$  be a bunch of subvarieties of  $V$  such that every point of  $\{\mathcal{E}\}$  is simple on  $V$ . Suppose there is a variety  $U$  and a subvariety  $W$  of  $U \times V$  with the following properties:

- i. If  $Q$  is a generic point on  $U$  over a common field of definition  $k$  for  $B$ ,  $V$ ,  $U$  and  $W$ , then the cycle  $W(Q) = A^*$  is defined, is of dimension  $n - b$ , and is such that every component of  $W(Q) \cap B$  is a point if this component is simple on  $V$ . Let  $A^* \cdot B = \sum_i R_i$  (it is not excluded that  $R_i = R_j$  for  $i \neq j$ ).
- ii. There is a point  $Q'$ , simple on  $U$ , such that the cycle  $W(Q')$  is defined and  $W(Q')$  is such that the maximal connected bunches of  $W(Q') \cap B$  are either entirely contained in  $\{\mathcal{E}\}$  or do not have any point in common with  $\{\mathcal{E}\}$ . Let  $W(Q') = \sum_\lambda \eta_\lambda A_\lambda$  (reduced expression), and let  $\mathcal{E}_{\lambda i}$  be those maximal connected bunches of  $A_\lambda \cap B$  which are in  $\{\mathcal{E}\}$ .

Suppose  $\sum_i R'_i$  is a specialization of  $\sum_i R_i$  over the specialization  $Q \rightarrow Q'$  with reference to  $k$ , let  $\sigma$  be the number of points of  $\sum_i R'_i$  which are in  $\{\mathcal{E}\}$ . Then  $\sigma = \sum_\lambda \eta_\lambda \sum_i i(A_\lambda \cap B, \mathcal{E}_{\lambda i}; V)$ .

*Proof.* We proceed by induction on the integer  $\max_{C \in \mathcal{E}} \dim C$ ; let this integer be denoted by  $d$ . The proof for the case  $d = 0$  is included in the considerations below. Furthermore, for convenience, we assume that  $k$  is such that  $Q'$  is rational in  $k$ . Write  $A^* = \sum_\gamma \alpha_\gamma A^*_\gamma$  with  $A^*_\gamma$   $k(Q)$ -prime rational; let  $A^*_\gamma \cdot B = \sum_i R_{\gamma i}$ , hence  $\sum_\gamma \alpha_\gamma \sum_i R_{\gamma i} = \sum_i R_i$ . Let

$$\beta = [X, Y, K, P'; B, V, k]$$

be a system having the properties of Proposition 1, where  $K$  is free with respect to  $k(Q)$  over  $k$ . Write  $B^* = Y(P)$  for  $P$  generic on  $X$  over  $K(Q)$ . By assumption on the system  $\beta$ , we have  $A^* \cdot B^* = \sum_{\gamma} \alpha_{\gamma} A^*_{\gamma} \cdot B^* = \sum_{\gamma} \alpha_{\gamma} \sum_i S_{\gamma i}$  with  $S_{\gamma i} \neq S_{\gamma j}$  and  $\sum_i S_{\gamma i}$  is  $K(Q, P)$ -prime rational. Let  $Y(P') = B + \sum \pi_{\mu} B_{\mu}$ . It follows by the properties of the system  $\beta$  (see Prop. 1iii.) that all the  $A^* \cdot B_{\mu}$  are defined (since  $A^* \cdot B$  is defined); write  $A^* \cdot B_{\mu} = \sum_{\mu, i} R_{\mu i}$ . Let  $\mathcal{D}_{\lambda \mu j}$  be those maximal connected bunches of the  $A_{\lambda} \cap B_{\mu}$  which have points in common with  $\{\mathcal{C}\}$ . It follows by the properties of the system  $\beta$  (see Prop. 1ii.) that for every  $\mathcal{D}_{\lambda \mu j}$ , either there is a  $\mathcal{C}_{\lambda i}$  such that  $\mathcal{D}_{\lambda \mu j} \prec \mathcal{C}_{\lambda i}$  or  $\mathcal{D}_{\lambda \mu j}$  is reduced to a point and hence is a proper component of  $A_{\lambda} \cap B_{\mu}$  on  $V$ .

Let  $E_{\gamma}$  be the locus of  $(P, Q, S_{\gamma i})$  over  $K$  ( $E_{\gamma}$  is independent of the index  $i$  since  $\sum_i S_{\gamma i}$  is  $K(P, Q)$ -prime rational);  $E_{\gamma} \subset U \times X \times V$ . The maximal connected bunches of  $E_{\gamma}\{P', Q'\}$  are contained in maximally connected unions of the maximal connected bunches of the  $A_{\lambda} \cap B$  and the  $A_{\lambda} \cap B_{\mu}$ ; therefore—as explained above—the maximal connected bunches of  $E_{\gamma}\{P', Q'\}$  are either entirely inside or entirely outside  $\{\mathcal{C}\}$ . Apply [II], Th. 3 to every variety  $E_{\gamma}$  and to those maximal connected bunches of  $E_{\gamma}\{P', Q'\}$  which are in  $\{\mathcal{C}\}$ ; since  $(P', Q')$  is simple on  $U \times X$ , it follows that the number of points which are in  $\{\mathcal{C}\}$  of a specialization of the cycle  $\sum_i S_i$  over the specialization  $(P, Q) \rightarrow (P', Q')$ , with reference to  $K$ , is always the same, independent of the specialization ( $\sum_i S_i = A^* \cdot B^*$ ). Consider first the specialization

$$(P, Q, \sum_i S_i) \rightarrow (P', Q, A^* \cdot [B + \sum_{\mu} \pi_{\mu} B_{\mu}]) \\ \rightarrow (P', Q', \sum_i S'_i = \sum_i R'_i + \sum_{\mu, i} \pi_{\mu} R'_{\mu i})$$

over  $K$ ; let  $n'$ ,  $\sigma$  and  $\sigma_{\mu}$  be, respectively, the number of points of  $\sum_i S'_i$ , of  $\sum_i R'_i$  and of  $\sum_i R'_{\mu i}$  which are in  $\{\mathcal{C}\}$ . Next, consider the specialization  $(P, Q, \sum_i S_i) \rightarrow (P, Q', [\sum_{\lambda} \eta_{\lambda} A_{\lambda}] \cdot B^*) \rightarrow (P', Q', \sum_i S''_i = \sum_{\lambda, i} \eta_{\lambda} T'_{\lambda i})$  over  $K$  (with obvious notations); let  $n''$  and  $\tau_{\lambda}$  be the number of points of  $\sum_i S''_i$  and  $\sum_i T'_{\lambda i}$  respectively, which are in  $\{\mathcal{C}\}$ . We have seen  $n' = n''$ ; hence

$$(1) \quad \sigma + \sum_{\mu} \pi_{\mu} \sigma_{\mu} = n' = n'' = \sum_{\lambda} \eta_{\lambda} \tau_{\lambda}.$$

Since—as we have seen above—every  $\mathcal{D}_{\lambda \mu j}$  either is strictly contained componentwise in a certain  $\mathcal{C}_{\lambda i}$  or is a proper component of intersection, we can apply, by our induction assumption, the theorem to every variety  $B_{\mu}$ , to the bunches  $\mathcal{D}_{\lambda \mu j}$  ( $\mu$  fixed), and to the cycle  $A^* \cdot B_{\mu}$  and its specialization  $\sum_i R'_{\mu i}$  over the specialization  $Q \rightarrow Q'$  with reference to  $K$ . This gives

$$(2) \quad \sigma_{\mu} = \sum_{\lambda} \eta_{\lambda} \sum_j i(A_{\lambda} \cap B_{\mu}, \mathcal{D}_{\lambda \mu j}; V).$$

(If the integer  $d$ , introduced in the beginning, is zero, then there are no

bunches  $\mathcal{D}_{\lambda\mu j}$ , as follows from Prop. 1iib. applied to the system  $\beta$ ). Next we use the definitions of the symbols  $i(B \cap A_\lambda, \mathcal{E}_\lambda; V)$ . A summation over all the bunches  $\mathcal{E}_\lambda$  ( $\lambda$  fixed) gives

$$(3) \quad \tau_\lambda = \sum_i i(B \cap A_\lambda, \mathcal{E}_\lambda; V) + \sum_\mu \pi_\mu \sum_j i(B_\mu \cap A_\lambda, \mathcal{D}_{\lambda\mu j}; V).$$

By (1), (2), (3) and the commutativity of the  $i$ -symbol, the required formula follows immediately.

**THEOREM 4 (Projection theorem).** *Let  $B^b$  be a subvariety of  $U^n \times V^m$ , where  $U$  and  $V$  are projective varieties. Let  $B_1$  be the projection of  $B$  on  $U$  and let  $\dim B = \dim B_1$ . Let  $A^a$  ( $a + b = n$ ) be a subvariety of  $U$ . Let  $\mathcal{E}$  be a maximal connected bunch of  $A \cap B_1$ , and let  $\mathcal{E}_\gamma$  be those maximal connected bunches of  $(A \times V) \cap B$  which project (as point-sets) into  $\{\mathcal{E}\}$ . Assume that all the points of  $\{\mathcal{E}\}$  and all the  $\{\mathcal{E}_\gamma\}$  are simple on  $U$ , respectively  $U \times V$ . Then*

$$\sum_\gamma i[(A \times V) \cap B, \mathcal{E}_\gamma; U \times V] = [B : B_1] i(A \cap B_1, \mathcal{E}; U).$$

*Proof.* As to the dimensions, all the symbols are defined. Furthermore, we note that the projections on  $U$  of the maximal connected bunches of  $(A \times V) \cap B$  are either entirely contained in  $\{\mathcal{E}\}$  or do not have any point in common with  $\{\mathcal{E}\}$ .

We proceed by induction on the maximum of the excess of  $\mathcal{E}$  and the excess of the  $\mathcal{E}_\gamma$ . If this is zero, then the theorem is true ([VI], Chap. VII, Th. 8).

Let  $k$  be an algebraically closed field of definition for  $U, V, A$  and  $B$ . Let  $\alpha = [X, Y, K, Q'; A, U, k]$  be a system having the properties mentioned in Prop. 1,  $Y(Q') = A + \sum_\lambda \eta_\lambda A_\lambda$ ; let  $\mathcal{D}_\lambda$  be those maximal connected bunches of  $A_\lambda \cap B_1$  which are in  $\{\mathcal{E}\}$ . We have by Prop. 1iib.,  $\mathcal{D}_\lambda < \mathcal{E}$ . For  $Q$  generic on  $X$  over  $K$ , we have that  $Y(Q) = A^*$ , with  $A^* \cdot B_1 = \sum_i R_i$ , is  $K(Q)$ -prime rational. Let  $\sum_i R'_i$  be a specialization of  $\sum_i R_i$  over the specialization  $Q \rightarrow Q'$ , with reference to  $K$ ; let  $\sigma$  be the number of points of  $\sum_i R'_i$  which are in  $\{\mathcal{E}\}$ . Then we have by the definition of the  $i$ -symbol.

$$(1) \quad i(A \cap B, \mathcal{E}; U) = \sigma - \sum_\lambda \eta_\lambda \sum_i i(A_\lambda \cap B_1, \mathcal{D}_\lambda; U).$$

Next, consider the variety  $X$  and the subvariety  $Y \times V$  of  $X \times U \times V$ . We will show that they have, with respect to the variety  $B$ , the variety  $U \times V$  and every one of the bunches  $\mathcal{E}_\gamma$ , the properties mentioned in Th. 3. Write  $Z = Y \times V$  and let again  $Q$  be generic on  $X$  over  $K$ . Then  $Z(Q) = Y(Q) \times V$  by [VI], Chap. VI, Th. 7.  $Z(Q)$  is of the required dimension  $n + m - b$ . Since  $Y(Q) \cdot B_1 = \sum_i R_i$ , with (by Prop. 1i.)  $R_i$  generic on  $B_1$  over  $K$ , it follows (since  $[B : B_1] \neq 0$ ) that  $[Y(Q) \times V] \cap B$

consists of generic points on  $B$ . Hence we can write  $Z(Q) \cdot B = \sum_i \sum_j T_{ij}$ ,  $j=1, \dots, [B:B_1]$ ,  $\text{pr}_U T_{ij} = R_i$ . Again by [VI], Chap. VI, Th. 7, we have  $Z(Q') = Y(Q') \times V = A \times V + \sum_\lambda \eta_\lambda (A_\lambda \times V)$ . Let  $\mathcal{D}_{\lambda i}$  be those maximal connected bunches of  $(A_\lambda \times V) \cap B$  which project into  $\{\mathcal{D}_{\lambda i}\}$  (as point sets); then (in view of the fact that  $\mathcal{D}_{\lambda i} \prec \mathcal{B}$  and by applying Prop. 1iib. to the projection on  $U$  of every component of the bunches  $\mathcal{B}_\gamma$ ) it follows that there is to every  $\mathcal{D}_{\lambda i}$  a certain  $\mathcal{B}_\gamma$  such that  $\mathcal{D}_{\lambda i} \prec \mathcal{B}_\gamma$ . Moreover, the bunches  $\mathcal{D}_{\lambda i}$  contain all the components of  $(A_\lambda \times V) \cap B$  which have points in common with the  $\{\mathcal{B}_\gamma\}$  for some  $\gamma$  since the  $\mathcal{D}_{\lambda i}$  contain all the components of  $A_\lambda \cap B_1$  which have points in common with  $\{\mathcal{B}\}$ .

Let  $\sum_{i,j} T'_{ij}$  be a specialization of  $\sum_{i,j} T_{ij}$  over the specialization  $(Q, \sum_i R_i) \rightarrow (Q', \sum_i R'_i)$  with reference to  $K$ . Then we have  $\text{pr}_U T'_{ij} = R'_i$ ; hence  $T'_{ij} \in \{\mathcal{B}_\gamma\}$  for some  $\gamma$  if and only if  $R'_i \in \{\mathcal{B}\}$ . Therefore, the number of points of  $\sum_{i,j} T'_{ij}$  which are in  $\{\mathcal{B}_\gamma\}$  for some  $\gamma$  is equal to  $[B:B_1]\sigma$ . An application of Th. 3 to the subvariety  $B$  of  $U \times V$ , to every one of the bunches  $\mathcal{B}_\gamma$ , to the varieties  $X$  and  $Y \times V$  on  $X \times U \times V$  and to the specialization  $(Q, \sum_{i,j} T_{ij}) \rightarrow (Q', \sum_{i,j} T'_{ij})$  over  $K$  gives

$$(2) \quad \sigma[B:B_1] = \sum_\gamma i[(A \times V) \cap B, \mathcal{B}_\gamma; U \times V] \\ + \sum_\lambda \eta_\lambda \sum_{i,h} i[(A_\lambda \times V) \cap B, \mathcal{D}_{\lambda i h}; U \times V].$$

Since, as we have seen above,  $\mathcal{D}_{\lambda i} \prec \mathcal{B}$  and every  $\mathcal{D}_{\lambda i h} \prec \mathcal{B}_\gamma$  for some  $\gamma$ , we can apply, by our induction hypothesis, the theorem to  $B$ , to  $A_\lambda \times V$ , and to the bunches  $\mathcal{D}_{\lambda i}$  and  $\mathcal{D}_{\lambda i h}$ . Hence

$$(3) \quad [B:B_1] i(A_\lambda \cap B_1, \mathcal{D}_{\lambda i}; U) = \sum_h i[(A_\lambda \times V) \cap B, \mathcal{D}_{\lambda i h}; U \times V]$$

Summation over all  $\lambda, i$  and substitution into (2) gives

$$[B:B_1] \cdot [\sigma - \sum_\lambda \eta_\lambda \sum_i i(A_\lambda \cap B_1, \mathcal{D}_{\lambda i}; V)] = \sum_\gamma i[(A \times V) \cap B, \mathcal{B}_\gamma; U \times V].$$

From this, one obtains, by using (1), the required formula.

**THEOREM 5 (Associativity formula).** *Let  $A^a, B^b$  and  $C^c$  be subvarieties of a projective variety  $V^n$  ( $a+b+c=2n$ ); let  $\mathcal{B}$  be a maximal connected bunch of  $A \cap B \cap C$  such that every point of  $\{\mathcal{B}\}$  is simple on  $V$ . Suppose that all the components  $D_\sigma$  of  $A \cap B$  which have points in common with  $\{\mathcal{B}\}$  are proper and, similarly, let all the components  $E_\tau$  of  $B \cap C$  which have points in common with  $\{\mathcal{B}\}$  be proper. Let  $\mathcal{D}_{\sigma j}$ , respectively  $\mathcal{E}_{\tau j}$ , be those maximal connected bunches of  $D_\sigma \cap C$ , respectively  $A \cap E_\tau$ , which have points in common with  $\{\mathcal{B}\}$ . Then*

$$\sum_\sigma i(A \cdot B, D_\sigma; V) \sum_j i(D_\sigma \cap C, \mathcal{D}_{\sigma j}; V) \\ = \sum_\tau i(B \cdot C, E_\tau; V) \sum_j i(A \cap E_\tau, \mathcal{E}_{\tau j}; V).$$

*Proof.* First of all we observe that the maximal connected bunches of  $D_\sigma \cap C$  and  $A \cap E_\tau$  are either entirely inside or entirely outside  $\{\mathcal{S}\}$ . We proceed by induction on the excess of  $\mathcal{S}$  (i.e., the maximum of the integers  $\dim G$  for all components  $G$  of  $\mathcal{S}$ ). If the excess  $\mathcal{S} = 0$  then the theorem is a special case of [VI], Chap. VI, Th. 5.

Let  $k$  be a field of definition for  $A, B, C$ , and  $V$ . Let

$$\alpha = [U, W, K_1, P'; A, V, k] \text{ and } \gamma = [X, Y, K_2, Q'; C, V, k]$$

be systems having the properties mentioned in Proposition 1; moreover, we assume that  $K_1$  and  $K_2$  are free with respect to each other over  $k$ . Let  $K$  be the algebraic closure of the compositum of  $K_1$  and  $K_2$ .

We have, by assumption,

$$W(P') = A + \sum_{\lambda=1} \eta_\lambda A_\lambda \text{ and } Y(Q') = C + \sum_{\mu=1} \pi_\mu C_\mu;$$

for convenience of notation, we will write  $W(P') = \sum_\lambda \eta_\lambda A_\lambda$ , where it is understood (during the proof of this theorem) that without further references, the summation starts with  $\lambda = 0$ , where  $\eta_0 = 1$  and  $A_0 = A$ . Similarly, we write  $Y(Q') = \sum_\mu \pi_\mu C_\mu$  with  $\pi_0 = 1$  and  $C_0 = C$ . Let  $D_{\lambda\sigma}$  and  $E_{\mu\tau}$  be those components of  $A_\lambda \cap B$ , respectively  $B \cap C_\mu$ , which have points in common with  $\{\mathcal{S}\}$ ; again we denote  $D_\sigma$  by  $D_{0\sigma}$  and  $E_\tau$  by  $E_{0\tau}$ . Since  $D_\sigma$  and  $E_\tau$  are proper components of  $A \cap B$ , respectively  $B \cap C$ , it follows by the properties of the systems  $\alpha$  and  $\gamma$  (see Prop. 1ii.) that all  $D_{\lambda\sigma}$  and  $E_{\mu\tau}$  are proper components of intersection of  $A_\lambda \cap B$ , respectively  $B \cap C_\mu$  on  $V$ . Furthermore, it follows also by the properties of the systems  $\alpha$  and  $\gamma$  that (see Prop. 1ii.) the maximal connected bunches of  $A_\lambda \cap B \cap C$ , of  $A \cap B \cap C_\mu$  and of the  $A_\lambda \cap B \cap C_\mu$  are either strictly contained in  $\{\mathcal{S}\}$ , or do not have any point in common with  $\{\mathcal{S}\}$ . Let  $\mathcal{D}_{\lambda\sigma\mu j}$ , respectively  $\mathcal{E}_{\mu\tau\lambda j}$ , be those maximal connected bunches of  $D_{\lambda\sigma} \cap C_\mu$ , respectively  $A_\lambda \cap E_{\mu\tau}$ , which have points in common with  $\{\mathcal{S}\}$ . (Similarly to the above, we choose notations such that  $\mathcal{D}_{0\sigma 0j} = \mathcal{D}_{\sigma j}$  and  $\mathcal{E}_{0\tau 0j} = \mathcal{E}_{\tau j}$ .)

Let  $P$  and  $Q$  be independent generic points on  $U$  and  $X$  over  $K$ . Write  $W(P) = A^*$  and  $Y(Q) = C^*$ ; by assumption on the systems  $\alpha$  and  $\gamma$  (and since  $K_1$  and  $K_2$  are free with respect to each other over  $k$ ), we have that  $A^* \cdot B \cdot C^* = \sum_i S_i$  is a  $K(P, Q)$ -rational cycle. Let  $H$  be the locus of  $(P, Q, S_i)$  over  $K$ ;  $H \subset U \times X \times V$ . Since  $S_i \in A^* \cdot B \cdot C^*$  it follows from the fact that  $A^* \rightarrow \sum_\lambda \eta_\lambda A_\lambda$  and  $C^* \rightarrow \sum_\mu \pi_\mu C_\mu$  over the specialization  $(P, Q) \rightarrow (P', Q')$  with reference to  $K$ , by [III], page 53g, that the maximal connected bunches of  $H\{P', Q'\}$  are contained in the maximal connected bunches of  $A_\lambda \cap B \cap C_\mu$  ( $\lambda = 0$  and  $\mu = 0$  included). By our observations, it follows

that those bunches are either entirely inside or entirely outside  $\{\mathcal{S}\}$ . Apply [II], Th. 3 to the variety  $H$  on  $U \times X \times V$ , to the simple point  $(P', Q')$  on  $U \times X$  and to those maximal connected bunches of  $H\{P', Q'\}$  which are in  $\{\mathcal{S}\}$ ; it then follows that the number of points of every specialized cycle of  $\sum_i S_i$ , over the specialization  $(P, Q) \rightarrow (P', Q')$  with reference to  $K$ , which are in  $\{\mathcal{S}\}$  is always the same. Consider the specialization  $(P, Q, \sum_i S_i) \rightarrow (P, Q', \dots) \rightarrow (P', Q', \sum_i S'_i)$  over  $K$ ; let  $n'$  be the number of points of  $\sum_i S'_i$  which are in  $\{\mathcal{S}\}$ . Let  $n''$  be the corresponding number for the cycle  $\sum_i S''_i$ , where  $(P, Q, \sum_i S'_i) \rightarrow (P', Q, \dots) \rightarrow (P', Q', \sum_i S''_i)$  over  $K$ . As we have seen,

$$(1) \quad n' = n''$$

Next we note that all the  $A^* \cdot E_{\mu\tau}$  are defined, as follows from the assumption on the system  $\alpha$  and from the fact that  $K_2$  (over which the  $E_{\mu\tau}$  are defined) and  $K_1$  are free with respect to each other over  $k$ . (Furthermore, we keep in mind that the components of the  $B \cap C_\mu$ , except the  $E_{\mu\tau}$ , are entirely outside  $\{\mathcal{S}\}$ .) Apply Th. 3 to every variety  $E_{\mu\tau}$ , to  $U$  and  $W$  on  $U \times V$  and to those maximal connected bunches of  $A_\lambda \cap E_{\mu\tau}$  which are in  $\{\mathcal{S}\}$ . We obtain after a summation

$$(2) \quad n' = \sum_{\mu, \tau} \pi_\mu \mathbf{i}(C_\mu \cdot B, E_{\mu\tau}; V) \cdot \sum_{\lambda, h} \eta_\lambda \mathbf{i}(A_\lambda \cap E_{\mu\tau}, \mathcal{E}_{\mu\tau\lambda h}; V).$$

Similarly, one has

$$(3) \quad n'' = \sum_{\lambda, \sigma} \eta_\lambda \mathbf{i}(A_\lambda \cdot B, D_{\lambda\sigma}; V) \cdot \sum_{\mu, j} \pi_\mu \mathbf{i}(D_{\lambda\sigma} \cap C_\mu, \mathcal{D}_{\lambda\sigma\mu j}; V)$$

Suppose we take a pair  $\lambda, \mu$  but not the pair  $\lambda=0$  and  $\mu=0$  (at the same time). As we have already observed above, those maximal connected bunches of  $A_\lambda \cap B \cap C_\mu$  which have points in common with  $\{\mathcal{S}\}$  are strictly contained componentwise in  $\{\mathcal{S}\}$ . Therefore, by our induction hypothesis, we can apply the theorem to every such bunch. After a summation (over all the maximal connected bunches of  $A_\lambda \cap B \cap C_\mu$  which are in  $\{\mathcal{S}\}$ ,  $\lambda$  and  $\mu$  fixed), we obtain

$$(4) \quad \sum_\sigma \mathbf{i}(A_\lambda \cdot B, D_{\lambda\sigma}; V) \cdot \sum_j \mathbf{i}(D_{\lambda\sigma} \cap C_\mu, \mathcal{D}_{\lambda\sigma\mu j}; V) \\ = \sum_\tau \mathbf{i}(B \cdot C_\mu, E_{\mu\tau}; V) \cdot \sum_h \mathbf{i}(A_\lambda \cap E_{\mu\tau}, \mathcal{E}_{\mu\tau\lambda h}; V).$$

(4) holds for all pairs  $(\lambda, \mu)$ , except for the pair  $(0, 0)$ . However then, from (1), (2), (3) and (4), and since  $\eta_0=1$  and  $\pi_0=1$ , it follows that (4) is also true for  $\lambda=0$  and  $\mu=0$ . Since  $A_0=A$ ,  $C_0=C$ ,  $D_{0\sigma}=D_\sigma$ ,  $E_{0\tau}=E_\tau$ ,  $\mathcal{D}_{0\sigma 0j}=\mathcal{D}_{\sigma j}$  and  $\mathcal{E}_{0\tau 0h}=\mathcal{E}_{\tau h}$ , this is the required result.

From the above theorems, and from the theorem corresponding to [VI],

Chap. VI, Th. 7 which can be proved by the above developed methods, many of the usual properties of the  $i$ -symbol can be derived, for instance, the analogous result to [VI], Chap. VI, Th. 9 (see the proof of that theorem in [VI]).

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## EXTENSION OF REPRESENTATIONS OF LIE GROUPS, II.\*

By G. D. Mostow.<sup>1</sup>

**1. Introduction.** This paper is a sequel to "Extension of Representations of Lie Groups and Lie Algebras, I" by G. Hochschild and G. D. Mostow (hereafter referred to as ERI). In ERI, some simple but basic constructions of representation spaces are introduced, and thereby one can analyze almost completely the problem of extending a (finite dimensional) representation of a normal analytic subgroup  $G$  of an analytic group  $L$  to a representations of  $L$  in the case that there is an analytic subgroup  $Q$  with  $L = QG$  and  $Q \cap G$  compact. Indeed, from the extension theorem in this special situation, one deduces quickly all the standard results on faithful representations of Lie groups (see ERI).

This paper is devoted to the more general extension problem of extending a representation from a normal closed connected subgroup  $G$  to an analytic group  $L$  with  $L \supset G$ . The analogue of this problem for Lie algebras presents no difficulties at all, once one has resolved the special case of extending from an ideal  $G$  to a semi-direct sum  $Q + G$ , because one can reconstruct a Lie algebra  $L$  from an ideal  $G$  by successive formation of semi-direct sums. By contrast, the extension methods of ERI do not suffice to settle the more general extension problem—essentially for topological reasons. (See Sec. 2, Remark 2.8). In addition to the methods of ERI, we require one additional extension method in order to be able to overcome the topological complications that arise. The method presented here relies heavily on the theory of algebraic groups and, in particular, on the author's previous results on fully reducible subgroups of algebraic groups. The main theorem of this paper (see Sec. 3) provides a complete solution to the extension problem for closed normal connected subgroups of analytic groups.

It is a pleasure to acknowledge my debt to Dr. Gerhard Hochschild with whom I have had many stimulating conversations on topics treated here.

**2. Preliminaries.** Let  $G$  be a Lie group. We denote by  $G_1$  the connected component of the identity element. We denote by  $G^*$  the Lie algebra

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of  $G$  and identify  $G^\cdot$  with the tangent space to  $G$  at the identity element. If  $\rho$  is a homomorphism of  $G$ , we denote by  $\rho^\cdot$  the differential of  $\rho$  at the identity element so that  $\rho^\cdot$  is the induced homomorphism of  $G^\cdot$ . We denote by  $\text{Aut } G$  and  $\text{Aut } G^\cdot$  the group of automorphisms of  $G$  and  $G^\cdot$  respectively. The map  $x \rightarrow x^\cdot$  of  $\text{Aut } G$  into  $\text{Aut } G^\cdot$  is one-to-one, and it is a homeomorphism when  $\text{Aut } G$  is given the compact open topology and  $\text{Aut } G^\cdot$  is given its topology relative to the full linear group or, equivalently, its Lie group topology [4]. If  $G$  is an analytic *real linear* group, that is, an analytic subgroup of the full linear group  $GL(V)$  of some real linear space  $V$ , then we shall have to distinguish between its analytic group topology and its topology relative to the full linear group  $GL(V)$ . Accordingly, if  $F \subset G$ , we denote by  $F^G$  the topological closure of  $F$  in the analytic group topology of  $G$  and by  $\bar{F}$  the closure in the full linear group. It is known that  $F^G \subset \bar{F}$  if  $F \subset G$ , so that  $(\bar{C} \cap G)^G = \bar{C} \cap G$  for any subset  $C$  of the full linear group. If  $G$  is a Lie subgroup of the full linear group  $GL(V)$ , we can identify the Lie algebra  $G^\cdot$  with a subset of the linear space  $E(V)$  of all endomorphisms on  $V$ , by the usual rule for identifying the tangent space to a submanifold of a linear space  $E$  with a subspace of  $E$ . When  $G^\cdot$  is thus identified with a subspace of  $E(V)$ , the adjoint operation  $\text{Ad } x$ , for  $x \in G$ , coincides with the restriction of the conjugation  $C(x): y \rightarrow xyx^{-1}$  ( $y \in E(V)$ ) to  $G^\cdot$ , because  $(C(x))^\cdot = C(x)$ .

We use the term "representation" to mean "finite dimensional representation." By an *algebraic group* we mean here a group  $G$  of invertible endomorphisms of a finite dimensional real or complex linear space  $V$ , such that  $G$  is the intersection of the full linear group with an algebraic manifold in the linear space  $E(V)$  of all endomorphisms of  $V$ . An *algebraic Lie algebra* is the Lie algebra of an algebraic group. By a *rational representation* of a subgroup  $G \subset GL(V)$  we mean a (finite dimensional) representation of  $G$  which is the restriction to  $G$  of a representation  $\rho$  of an algebraic group  $L$  such that  $L \supset G$  and  $\rho$  is a rational map. If  $G \subset GL(V)$ , we denote by  $G^*$  the topologically connected component of the identity of the smallest algebraic group containing  $G$ . In general,  $G^*$  is not an algebraic group. If  $\rho$  is a rational representation of an algebraic group  $L$ , then  $\rho(L)$  need not be algebraic. However  $\rho^\cdot(L^\cdot)$  is an algebraic Lie algebra ([2], p. 140). It follows immediately that  $\rho(G^*) = \rho(G)^*$  for any rational representation of  $G$ .

If  $t$  is an endomorphism of any finite dimensional linear space  $V$  over a perfect field, there is a unique semi-simple (i.e. fully reducible) element  $s$  and a unique nilpotent element  $n$  such that  $t = s + n$  and  $sn = ns$ . If  $t$  is

invertible, then  $s$  is also invertible and  $t = su$  where  $u$  is the *unipotent* (i.e.,  $u - 1$  is nilpotent) element  $1 + s^{-1}n$ . The representation  $t = su$  with  $s$  semi-simple,  $u$  unipotent, and  $su = us$  is unique; it is called the *Jordan product decomposition* (see [2], p. 71, Th. 7). The endomorphism  $s$  is called the *semi-simple part* of  $t$  and is denoted by  $t'$ . An algebraic group which contains  $t$  contains  $t'$  also ([2], p. 184). A rational representation sends semi-simple endomorphisms into semi-simple endomorphisms, unipotent endomorphisms into unipotent endomorphisms ([8]), and therefore  $\rho(t') = \rho(t)'$ .

A set  $U$  of endomorphisms of a linear space  $V$  is called a *unipotent family* if there is a series of subspaces  $V_0 = V_1 \subset \cdots \subset V_n = V$ , each invariant under  $U$ , such that the  $V_i/V_{i-1}$  part of  $U$  consists of the identity element,  $i = 1, \dots, n$ . A representation  $\rho$  of a group  $G$  is called *unipotent* if  $\rho(G)$  is a unipotent family. A group of unipotent endomorphisms of a linear space is a unipotent family ([5], Sec. 2).

*Remark 2.1.* If  $M$  and  $N$  are algebraic Lie algebras with  $[M, N] \subset N$ , then  $M + N$  is algebraic ([8], Sec. 4.4, p. 209). From this it follows that if  $M$  and  $N$  are analytic linear groups with  $N$  invariant under conjugations from  $M$  and  $N = N^*$ , then  $(MN)^* = M^*N$ .

*Remark 2.2.* A Lie algebra of nilpotent endomorphisms is algebraic ([2], pp. 181-183). An algebraic group of unipotent elements is algebraically-connected (or "algebraically-irreducible"; i.e., the associated ideal is prime) ([2], p. 183) and is, in fact, topologically connected since each element in it is the exponential of an element from the Lie algebra. In particular, an analytic group of unipotent elements is algebraic and a fortiori closed.

*Remark 2.3.* Let  $G$  be an algebraic Lie group, let  $M$  be any maximal fully reducible subgroup of  $G$  and let  $U$  be the maximum normal algebraically-connected subgroup of unipotent elements in  $G$ . Then  $G = M \cdot U$  (semi-direct) ([8], Th. 7.1). If  $G$  is algebraically-connected, then any fully reducible subgroup of  $G$  is conjugate to a subgroup of  $M$  by an inner automorphism from  $(U \cap G')_1$  ([8], Th. 7.1, Th. 4.1), where  $G'$  denotes the commutator subgroup of  $G$ . By Remark 2.2,  $U^* = U_1 = U$  and therefore  $G_1 = M_1 \cdot U$  (semi-direct). It follows from the foregoing that any fully reducible subgroup of  $G_1$  is conjugate to a subgroup of  $M_1$  by an inner automorphism from  $(U \cap G')_1$ .

If  $P$  is a fully reducible subgroup of  $G_1$  and  $U$  is a normal analytic group of unipotent elements with  $G_1 = PU$ , then  $P$  is maximal fully reducible in  $G_1$ . For let  $Q$  be a fully reducible subgroup of  $G$  containing  $P$ . Then

$Q = P(Q \cap U)$ . But  $Q \cap U$  is normal in  $Q$  and therefore fully reducible. Simultaneously, it is a unipotent family. Hence it consists of the identity alone and  $P = Q$ . A similar argument shows that  $U$  is the maximum analytic normal subgroup of unipotent elements.

A rational representation carries fully reducible groups into fully reducible groups ([8]).

*Remark 2.4.* If  $S$  is an analytic semi-simple linear group, then  $S^* = S$ . For let  $S^\cdot$  denote the Lie algebra of  $S$ . The commutator subalgebra  $[S^\cdot, S^\cdot]$  is an algebraic Lie algebra ([2], p. 177). Hence  $S^\cdot = [S^\cdot, S^\cdot]$  is algebraic and therefore  $S$  is the connected component of the identity of the algebraic group hull of  $S$ . Consequently  $S^* = S$ . In particular,  $S$  is closed in the full linear group. The center of  $S$  is finite, as can be seen in a variety of ways.

*Remark 2.5.* Let  $L$  be an analytic group and let  $P$  denote the intersection of the kernels of all finite dimensional representations of  $L$ . Then  $L/P$  has faithful representation, by a result of Goto (see [3] or [5]).

*Remark 2.6.* Let  $L$  be an analytic group. As is well-known,  $L$  is topologically the direct product of a maximal compact subgroup and a euclidean space; moreover, any compact subgroup is conjugate to a subgroup of any maximal one. If  $G$  is a closed normal analytic subgroup, and  $L/G$  is compact, then  $L = QG$  for any maximal compact subgroup  $Q$ . This is a result of Malcev ([7], Th. 10, p. 180), but since we require it only in the special case that  $L/G$  is solvable, we present the following brief proof for the reader's convenience. The group  $L/G$ , being a compact solvable analytic group, is abelian, as is seen from considering its adjoint representation. Therefore  $QG$  is normal, and  $L/QG = (L/G)/(QG/G)$  is again a toroid. But  $L/QG = (L/Q)(QG/Q)$  is the image of the euclidean space  $L/Q$  by a continuous and open mapping with the inverse image of each point homeomorphic to the connected set  $QG/Q$ . Therefore  $L/QG$  is a simply connected toroid. Hence it is a point and  $L = QG$ . A part of our argument establishes that  $L/F$  is simply connected if  $F$  is a closed analytic subgroup containing a maximal compact subgroup of  $L$ .

*Remark 2.7.* Let  $\rho$  be a representation of an analytic group  $G$  on a real or complex linear space  $V$ . Let  $V_0 \subset V_1 \subset \cdots \subset V_n = V$  be a sequence of linear subspaces of  $V$  invariant under  $\rho(G)$  such that the  $V_i/V_{i-1}$  part of  $\rho$  is irreducible,  $i=1, \cdots, n$ . Let  $\rho'$  denote the representation induced by  $\rho$  on the direct sum  $V_1/V_0 + \cdots + V_n/V_{n-1}$ . Then  $\rho'$  is semi-simple

(i.e. fully reducible) and it is independent of the original choice of the subspaces  $\{V_i\}$  up to an isomorphism ([5] or [6]);  $\rho'$  is called the *associated semi-simple representation* of  $\rho$ . We consistently denote the commutator subgroup of  $L$  by  $L'$  or by  $[L, L]$ . By the *radical* of an analytic group  $L$ , we consistently mean the analytic subgroup  $T$  whose Lie algebra  $T'$  is the radical of the Lie algebra  $G'$ .

Let  $\rho$  be a representation of an analytic group  $G$  on a real or complex linear space  $V$ , and let  $G$  be a subgroup of an analytic group  $L$ . Let  $\sigma$  be a representation of  $L$  on a real or complex linear space  $W$ .  $\sigma$  is called an *extension* of  $\rho$  if there is a monomorphism  $\theta$  of  $V$  into  $W$  such that  $\theta \cdot \rho(g) = \sigma(g) \cdot \theta$  for all  $g \in G$ .

The main extension results in ERI can be stated as follows: *Let  $G$  be a closed normal analytic subgroup of an analytic group  $L$ . Let  $Q$  be an analytic subgroup of  $L$  such that  $Q \cap G$  is compact. Assume that there is a representation of  $Q$  which is faithful on  $Q \cap G$ . Let  $\rho$  be a representation of  $G$ . A necessary and sufficient condition that  $\rho$  can be extended to a representation of the analytic group  $QG$  is the following Condition a),*

Condition a).  $\rho'(xy^{-1}x^{-1}) = 1$  for all  $y$  in  $QG$  and  $x$  in the radical of  $G$ .

Moreover, the extended representation  $\sigma$  can be chosen in such a way that the kernel of  $\sigma'$  contains the kernel of  $\rho'$  (ERI Theorem 3.2). If  $Q$  satisfies the additional Condition b),

Condition b). (i)  $\rho'(qqq^{-1}g^{-1}) = 1$  for all  $q \in Q$  and  $g \in G$ ; and (ii) the  $G$ -part of  $\text{Ad } q$  is unipotent for all  $q \in Q$ , then  $\sigma$  can be so chosen that the kernel of  $\sigma'$  contains  $Q$  as well as the kernel of  $\rho'$ . (ERI, Section 6)

Condition a) is clearly implied by

Condition a').  $\rho'$  is trivial on  $N \cap G$ , where  $N$  is the radical of the commutator subgroup of  $L$ . Conversely, if  $\rho$  can be extended to a representation  $\sigma$  of  $L$ , then  $\rho'$  is trivial on  $N \cap G$ . For  $\sigma$  is unipotent on  $N$ , as is well known, so that  $\sigma'$  is trivial on the normal subgroup  $N$  (ERI, Sec. 2), and consequently  $\rho'$  is trivial on  $N \cap G$ .

If  $Q \subset N$ , and  $\rho$  satisfies Condition a'), then  $Q$  satisfies Condition b). We can now assert the following remark.

**Remark 2.8.** *Let  $L, G, N, Q$ , be as in Remark 2.7. Assume that  $\rho'$  is trivial on  $N \cap G$ . Then  $\rho$  can be extended to a representation  $\sigma$  of  $QG$*

such that the kernel of  $\sigma'$  contains the kernel of  $\rho'$  and also the subgroup  $Q$  if  $Q \subset N$ .

We note that if  $Q \subset N$ , then  $\sigma'$  is trivial on  $Q(N \cap G) = N \cap QG$ . This raises the possibility of extending  $\rho$  from  $G$  to  $L$  in two stages: from  $G$  to  $NG$ , by recursively selecting  $Q \subset N$ ; and then from  $NG$  to  $L$ . The obstruction to this procedure occurs in two places. In the first place  $NG$  need not be closed; the passage from  $NG$  to  $L$  suggests that an additional condition must be imposed on  $\rho$ . In the second place, even assuming that  $NG$  is closed, it may be possible to select a non trivial  $Q \subset N$  so that  $Q \cap G$  is compact. The essential contribution of this paper is to show how to extend  $\rho$  from  $G$  to  $\overline{GN}$ .

### 3. A determination of $\rho'$ .

**LEMMA 3.1.** *Let  $G$  be an analytic group. For  $x \in G$ , let  $\text{Ad } x$  denote the differential at the identity element of the inner automorphism  $g \rightarrow xgx^{-1}$ . Then every element in  $(\text{Ad } G)^*$  is the differential at the identity of an automorphism of  $G$  which keeps fixed each central element of  $G$ .*

*Proof.* Clearly, no generality is lost in assuming  $G$  to be simply connected. By Ado's theorem, there exists a faithful representation of the Lie algebra of  $G$ . Let  $\rho$  denote the corresponding representation of  $G$  and let  $K$  denote its kernel.

Let  $F$  denote the tangent space to  $\rho(G)$  at the identity element. We identify it (see Sec. 2) with a Lie subalgebra of the Lie algebra  $E$  of all endomorphisms of the underlying linear space  $V$ . The conjugations  $C(x)$ ,  $y \rightarrow xyx^{-1}$ , of  $E$  clearly keep  $F$  invariant when  $x \in \rho(G)$ , and consequently  $F$  is invariant under the conjugations from the algebraic group hull of  $\rho(G)$ . Define  $\beta(x)$  as the restriction of  $C(x)$  to  $F$  on the group  $A$  of all automorphisms of  $V$  with  $C(x)F = F$ . Clearly,  $A$  is an algebraic group,  $\beta$  is a rational representation and  $\rho(G) \subset A$ . Therefore  $\rho(G)^* \subset A$  and  $\beta(\rho(G)^*) = (\beta(\rho(G)))^*$ . It is easily verified that  $\beta(x)$  coincides with the differential at the identity element of the restriction of  $C(x)$  to  $\rho(G)$  and that  $C(x)\rho(G) = \rho(G)$  whenever  $C(x)F = F$ . Thus  $\beta(\rho(G)) = \text{Ad } \rho(G)$  and  $\beta(\rho(G)^*) = (\text{Ad } \rho(G))^*$ . But  $\rho(G)^*$  is contained in the centralizer of the center of  $\rho(G)$ . Thus each automorphism in  $(\text{Ad } \rho(G))^*$  is induced by an automorphism of  $\rho(G)$  keeping fixed each central element. Lifting these automorphisms of  $\rho(G)$  to a set of automorphisms,  $A$ , of the simply connected group  $G$ , we find that each element of  $(\text{Ad } G)^*$  is the differential at the identity element of an automorphism keeping invariant the discrete

set  $zK$  for each  $z$  in the center of  $G$ . Since the set  $A$  is a connected family of transformations of  $G$  and contains the identity, it leaves each element of  $zK$  fixed. Proof of Lemma 3.1 is now complete.

LEMMA 3.2. *Let  $G$  be an analytic linear group and  $N$  a normal subgroup closed in the analytic group topology of  $G$ . Then  $N$  is normal in  $G^*$ .*

*Proof.* The group  $N_1$  is connected and closed in  $G$  and is therefore analytic. Let  $G^*$  and  $N^*$  denote the Lie algebras of  $G$  and  $N$  respectively; we identify their elements with endomorphisms of the underlying linear space  $V$  in the usual way. For any automorphism  $x$  of  $V$ , let  $C(x)$  denote conjugation by  $x$ , and let  $A$  be the set of all automorphisms  $x$  of  $V$  such that  $C(x)N^* = N^*$ . Clearly  $A$  is an algebraic group. It is also clear that  $xN_1x^{-1} = N_1$  if and only if  $C(x)N^* = N^*$ . Accordingly,  $G \subset A$ . From this,  $G^* \subset A$ . Hence  $N_1$  is normal in  $G^*$ . By the same reasoning,  $G$  is normal in  $G^*$ . Let  $\beta(x)$  denote the  $G^*/N^*$  part of  $C(x)$  for  $x \in G^*$ . Then  $\beta$  is a rational representation of  $G^*$  and thus  $\beta(G^*) = \beta(G)^* = (\text{Ad } G/N_1)^*$ . Let  $\alpha(x)$  denote the automorphism of  $G/N_1$  that is induced by  $C(x)$  for  $x \in G^*$ . Then  $\alpha(G^*) = \beta(G^*) = (\text{Ad } G/N_1)^*$ . Applying Lemma 3.1, each element in  $\alpha(G)^*$  is the differential of an automorphism of  $G/N_1$  keeping each central element fixed. Since  $x \rightarrow x^*$  is a monomorphism of  $\text{Aut } G/N_1$  into  $\text{Aut}(G/N_1)^*$ , each automorphism in  $\alpha(G^*)$  keeps each central element of  $G/N_1$  fixed. Since  $N/N_1$  is a discrete normal subgroup in  $G/N_1$ , it is central. Hence  $\alpha(G^*)(N/N_1) = N/N_1$  and  $C(G^*) \cdot N = N$ . That is,  $N$  is normal in  $G^*$ .

*Note.* It is curious that Lemma 3.2 can be proved trivially when  $N$  is connected and requires so complicated a proof for  $N$  not connected.

LEMMA 3.3. *Let  $L$  be an analytic linear group, let  $G$  be a normal analytic subgroup closed in  $L$ , let  $T$  be the radical of  $G$ , let  $H$  be an analytic subgroup whose Lie algebra is a Cartan subalgebra of the Lie algebra of  $T$ , let  $N$  be the radical of the commutator subgroup  $L'$ , and let  $D$  be a maximal compact subgroup of  $(H^* \cap (TN)^L)_1$ . Then*

1.  $D$  is a maximal compact subgroup in  $DTN$ .
2.  $(TN)^L = DTN$ .
3.  $(GN)^L = DGN$ .
4.  $HN \cap D$  is dense in  $D$ .
5.  $DG$  contains a maximal compact subgroup of  $(GN)^L$ , and  $N \cap DG$  is connected.

*Proof.* It is known (see [9] Lemma 2, p. 7) that  $T = HT'$  and, by definition,  $T' \subset N$ . Hence  $TN = HN$  and  $(TN)^L \subset (TN)^* = (HN)^*$ . Now the Lie algebra of  $N$  consists of nilpotent endomorphisms, as is well known. Hence  $N^* = N$  by Remark 2.2 and  $(HN)^* = H^*N$  by Remark 2.1. Now  $H^* = M \cdot U$  where  $M$  is a maximal fully reducible subgroup of  $H^*$  containing  $D$  and  $U$  is the maximum analytic normal subgroup of unipotent elements. Hence  $(TN)^L \subset (M \cdot U)N = M(UN)$ , where the elements of  $UN$  are unipotent by Lie's theorem on simultaneous triangularizing. Therefore  $M$  is a maximal fully reducible subgroup in  $(TN)^L$ , by Remark 2.3. Let  $C$  be a connected compact subgroup of  $(TN)^L$  with  $C \supset D$ . By Remark 2.3, there is an element  $x \in ((TN)^* \cap N)_1 = ((TN)' \cap N)_1$  such that  $xCx^{-1} \subset M$ . Then, for every  $d \in D$ ,  $xdx^{-1} = m(d) \in M$  implies  $dx^{-1} = x^{-1}m = m(m^{-1}x^{-1}m)$ . Hence  $d = m(d) = xdx^{-1}$  and  $xDx^{-1} = D$ . Therefore  $D \subset xCx^{-1} \subset (M \cap (TN)^L)_1 \subset (H^* \cap (TN)^L)_1$ , and, by its maximality,  $D = xCx^{-1}$ . Hence  $D = C$ , and  $D$  is a maximal compact subgroup in  $(TN)^L$ , because a maximal compact subgroup of  $(TN)^L$  is connected.

*Proof of 2.* Let  $J$  denote the simply connected covering group of  $(TN)^L$ , let  $\pi$  denote the covering homomorphism, and let  $K = \pi^{-1}(D)$ . Then  $J/K = (TN)^L/D$ , and the latter is simply connected since an analytic group is topologically the direct product of any maximal compact group and a euclidean space. Consequently,  $K$  is connected. Hence  $\pi^{-1}(DTN)$ , being a connected normal analytic subgroup of a simply connected Lie group, is closed. It follows immediately that  $DTN$  is closed in  $L$  and equals  $(TN)^L$ .

*Proof of 3.* Let  $SR$  be a Levi decomposition for  $(GN)^L$ ,  $S$  being semi-simple and  $R$  the radical.  $S \cap R$  is a discrete normal subgroup of the analytic group  $S$  and is therefore central in  $S$ . Inasmuch as the center of a semi-simple analytic linear group is finite,  $S \cap R$  is finite. Let  $\lambda$  denote the epimorphism  $s \cdot r \rightarrow sr$  of the semi-direct product  $S \cdot R$  onto  $SR$ . The kernel of  $\lambda$  consists of all elements  $x \cdot x^{-1}$  with  $x \in S \cap R$  and is thus finite. Consequently, an analytic subgroup  $A$  which contains  $S$  is closed in  $SR$  if and only if the connected component  $(A \cap R)_1$  is closed in  $R$ .

The semi-simple analytic group  $S$  is contained in  $DGN$  since

$$S = [S, S] \subset [(GN)^L, (GN)^L] \subset [(GN)^*, (GN)^*] \subset [GN, GN] \subset GN.$$

Moreover,  $(DGN \cap R)_1 = DTN$  and is closed in  $L$  by part 2 above. Hence  $(DTN)^R = DTN$ , and  $(DGN)^L = DGN$ .

*Proof of 4.* Recall that  $N$  is the radical of the commutator group  $L'$  and therefore  $DHN/N$  is an abelian analytic group. Therefore it is a direct

product  $C \cdot W$ , where  $C$  is its maximum compact subgroup and  $W$  is a vector group which clearly can be taken to be a subgroup of  $HN/N$ . It is known (Remark 2.6) that the preimage of  $C$  contains a compact subgroup mapping onto  $C$ . Since  $D$  is a maximal compact subgroup of  $DHN$ , we deduce that  $D$  projects onto  $C$ . Let  $V$  denote the preimage of  $W$ . Then  $DHN = DV$  with  $V$  closed and simply connected, and  $D \cap V \subset N$ . Since  $D \cap V$  is a compact group of unipotent elements, it contains the identity alone. Hence the epimorphism of the semi-direct product  $D \cdot V$  onto  $DV$  is a monomorphism, and, consequently, it is an isomorphism (in the topological sense as well, of course). Therefore

$$DV = (DHN) = (HN)^L = (HN \cap DV)^L = ((HN \cap D)V)^L = (HN \cap D)^L V.$$

It follows at once that  $D = (HN \cap D)^L$ .

*Proof of 5.* Let  $\mathfrak{P}$  and  $\mathfrak{Q}$  denote the Lie algebras of  $DG$  and its radical  $DT$  respectively. The  $\mathfrak{P}/\mathfrak{Q}$  part of the image  $\text{Ad}(DT)$  under the adjoint representation is trivial. Since  $\text{Ad } D$  is fully reducible, there is a complementary subspace to  $\mathfrak{Q}$  in  $\mathfrak{P}$  which is left fixed under  $\text{Ad } D$ . Let  $\mathfrak{Z}$  denote the subalgebra generated by this subspace, and let  $Z$  denote the corresponding analytic subgroup of  $DG$ . Clearly the elements of  $Z$  are fixed under inner automorphisms from  $D$ , and  $DG = ZDT$ . Let  $\mathfrak{S}$  be a semi-simple algebra occurring in a Levi-decomposition of  $\mathfrak{Z}$  and let  $S$  denote the corresponding analytic group. Since  $\mathfrak{P}/\mathfrak{Q} = \mathfrak{Z}/(\mathfrak{Z} \cap \mathfrak{Q})$  is semi-simple,  $\mathfrak{Z} = \mathfrak{S} + (\mathfrak{Z} \cap \mathfrak{P})$  and  $DG = SDT$ . Now select a maximal compact subgroup  $C$  of  $S$ .

We prove that  $CD$  is a maximal compact subgroup of  $DGN$ . Let  $R$  denote the radical of  $DGN$ , and  $\lambda$  denote the epimorphism of the semi-direct product  $S \cdot R$  onto  $SR$ . Then, as in the proof of 3, the kernel of  $\lambda$  is finite, so that a subgroup in  $SR$  is compact if and only if its inverse image in  $S \cdot R$  is compact. Consequently, to prove that  $CD$  is a maximal compact subgroup of  $SR$ , it suffices to prove that  $C \cdot D$  is a maximal compact subgroup of  $S \cdot R$ . Let  $J$  be a compact subgroup of  $S \cdot R$  containing  $C \cdot D$ . The image of  $J$  under the epimorphism  $s \cdot r \rightarrow s$  of  $S \cdot R$  onto  $S$  is a compact subgroup of  $S$  containing  $C$ . Therefore it coincides with  $C$  and  $J = C \cdot (J \cap R)$ . But  $J \cap R$  is a compact subgroup of  $R = DTN$ . By part 1,  $J \cap R = D$ , and therefore  $J = C \cdot D$ . This proves that  $C \cdot D$  is maximal compact in  $S \cdot R$ , and therefore  $CD$  is a maximal compact subgroup in  $DGN$ . Since  $DG \supset CD$ , the first assertion in 5. is proved.

In order to prove that  $N \cap DG$  is connected, we observe that there is a homeomorphism of  $N/(N \cap DG)$  onto  $DGN/DG$ . Let  $K$  denote  $CD$ . Then

$DGN/DG = (DGN/K)/(DG/K)$ . Since  $K$  is maximal compact in  $DGN$ , the space  $DGN/K$  is homeomorphic to euclidean space. The natural map of  $DGN/K$  onto  $(DGN/K)/(DG/K)$  is a continuous and open map with the inverse image of each point connected. Therefore the image  $(DGN/K)/(DG/K)$  is simply connected. Therefore

$$N/(N \cap DG) = DGN/DG = (DGN/K)/(DG/K)$$

is simply connected. It follows that  $N \cap DG$  is connected. Proof of Lemma 3.3 is now complete.

LEMMA 3.4. *Let  $\rho$  be a representation of a group  $G$  by automorphisms of a linear space  $V$  over a field of characteristic zero. Let  $M$  and  $U$  be a maximal fully reducible subgroup and the maximum normal subgroup of unipotent elements, respectively, in  $F$ , the smallest algebraic group in  $GL(V)$  containing  $\rho(G)$ . Let  $\sigma(g)$  denote the image of  $\rho(g)$  under the map  $\mu: \mu u \rightarrow m$  of  $MU$  onto  $M$ . Then  $\sigma$  is a representation of  $G$  equivalent to  $\rho'$ .*

*Proof.* That  $\sigma$  is a representation of  $G$  follows from the fact that  $\mu u \rightarrow m$  is a homomorphism of the algebraic group hull  $F$  (see Remark 2.3 above). The set  $\sigma(G)$ , being in  $F$ , keeps invariant any linear subspace of  $V$  that is invariant under  $\rho(G)$ . It suffices, therefore, to verify that if  $X$  and  $Y$  are linear subspaces invariant under  $\rho(G)$  with  $X \supset Y$ , and if  $\rho_{x/y}$ , the  $X/Y$  part of  $\rho$ , is irreducible, then  $\rho_{x/y}$  coincides with the  $X/Y$  part of  $\sigma$ . In that case  $\rho_{x/y}(U) = (1)$ ; for the subspace of elements in  $X/Y$  that are fixed under  $U$  must be invariant under  $G$  and, therefore, must coincide with  $X/Y$ . Consequently  $\rho_{x/y}(g) = \rho_{x/y}(g)u_{x/y} = (\rho(g)u)_{x/y} = \sigma(g)_{x/y}$ , where  $u = \rho(g)^{-1}\sigma(g) \in U$ . The proof is now complete.

LEMMA 3.5. *Let  $G$  be an analytic group and  $T$  its radical. Let  $H$  be an analytic subgroup whose Lie algebra is a Cartan subalgebra of the Lie algebra of  $T$ , and let  $D$  be a compact subgroup of  $G$  which lies in the centralizer of  $H$ . Let  $\rho$  be a representation of  $G$ . Then the associated semi-simple representation  $\rho'$  is equivalent to a representation  $\sigma$  having the same representation space that  $\rho$  has and such that 1)  $\sigma(g) = \rho(g)'$  for all  $g \in H$ ; and 2)  $\sigma(d) = \rho(d)$  for  $d \in D$ .*

*Proof.* The group  $\rho(D)$  is compact and therefore fully reducible. Furthermore,  $\rho(D)$  lies in the centralizer of  $\rho(H)$  and, therefore, in the centralizer of  $\rho(H)^*$ . Let  $A$  be a maximal fully reducible subgroup of  $\rho(H)^*$ . It is well known, and it is easily seen, that  $\rho(D) \cdot A$  is fully reducible. Let  $M$  be a maximal fully reducible subgroup of  $\rho(G)^*$  containing  $\rho(D) \cdot A$ .

and let  $U$  be the maximum normal subgroup of unipotent element in  $\rho(G)^*$ . Then  $\rho(G)^* = MU$  (semi-direct) (Remarks 2.2 and 2.3).

We assert first that  $A$  is central in  $\rho(H)^*$ . For let  $\mathcal{L}$  and  $\mathcal{B}$  denote the Lie algebras of  $\rho(H)$  and  $\rho(H)^*$  respectively. We identify these in the usual way with Lie subalgebras of the algebra  $E$  all endomorphisms of the underlying linear space. Let  $C(x)$  denote the conjugation  $y \rightarrow xyx^{-1}$  of  $E$  whenever  $x$  is an invertible element in  $E$ . Since the Lie algebra of  $H$  is nilpotent, the same is true of  $\mathcal{L}$  and therefore of  $\mathcal{B}$ . Let  $\beta(x)$  denote the restriction of  $C(x)$  to  $\mathcal{B}$  for  $x \in \rho(H)^*$ . Clearly  $\beta$  coincides with the adjoint representation of  $\rho(H)^*$ , and therefore it is unipotent. Hence  $\beta(A)$  is a unipotent family. On the other hand,  $\beta$  is a rational representation, and therefore  $\beta(A)$  is a fully reducible family. It follows immediately that  $\beta(A)$  consists only of the identity element. Thus  $\text{Ad } A$  is trivial on  $\mathcal{B}$ , and consequently  $A$  commutes with the elements of  $\rho(H)^*$ . Since  $A \subset \rho(H)^*$ ,  $A$  is a commutative fully reducible group. From this it follows that all its elements are fully reducible. Moreover, if  $a \in A$  and  $u \in \rho(H)^* \cap U$ , then  $(au)' = a$  since the Jordan product decomposition is unique.

We now define  $\mu$  to be the epimorphism  $mu \rightarrow m$  of  $\rho(G)^*$  onto  $M$ , and we set  $\sigma = \mu \cdot \rho$ . Then for  $h \in H$ ,  $\sigma(h) = \mu(\rho(h)) = \rho(h)'$ , and for  $d \in D$ ,  $\sigma(d) = \mu(\rho(d)) = \rho(d)$ . Also,  $\sigma$  is equivalent to  $\rho'$ , by Lemma 3.4. Proof of Lemma 3.5 is now complete.

*Note.* The proof of Lemma 3.5 establishes the fact that the totality of semi-simple elements in a nilpotent algebraic group  $B$  is central in  $B$  and therefore coincides with the maximal fully reducible subgroup of  $B$ .

**4. The first partial extension.** This section and the one following are devoted to a proof of the following extension theorem.

**THEOREM 4.1.** *Let  $L$  be an analytic group,  $G$  a closed normal analytic subgroup, and let  $\rho$  be a finite dimensional representation of  $G$ . Let  $\rho'$  denote the semi-simple representation of  $G$  associated with  $\rho$ , and let  $N$  denote the radical of the commutator subgroup  $L'$ . In order that the representation  $\rho$  can be extended to a finite dimensional representation of  $L$  (see Section 2, Remark 2.7, for definition), the following three conditions are necessary and sufficient:*

- 1)  $\rho'$  is trivial on  $G \cap N$ .
- 2) The representation  $\sigma$  of the analytic group  $GN$  defined by  $\sigma(gu) = \rho'(g)$  (the existence of  $\sigma$  being assured by Condition 1) is continuous in the topology of  $GN$  relative to  $L$ .

3) Let  $P$  denote the intersection of the kernels of all finite dimensional representations of  $L$ . Then  $\rho$  is trivial on  $P \cap G$ .

The necessity of these conditions is fairly clear (see Remark 2.7). We prove sufficiency in two stages. In this section, we show how to extend  $\rho$  from  $G$  to a certain analytic subgroup  $F$  of  $\overline{GN}$  that contains a maximal compact subgroup of  $\overline{GN}$ . In the following section, we apply the results of ERI, to extend the rest of the way to  $L$ . The first extension relies heavily on the representation  $\rho'$  as described in Lemma 3.5 above.

LEMMA 4.1. *In addition to the assumptions of Theorem 4.1, assume that  $L$  is a linear analytic group. Then there exists a closed connected subgroup  $F$  of  $L$  such that 1)  $F$  contains a maximal compact subgroup of  $\overline{GN}$ ,  $FN = \overline{GN}$  and  $G \subset F \subset G^*$ ; 2)  $\rho$  can be extended to a representation  $\theta$  of  $F$  having the same representation space as  $\rho$ ; 3) The associated semi-simple representation  $\theta'$  is trivial on  $N \cap F$ .*

*Proof.* We take care to distinguish between the analytic group topology of  $L$  and its topology relative to the full linear group. Accordingly, the  $GN$  of Lemma 4.1 is hereafter denoted by  $(GN)^L$ , following the notation of Section 2.

Let  $T$  denote the radical of  $G$ . Let  $H$  be an analytic subgroup of  $G$  whose Lie algebra is a Cartan subalgebra of the Lie algebra of  $T$ . Let  $D$  be a maximal compact subgroup in  $(H^* \cap (TN)^L)_1$ . By the note following Lemma 3.5,  $D$  is central in  $H^*$  and, therefore, is in the centralizer of  $H$ . Now select the associated semi-simple representation  $\rho'$  as in Lemma 3.5 with  $D \cap G$  as the " $D$ " of that lemma. By hypothesis 2 of Theorem 4.1, the representation  $\sigma$  is continuous on  $GN$  in the topology relative to  $L$ . It follows quite directly that  $\sigma$  is the restriction to  $GN$  of a representation  $\tau$  of  $(GN)^L$ . If  $d \in D \cap G$  we have  $\tau(d) = \sigma(d) = \rho(d)$ , by Lemma 3.5. On the semi-direct product  $D \cdot G$  we define a map  $\phi$  by the formula  $\phi(d \cdot g) = \tau(d)\rho(g)$ . The kernel of the epimorphism  $d \cdot g \rightarrow dg$  of  $D \cdot G$  onto  $DG$  consists of all elements  $d \cdot d^{-1}$  with  $d \in D \cap G$ . Thus  $\phi$  maps this kernel into the identity.

We show next that  $\phi$  is a homomorphism. To do so, we define five homomorphisms  $\alpha, \beta, \bar{\beta}, \gamma, \delta$ . Let  $\mathcal{E}$  and  $\mathcal{F}$  denote the Lie algebras of all endomorphisms on the linear spaces on which  $G$  and  $\rho(G)$  operate respectively. Let  $C(x)$  denote conjugation by  $x$  if  $x$  is an invertible endomorphism. We identify the Lie algebras of all-linear Lie groups with Lie subalgebras of the full linear algebra in the usual way. We shall consistently denote the

Lie algebra of a Lie group by affixing a dot superscript to the symbol for the group. We denote by  $\delta$  the map which assigns to an automorphism  $t$  of any analytic group  $S$ , its differential  $t'$  at the identity element. The map  $\delta$  is an isomorphism of  $\text{Aut } S$ , the automorphism group of  $S$ , into  $\text{Aut } S'$ , the automorphism group of its Lie algebra  $S'$ . Strictly speaking,  $\delta$  depends on  $S$  but we shall employ the same letter  $\delta$  for all analytic groups.

The kernel  $K$  of  $\rho$  is normal in  $G$ . By Lemma 3.2,  $K$  is normal in  $G^*$ . For  $g \in G^*$ , let  $\alpha(g)$  denote the operation of  $g$  on  $G/K$  that is induced by the conjugation  $C(g)$ . Clearly  $\delta \cdot \alpha(g)$  coincides with the  $G'/K'$  part of  $C(g)$ . Therefore  $\delta \cdot \alpha$  is a rational representation.

The map  $\beta$  (resp.  $\bar{\beta}$ ) is the isomorphism between  $\text{Aut } G/K$  and  $\text{Aut } \rho(G)$  (resp.  $\text{Aut } G'/K'$  and  $\text{Aut } \rho(G)'$ ) that is induced by  $\rho$ . The map  $\gamma$  is defined for all  $y \in \rho(G)^*$  by making  $\gamma(y)$  the operation of  $y$  on  $\rho(G)$  that is induced by  $C(y)$ . Just as in the case of  $\delta \cdot \alpha$ ,  $\gamma$  is a rational representation of  $\rho(G)^*$  in  $\text{Aut } (\rho(G))'$ . In addition,  $\beta$  is rational.

We shall prove that  $\beta \cdot \alpha(d) = \gamma \cdot \tau(d)$  for all  $d \in D$ .

$$\begin{array}{ccccc} G^* \supset D & \xrightarrow{\alpha} & \text{Aut } G/K & \xrightarrow{\delta} & \text{Aut } G'/K' \\ \downarrow \tau & & \downarrow \beta & & \downarrow \bar{\beta} \\ \rho(G)^* & \xrightarrow{\gamma} & \text{Aut } \rho(G) & \xrightarrow{\delta} & \text{Aut } \rho(G)' \end{array}$$

For  $d \in HN \cap D$ , we have  $d = hn$ ,  $h \in H$ ,  $n \in N$ ,  $\tau(d) = \tau(h)\tau(n) = \tau(h) = \rho(h)'$ , by Lemma 3.5. Therefore  $\delta \cdot \gamma \cdot \tau(d) = \delta \cdot \gamma(\rho(h)') = (\delta \cdot \gamma(\rho(h)))'$ , since a rational representation carries the semi-simple part of an endomorphism into the semi-simple part of its image (see Section 2). We observe that  $d = h'$ . For let  $A$  be a maximal fully reducible subgroup in  $H^*$ , and let  $U$  denote the maximal normal analytic subgroup of unipotent elements. Then  $H^* = AU$ , and  $(HN)^* = H^*N = A(UN)$  with  $A(UN)$  isomorphic to the semi-direct product  $A \cdot (UN)$  (Remarks 2.1 and 2.3). Since  $A$  contains all the semi-simple elements of  $H^*$  (see note following Lemma 3.5), it follows that  $d \in A$  and also  $h' \in A$ . We have  $d = hn = h'(h'^{-1}hn)$ , where  $h'^{-1}hn \in UN$ . From the unique representation of elements in  $A \cdot (UN)$ , we infer that  $d = h'$ .

Next observe that  $\rho$  is a homomorphism of  $G$ , so that by definition of  $\beta$ ,  $\gamma \cdot \rho = \beta \cdot \alpha$  on  $G$ . Also,  $\delta \cdot \beta = \bar{\beta} \cdot \delta$ . Therefore

$$\begin{aligned} (\delta \cdot \gamma(\rho(h)))' &= (\delta \cdot \gamma \cdot \rho(h))' = (\delta \cdot \beta \cdot \alpha(h))' = ((\bar{\beta} \cdot \delta) \cdot \alpha(h))' \\ &= (\bar{\beta} \cdot (\delta \cdot \alpha)(h))' = \bar{\beta}((\delta \cdot \alpha(h))') = \bar{\beta}((\delta \cdot \alpha)(h')) \\ &= \bar{\beta} \cdot \delta \cdot \alpha(h') = \bar{\beta} \cdot \delta \cdot \alpha(d) = \delta \cdot \beta \cdot \alpha(d). \end{aligned}$$

Therefore  $\delta \cdot \gamma \cdot \tau(d) = \delta \cdot \beta \cdot \alpha(d)$  for  $d \in HN \cap D$ . Since  $\delta$  is a monomorphism, we get  $\gamma \cdot \tau(d) = \beta \cdot \alpha(d)$  for  $d \in HN \cap D$ . Since  $HN \cap D$  is dense in  $D$ , we get  $\gamma \cdot \tau = \beta \cdot \alpha$  throughout  $D$ .

Let  $d_1 \cdot g_1$  and  $d_2 \cdot g_2$  be elements of the semi-direct product  $D \cdot G$ . Then

$$\begin{aligned}\phi(d_1 \cdot g_1) \phi(d_2 \cdot g_2) &= \tau(d_1) \rho(g_1) \tau(d_2) \rho(g_2) \\ &= \tau(d_1) \tau(d_2) (\tau(d_2)^{-1} \rho(g_1) \tau(d_2)) \rho(g_2) \\ &= \tau(d_1 d_2) (\gamma \cdot \tau(d_2) (\rho(g_1))) \rho(g_2) = \tau(d_1 d_2) (\beta \cdot \alpha(d_2) (\rho(g_1))) \rho(g_2).\end{aligned}$$

Now by definition of  $\beta$ ,  $\beta \cdot \alpha(d_2) (\rho(g_1)) = \rho(d_2^{-1} g_1 d_2)$ . Thus

$$\begin{aligned}\phi(d_1 \cdot g_1) \phi(d_2 \cdot g_1) &= \tau(d_1 d_2) \rho(d_2^{-1} g_1 d_2) \rho(g_2) \\ &= \tau(d_1 d_2) \rho(d_2^{-1} g_1 d_2 g_2) = \phi((d_1 \cdot g_1) (d_2 \cdot g_2)).\end{aligned}$$

Thus  $\phi$  preserves products and is indeed a homomorphism of the analytic group  $D \cdot G$ . We have seen that the kernel of  $\phi$  contains the kernel of the epimorphism  $D \cdot G \rightarrow DG$ . Therefore  $\phi$  induces a representation  $\theta$  of  $DG$ , and, obviously,  $\theta$  is an extension of  $\rho$ .

Next we prove that  $\theta'(N \cap DG) = 1$ . By part 5 of Lemma 3.3,  $N \cap DG$  is connected. Therefore  $N \cap DG$ , being a solvable normal analytic subgroup, is in the radical of  $DG$ . Hence  $N \subset DT$ . But  $T = HT'$ . Thus  $N \cap DG \subset DHT'$ . Let  $n \in N \cap DG$ . Then  $n = dhu$  with  $d \in D$ ,  $h \in H$ , and  $u \in T'$ . We have  $1 = \tau(n) = \tau(d) \tau(h) \tau(u) = \tau(d) \rho'(h) = \tau(d) \rho(h)'$ . Therefore

$$\begin{aligned}\phi(n) &= \phi(d) \phi(h) \phi(u) = \tau(d) \rho(h) \rho(u) \\ &= \tau(d) \rho(h)'^{-1} \rho(h) \rho(u) = ((\rho(h)')^{-1} \rho(h)) \rho(u).\end{aligned}$$

Now by hypothesis 1 of Theorem 4.1,  $\rho'(u) = 1$ , so that  $\rho(u)$  is unipotent. Furthermore,  $(\rho(h)')^{-1} \rho(h)$  is unipotent by definition of  $\rho(h)'$ . In addition  $(\rho(h)'^{-1}) \rho(h)$  and  $\rho(h)$  are contained in the algebraic group hull  $J$  of the solvable analytic group  $\rho(T)$ . Since  $J$  is solvable and algebraically connected, it follows from Lie's theorem on simultaneous triangularizing of solvable linear Lie algebras that the product of unipotent endomorphisms in  $J$  is unipotent. Therefore  $\theta(n) = ((\rho(h)'^{-1} \rho(h)) \rho(u))$  is unipotent for all  $n \in N \cap DG$ . It follows that  $\theta$  is unipotent on  $N \cap DG$  (Section 2 of ERI). Since  $N \cap DG$  is normal in  $DG$ ,  $\theta'$  is trivial on  $N \cap DG$  (see Section 2 of ERI).

In order to complete the proof of Lemma 4.1, we need only mention that  $F$  should be selected as  $DG$ .

**5. Completion of proof.** We proceed under the assumptions of Theorem 4.1. Our first step is to replace  $L$  by  $L/P$  and to replace  $\rho$  by

the induced representation of  $GP/P$  whose existence is assured by hypothesis 3. After this substitution is made, hypothesis 3 is satisfied trivially. Indeed, the group  $L$  is then separated by its finite dimensional representations and, therefore, has a *faithful* finite dimensional representation (Remark 2.5). Consequently, we lose no generality when we assume, in addition to the hypotheses of Theorem 4.1, that  $L$  is an analytic *linear* group. We can then apply Lemma 4.1, and assume that the representation  $\rho$  has been extended to an analytic subgroup  $F$  of  $L$ , such that  $G \subset F \subset G^*$ ,  $FN = (GN)^L$ , and  $F$  contains a maximal compact subgroup of  $(GN)^L$ . We denote the extended representation by  $\rho$  also.

We assert that there is a normal series  $F_0 = F \subset F_1 \subset \cdots \subset F_r = (GN)^L$  with  $F_i$  a closed normal analytic subgroup in  $F_{i+1}$ , and  $F_{i+1}/F_i$  isomorphic to a 1-dimensional vector group ( $i = 1, \dots, r-1$ ). For the commutator subgroup of  $G^*$  coincides with the commutator subgroup of  $G$  (see [2], p. 173, Th. 13, for the Lie algebra formulation). Therefore  $A = (G^* \cap (GN)^L)_1/F$  is abelian. Since  $F$  contains a maximal compact subgroup of  $(G^* \cap (GN)^L)_1$ , the quotient space  $A$  is simply connected (see Remark 2.6). A composition series of analytic subgroups for  $A$  induces a normal series of analytic subgroups  $F_0 = F \subset F_1 \subset \cdots \subset F_s = (G^* \cap (GN)^L)_1$  with each  $F_{i+1}/F_i$  a one dimensional vector group. We next observe that  $(G^* \cap (GN)^L)_1$  is a closed normal connected subgroup of  $(GN)^L$  and the factor group is

$$FN/(G^* \cap FN)_1 = N/(N \cap (G^* \cap FN)_1),$$

which is solvable. Inasmuch as  $(G^* \cap FN)_1$  contains a maximal compact subgroup of  $FN$ , the quotient space is simply connected (Remark 2.6). A composition series of analytic subgroups for  $FN/(G^* \cap FN)_1$  induces a normal series of analytic subgroups  $F_s = (G^* \cap (GN)^L)_1 \subset F_{s+1} \subset \cdots \subset F_r = (GN)^L$ , and clearly  $F_{i+1}/F_i$  is a one dimensional vector group ( $i = s, s+1, \dots, r-1$ ).

Next, we select in each  $F_{i+1}$  a one parameter subgroup  $P_i$  satisfying a)  $P_i$  is not entirely in  $F_i$ , and b)  $P_i \subset F_{i+1} \cap N$  if  $(F_{i+1} \cap N)_1 \neq (F_i \cap N)_1$ . Condition a) implies that  $P_i F_i = F_{i+1}$ . Since  $P_i/P_i \cap P_i F_i/F_i = F_{i+1}/F_i$  is simply connected,  $P_i \cap F_i$  consists of the identity alone. Therefore the epimorphism of the semi-direct product  $P_i \cdot F_i$  onto  $P_i F_i$  is an isomorphism. We assert that  $N \cap F_{i+1} = (N \cap P_i)(N \cap F_i)$  ( $i = 0, \dots, r-1$ ). For  $N/(N \cap F_i) = NF_i/F_i = FN/F_i$ , and the latter is simply connected since  $F_i$  contains a maximal compact subgroup of  $FN$ . Hence  $F_i \cap N = (F_i \cap N)_1$ ,  $N \cap P_i = (1)$  or  $P_i$ , and  $N \cap F_{i+1} = (N \cap P_i)(N \cap F_{i+1})$  by Condition b),  $i = 0, \dots, r-1$ . It follows immediately that

$$N = N \cap (GN)^L = (N \cap P_{r-1})(N \cap P_{r-2}) \cdots (N \cap P_0)(N \cap F).$$

Our next step is to apply Remark 2.8 of Sec. 2, in turn, to the semi-direct products  $P_0 \cdot F_0, P_1 \cdot F_1, \dots, P_{r-1} \cdot F_{r-1}$  and thus to extend the representation  $\rho$  from  $F_0 = F$  to  $P_{r-1}F_{r-1} = F_r = (GN)^L$ . The construction of the extension provides a representation of  $(GN)^L$  unipotent on  $N$  (Remark 2.8).

**LEMMA 5.1.** *Under the hypotheses of Lemma 4.1, there is a normal series of closed analytic subgroups  $L_0 = (GN)^L \subset L_1 \subset \dots \subset L_j = L$ , and a closed analytic subgroup  $Q_i \subset L_{i+1}$  such that  $L_{i+1} = Q_i L_i$  and either  $Q_i \cap L_i$  contains only the identity element or  $Q_i$  is compact ( $i = 0, \dots, j-1$ ).*

*Proof.* Let  $R$  denote the radical of  $L$  and let  $S$  be a maximal semi-simple analytic subgroup of  $L$ . Then  $S \cap R$  is a discrete normal subgroup of  $S$  and is therefore central. Since the center of a semi-simple analytic linear group is finite,  $S \cap R$  is finite. Therefore the epimorphism  $s \cdot r \rightarrow sr$  of the semi-direct product  $S \cdot R$  is a finite covering map. It follows easily from this that an analytic subgroup  $J$  of  $L$  which contains  $R$  is closed in  $L$  if and only if the connected component  $(J \cap S)_1$  is closed. Now  $(L_0 R \cap S)_1$  is a normal analytic subgroup of  $S$  and is therefore semi-simple. But a semi-simple analytic linear group is closed in the full linear group. Hence  $(L_0 R \cap S)_1$  is closed and  $L_0 R$  is closed in the analytic group  $L$ .

The factor group  $L_0 R / L_0$  is a solvable analytic group and therefore has a normal series of closed analytic subgroups whose successive quotients are one-dimensional analytic groups. This normal series provides a normal series of analytic subgroups  $L_0 = L \subset L_1 \subset \dots \subset L_{j-1} = L_0 R$  with  $L_{i+1}/L_i$  a one-dimensional analytic group ( $i = 0, \dots, j-2$ ). In case  $L_{i+1}/L_i$  is simply connected, we select  $Q_i$  to be any one parameter group in  $L_{i+1}$  that is not entirely in  $L_i$ . Then  $L_{i+1} = Q_i L_i$ ,  $Q_i \cap L_i$  is discrete in the analytic group  $Q_i$ , and  $Q_i / (Q_i \cap L_i) = L_{i+1}/L_i$  is simply connected. It follows that  $Q_i \cap L_i$  is connected as well as discrete, and therefore it consists only of the identity. Therefore the epimorphism  $x \cdot y \rightarrow xy$  of the semi-direct product  $Q_i L_i$  onto each such  $L_{i+1}$  is an isomorphism and  $Q_i$  is closed in  $L_{i+1}$ . In the other case that  $L_{i+1}/L_i$  is not simply connected, it is compact. Selecting  $Q_i$  to be a maximal compact subgroup of  $L_{i+1}$ , we obtain  $L_{i+1} = Q_i L_i$  (Remark 2.8).

Consider finally the factor group  $L/L_0 R = SR/L_0 R = S/(S \cap L_0 R)$ . The Lie algebra of  $(S \cap L_0 R)_1$  is an ideal of the Lie algebra  $S^*$  of  $S$ . Since  $S^*$  is a semi-simple Lie algebra, it contains an ideal  $Q^*$  complementary to the Lie algebra of  $(L_0 R \cap S)_1$ . Let  $Q$  be the analytic subgroup with Lie algebra  $Q^*$ . Then  $Q$  is a semi-simple analytic group and is therefore closed. Furthermore,  $(Q \cap L_0 R \cap S)_1 = (Q \cap (L_0 R \cap S)_1)_1$  has only zero in its Lie

algebra so that  $Q \cap L_0 R = Q \cap L_0 R \cap S$  is a discrete normal subgroup of  $Q$ . Hence it is central in  $Q$  and finite. Take  $Q_j = Q$  and  $L_j = L$ . The proof of Lemma 5.1 is now complete.

We now conclude the proof of Theorem 4.1. The representation has already been extended to  $\overline{GN} = L_0$ . Since  $N \subset \overline{GN}$ , the hypothesis on  $\rho$  made in Remark 2.8 is trivially satisfied. Since either  $Q_i$  is compact or  $Q_i \cap L_i$  contains only the identity, the hypothesis on  $Q$  is satisfied. Thus  $\rho$  can be extended in  $j$  steps from  $L_0$  to  $L_j$  via  $L_1, L_2, \dots, L_{j-1}$ . The proof of Theorem 4.1 is now complete.

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# SATAKE'S COMPACTIFICATION OF $V_n^*$ \*

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In a recent paper [13], Satake has obtained a compactification of the quotient space  $V_n = H_n/M_n$ , where  $H_n$  is the generalized upper half-plane of degree  $n$ , i.e., the space of symmetric matrices  $Z = X + iY$ , where  $Y$  is positive definite, and  $M_n$  the generalized modular group. This compactification  $V_n^*$  is the union  $V_n \cup V_{n-1} \cup \dots \cup V_0$  supplied with a certain topology and analytic structure, concerning which Satake states that it should not be hard to prove that, with this analytic structure,  $V_n^*$  is a general analytic space [13]. It is the purpose of this paper to prove that this is indeed the case and that, moreover,  $V_n^*$  is isomorphic, as a general analytic space, to a normal projective variety. One of the consequences of our result will be that if  $f$  is a meromorphic function in  $H_n$  invariant under  $M_n$  and if  $n \geq 2$ , then  $f$  is the quotient of automorphic forms.

In what follows, we refer to the notation of [13]. It is trivial that  $V_0^*$  is a general analytic space, and classical that  $V_1^*$  is such. If  $x \in V_n^*$ , we say that  $V_{n-1}^*$  has a local basis in  $V_n^*$  at  $x$  if there exists a finite number of analytic functions (for the given analytic structure)  $f_1, \dots, f_q$  on a neighborhood  $\mathfrak{U}$  of  $x$  in  $V_n^*$  such that  $V(f_1, \dots, f_q) \cap \mathfrak{U} = V_{n-1}^* \cap \mathfrak{U}$ , where  $V(f_1, \dots, f_q)$  is the variety of zeros of  $f_1, \dots, f_q$ ;  $f_1, \dots, f_q$  is called a local basis for  $V_{n-1}^*$  at  $x$ . We shall first show that if  $V_{n-1}^*$  has a local basis at each  $x \in V_n^*$ , and if  $V_{n-1}^*$  is a general analytic space, then  $V_n^*$  is a general analytic space.

In what follows, if  $F$  is a family or collection of functions defined on some set  $D$ , we let  $V(F)$  denote the set of common zeros in  $D$  of the elements of  $F$ . If the elements of  $F$  are analytic functions,  $V(F)$  is an analytic variety.

To begin with, it is well known that if  $M_{nx}$  is the subgroup of  $M_n$  leaving fixed  $x \in H_n$ , then  $\text{l.c.m.}_{x \in H_n}(\text{ord } M_{nx})$  is a finite positive integer, which we denote by  $q_n$ . Secondly,  $H_n$  is pseudo-conformally equivalent to a bounded domain  $D_n$  in  $C^{n(n+1)/2}$ , and since the automorphic forms of weight

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$m$  on  $H_n$  with respect to  $M_n$  can be identified with the automorphic forms on  $D_n$  of a certain weight, we can apply the results of [15], exposé I, on automorphic forms on a bounded domain to obtain immediately

LEMMA I. *There exists a positive integer  $m_n$  such that the following holds: Let  $x_1, \dots, x_r \in H_n$  correspond to  $r$  distinct points of  $H_n/M_n$  under the canonical mapping  $\phi_n$  of  $H_n$  onto  $H_n/M_n$ . Let  $a$  be a given positive integer. Then for all sufficiently large positive multiples  $m$  of  $m_n$ , there exists an automorphic form of weight  $m$  on  $H_n$  having prescribed power series developments up to and including the terms of degree  $a$  at each of the points  $x_1, \dots, x_r$ .*

It is clear that we may assume  $2 = m_0 | m_1 | m_2 | \dots$ . It is known (Satz 1 of [6]) that for  $n > 1$ , every automorphic form is bounded on the fundamental domain  $F_n$ , i.e., integral. Moreover, if  $\Phi_r^n$  is the homomorphism defined on p. 278 of [13], and if  $\mathcal{M}_n(m)$  and  $\mathcal{M}_r(m)$  are, respectively, the modules of integral automorphic forms of weight  $m$  on  $V_n^*$  and  $V_r^*$ , then for sufficiently large  $m$ ,  $\Phi_r^n: \mathcal{M}_n(m) \rightarrow \mathcal{M}_r(m)$  is onto by Satz 4 of [8]. For all sufficiently large multiples  $m$  of  $m_n$ ,  $V(\mathcal{M}_n(m))$  is empty; this is trivial for  $V_0^*$ , classical for  $V_1^*$ , and follows for general  $n$  by use of an obvious induction using LEMMA I and the fact that  $\Phi_{n-1}^n$  is onto. In the future we will always assume  $m_n$  has been chosen such that  $V(\mathcal{M}_n(m_n))$  is empty.

LEMMA II. *Let  $x \in V_n^*$ . Then there exists a neighborhood  $\mathfrak{U}$  of  $x$  in  $V_n^*$  such that if  $x_1, x_2 \in V_n \cap \mathfrak{U}$ , there exists a holomorphic function  $f$  on  $\mathfrak{U}$  such that  $f(x_1) \neq f(x_2)$ .  $f$  may be chosen as a quotient of modular forms.*

*Proof.* Let  $d \in \mathcal{M}_n(m_n)$  be such that  $d(x) \neq 0$ . Then let  $\mathfrak{U}$  be a neighborhood of  $x$  such that  $d \neq 0$  on  $\mathfrak{U}$ . Let  $x_1, x_2 \in \mathfrak{U} \cap V_n$ . By LEMMA I there exists an automorphic form  $g \in \mathcal{M}_n(m)$ ,  $m_n | m$ , such that  $g(x_1) = 0$ ,  $g(x_2) = 1$ . Let  $f = g/(d^{m/m_n})$ . It is clear that  $f(x_1) = 0 \neq f(x_2)$  and that  $f$  is holomorphic (in the given analytic structure) on  $\mathfrak{U}$ .

We shall eventually prove by induction that if  $V_{n-1}^*$  is a general analytic space, then  $V_n^*$  is one. Suppose, then, that  $V_{n-1}^*$  is a general analytic space, let  $x \in V_{n-1}^*$ , and suppose  $f$  to be holomorphic on  $V_{n-1}^*$  in a neighborhood  $\mathfrak{U}^*$  of  $x$ . We extend  $f$  to a holomorphic function on a small neighborhood of  $x$  on  $V_n^*$  as follows: If  $Z \in F_n$ , let  $Z_1$  be the matrix composed of the first  $n-1$  rows and  $n-1$  columns of  $Z$ ; then  $Z_1 \in F_{n-1}$ , and we define  $f(Z) = f(Z_1)$ . It is easy to see from Lemma 2' of [13] that  $f$  is a holo-

morphic function (in the analytic structure prescribed in [13]) in a small neighborhood of  $x$  on  $V_n^*$ . Since  $V_{n-1}^*$  is supposed to be a general analytic space, there exist holomorphic functions  $f_{q+1}, \dots, f_p$  in a neighborhood  $\mathfrak{U}_1$  of  $x$  on  $V_{n-1}^*$  such that  $V(f_{q+1}, \dots, f_p) = x$ , and these functions can be extended, as above, to holomorphic functions  $f_{q+1}, \dots, f_p$  on a small neighborhood  $\mathfrak{U}_2$  of  $x$  on  $V_n^*$  such that  $V(f_{q+1}, \dots, f_p) \cap V_{n-1}^* = x$ .

By definition, a general analytic space is a pair  $(V, \mathfrak{A})$ , where  $V$  is a Hausdorff space and  $\mathfrak{A}$  a distinguished subsheaf of the sheaf of germs of continuous, complex-valued functions on  $V$  subject to the following condition: If  $x \in V$ , there exists a neighborhood  $\mathfrak{U}$  of  $x$  on  $V$ , an analytic subvariety  $W$  of some open domain in  $C^m$  ( $m$  depending on  $x$  and  $\mathfrak{U}$ ), and a homeomorphism  $h$  of  $\mathfrak{U}$  onto  $W$  such that the sheaf, on  $\mathfrak{U}$ , of germs of continuous functions of the form  $f \circ h$ , where  $f$  is a germ of an analytic function in a neighborhood of some  $y \in W$ , precisely coincides with  $\mathfrak{A}$ .  $(V, \mathfrak{A})$  is said to be normal if each  $W$  is analytically normal at each point. In general, given a Hausdorff space  $V$  and a distinguished subsheaf  $\mathfrak{A}$  of the sheaf of germs of continuous, complex-valued functions on  $V$ , it may be of interest to decide whether  $(V, \mathfrak{A})$  (or more loosely speaking,  $V$ , when  $\mathfrak{A}$  is understood) is a general analytic space; at any rate, the cross-sections of  $\mathfrak{A}$  over any open set are to be identified with certain continuous functions, called  $\mathfrak{A}$ -functions, and if  $V'$  is a subset of  $V$ ,  $V'$  is called an  $\mathfrak{A}$ -variety if for any  $x \in V'$ , there exists a neighborhood  $\mathfrak{U}$  of  $x$  and a finite number of  $\mathfrak{A}$ -functions  $f_1, \dots, f_q$  on  $\mathfrak{U}$  such that  $V(f_1, \dots, f_q) = V'$ , in other words,  $f_1, \dots, f_q$  are a local basis for  $V'$  at  $x$ . (The terms  $\mathfrak{A}$ -function and  $\mathfrak{A}$ -variety are from an unpublished manuscript of H. Grauert and R. Remmert entitled "Bilder und Urbilder analytischer Garben.") Using this terminology, we have

**THEOREM 1.** *Let  $V$  be a Hausdorff space supplied with a distinguished subsheaf  $\mathfrak{A}$  of the sheaf of germs of continuous functions. Let  $V'$  be an  $\mathfrak{A}$ -variety on  $V$  and let  $V'$  be supplied with a distinguished subsheaf  $\mathfrak{A}'$  of the sheaf of germs of continuous functions on  $V'$ . Suppose that  $(V - V', \mathfrak{A} \mid (V - V'))$  is a normal general analytic space and that  $(V', \mathfrak{A}')$  is a general analytic space, respectively of dimensions  $n$  and  $m$ ,  $m < n$ . Suppose, moreover, that the following conditions are satisfied:*

- (1) *If  $V^0$  is the set of regular points of  $V - V'$ , each  $x \in V'$  has a neighborhood basis  $\{\mathfrak{U}_\alpha\}$  such that each  $\mathfrak{U}_\alpha \cap V^0$  is non-empty and connected.*
- (2) *If  $f$  is a  $\mathfrak{A}$ -function,  $f \mid V'$  is an  $\mathfrak{A}'$ -function.*
- (3) *If  $f$  is continuous and if  $f \mid (V - V')$  is an  $\mathfrak{A}$ -function, then  $f$  is an  $\mathfrak{A}$ -function.*

(4) If  $x \in V'$ , there exists a neighborhood  $\mathfrak{U}$  of  $x$  such that (i) There exist a finite number of  $\mathfrak{M}$ -functions  $f_{q+1}, \dots, f_p$  on  $\mathfrak{U}$  such that  $V(f_{q+1}, \dots, f_p) \cap V' = x$ , and (ii) If  $x_1, x_2 \in \mathfrak{U} - V'$ ,  $x_1 \neq x_2$ , there exists an  $\mathfrak{M}$ -function  $f$  on  $\mathfrak{U}$  such that  $f(x_1) \neq f(x_2)$ .

Then  $(V, \mathfrak{M})$  is a normal general analytic space.

*Proof.* Let  $x \in V'$ , let  $\mathfrak{U}_3$  be a neighborhood of  $x$  satisfying 4(i) and 4(ii). We may assume  $\mathfrak{U}_3$  chosen so that there are  $\mathfrak{M}$ -functions  $f_1, \dots, f_q$  on  $\mathfrak{U}_3$  such that  $V(f_1, \dots, f_q) = V'$ , let  $f_{q+1}, \dots, f_p$  satisfy 4(i), and let  $f_{p+1}, \dots, f_s$  be arbitrary  $\mathfrak{M}$ -functions on  $\mathfrak{U}_3$ . Let  $f: \mathfrak{U}_3 \rightarrow C^s$  be defined by  $f(y) = (f_1(y), \dots, f_s(y))$ . We assume in what follows that all neighborhoods  $\mathfrak{U}_3, \mathfrak{U}_5$ , etc., are such that  $\mathfrak{U}_i$  is open and  $\mathfrak{U}_i \cap V_n$  is connected,  $i = 1, 2, \dots$ . THEOREM 1 is proved by means of two lemmas.

LEMMA III. There exists a neighborhood  $\mathfrak{U}_7$  of  $x$  and an integer  $\beta \geq 0$  such that

- (1)  $f(\mathfrak{U}_7)$  is an irreducible analytic variety at  $f(x)$ .
- (2)  $f^{-1}(f(x)) \cap \mathfrak{U}_7 = x$ .
- (3)  $f(V' \cap \mathfrak{U}_7)$  is a subvariety of  $f(\mathfrak{U}_7)$ , and there exists a proper subvariety  $\mathcal{S}$  of  $f(\mathfrak{U}_7) - f(V' \cap \mathfrak{U}_7)$  such that if  $y \in f(\mathfrak{U}_7) - f(V' \cap \mathfrak{U}_7) - \mathcal{S}$ ,  $f^{-1}(y)$  consists of exactly  $\beta$  points, and  $f$  is biregular at each  $y' \in f^{-1}(y)$ .

*Proof.* In what follows we use a superscript  $c$  to denote "closure of." First of all,  $V(f_1, \dots, f_q) \cap \mathfrak{U}_3 = V' \cap \mathfrak{U}_3$  and  $V(f_1, \dots, f_p) \cap \mathfrak{U}_3 = x$ . Let  $\mathfrak{U}_5$  be a neighborhood of  $x$  such that  $\mathfrak{U}_5^c \subset \mathfrak{U}_3$ . Then there exist neighborhoods  $\mathfrak{U}_6$  and  $\mathfrak{U}_7$  of  $x$ , and a neighborhood  $N_1$  of  $f(x)$  such that (a)  $\mathfrak{U}_7^c \subset \mathfrak{U}_5$ ,  $\mathfrak{U}_6^c \subset \mathfrak{U}_5$ , and (b)  $f^{-1}(N_1^c) \cap \mathfrak{U}_5^c \subset \mathfrak{U}_7$ . These facts follow easily from the fact that  $f^{-1}(f(x)) \cap \mathfrak{U}_3 = x$  and from the continuity of  $f$ . Moreover, let  $\{\mathfrak{U}_8(\delta)\}$  be a basis of neighborhoods of the set  $V' \cap \mathfrak{U}_3$  closed in  $\mathfrak{U}_3$ . Then for each  $\delta$ ,  $f(\mathfrak{U}_5^c - \mathfrak{U}_8(\delta))$  is a compact set not meeting  $f(V' \cap \mathfrak{U}_5^c)$  since  $V(f_1, \dots, f_q) \cap \mathfrak{U}_3 = V' \cap \mathfrak{U}_3$ . Therefore,

$$\text{dist}(f(\mathfrak{U}_5^c - \mathfrak{U}_8(\delta)), f(V' \cap \mathfrak{U}_5^c)) = \sigma > 0.$$

Moreover, since  $f$  is continuous, for any neighborhood  $N_2$  of  $f(V' \cap \mathfrak{U}_5^c)$ , there exists a  $\mathfrak{U}_8(\delta)$  such that  $f(\mathfrak{U}_8(\delta) \cap \mathfrak{U}_5^c) \cap N_1 \subset N_1 \cap N_2$ .

By (b),  $f(\mathfrak{U}_7)$  is relatively closed in  $N_1$ . Since  $V'$  is a general analytic space, and by the choice of  $f_1, \dots, f_s$ , we may assume<sup>(1)</sup> that  $\mathfrak{U}_7$  (and corre-

<sup>(1)</sup> Let  $W$  be a purely  $k$ -dimensional general analytic space and let  $f: W \rightarrow C^m$  be a mapping whose coordinates  $f_1, \dots, f_m$  are analytic functions on  $W$ . If  $f^{-1}(x)$  is discrete

spondingly  $N_1$ ) is chosen so small that  $f(\mathfrak{U}_7 \cap V')$  is an irreducible subvariety  $\mathfrak{F}$  of dimension  $= \dim V'$  of  $N_1$ .  $N_1 - \mathfrak{F}$  is connected. Let  $a \in N_1 - \mathfrak{F}$ . If  $a \in N_1 - f(\mathfrak{U}_7)$ , there exists a neighborhood of  $a$  not meeting  $f(\mathfrak{U}_7)$ . If  $a \in f(\mathfrak{U}_7) - \mathfrak{F}$ ,  $f^{-1}(a) \cap (\mathfrak{U}_6 - V')$  is a subvariety  $\mathfrak{S}_a$  of  $\mathfrak{U}_6$ , and there exists a neighborhood  $\mathfrak{U}_8(\delta)$  of  $V'$  not meeting  $f^{-1}(a) \cap \mathfrak{U}_7 = f^{-1}(a) \cap \mathfrak{U}_6$ . Therefore  $f^{-1}(a) \cap \mathfrak{U}_6 \subset \mathfrak{U}_7 - \mathfrak{U}_8(\delta)$ . Thus  $f^{-1}(a) \cap \mathfrak{U}_6$  is a compact, complex analytic subvariety of  $\mathfrak{U}_6 - V'$ . Let  $y_1, y_2 \in f^{-1}(a) \cap \mathfrak{U}_6$ ,  $y_1 \neq y_2$ . Then there exists an  $\mathfrak{M}$ -function  $g$  on  $\mathfrak{U}_6$  such that  $g(y_1) \neq g(y_2)$ , whereas  $g$ , being holomorphic on  $\mathfrak{U}_6 - V'$ , is constant on each component of  $f^{-1}(a) \cap \mathfrak{U}_6$ . Therefore  $f^{-1}(a)$  is a discrete, finite set of points,  $a_1, \dots, a_p$ . Let  $A_1, \dots, A_p$  be any neighborhoods of  $a_1, \dots, a_p$  on  $V - V'$ . Then there exists a neighborhood  $A$  of  $a$  such that  $f^{-1}(A) \cap \mathfrak{U}_6 \subset \cup A_i$ ; for, otherwise, we could find  $b \in \mathfrak{U}_6$ ,  $b \neq a_i$ ,  $i = 1, \dots, p$ , such that  $f(b) = a$ , which is impossible. Therefore,  $A \cap f(\mathfrak{U}_7) \subset f(A_1) \cup \dots \cup f(A_p)$ . We can choose  $A_i$ ,  $i = 1, \dots, p$ , so small that each  $f(A_i)$  is an analytic variety at  $a$ . Therefore  $f(\mathfrak{U}_7)$  is an analytic variety in a neighborhood of  $a$ . Therefore  $f(\mathfrak{U}_7) - \mathfrak{F}$  is a complex analytic subvariety, evidently of dimension  $= \dim V$ , of  $N_1 - \mathfrak{F}$ . Therefore, since  $\mathfrak{F}$  is of dimension  $= \dim V' < \dim V$ ,  $f(\mathfrak{U}_7)$  is an analytic subvariety of  $N_1$  of dimension  $= \dim V$  by a result of Remmert and Stein [7]. We see, moreover, reasoning as above, that if  $J$  is the (necessarily proper<sup>(1)</sup>) subvariety of  $\mathfrak{U}_7 - V'$  on which the matrix  $(\partial f_i / \partial z_{ij})$  has rank  $< \dim V$ , then  $f(J)$  is a subvariety of  $f(\mathfrak{U}_7) - \mathfrak{F}$ . Let  $S$  be the set of singular points of  $f(\mathfrak{U}_7)$  and let  $S' = (f(\mathfrak{U}_7) - \mathfrak{F}) \cap S$ .  $\dim S' < \dim f(\mathfrak{U}_7)$ . Therefore  $f^{-1}(S')$  is a proper subvariety of  $\mathfrak{U}_7 - V'$ . Therefore  $\mathfrak{U}_7 - V' - f^{-1}(S') - J$  is connected and everywhere dense, however small  $\mathfrak{U}_7$  may be chosen, and consequently  $f(\mathfrak{U}_7 - V' - f^{-1}(S') - J)$  is a connected, dense set of regular points. Hence  $f(\mathfrak{U}_7)$  is irreducible at  $f(x)$ . Let  $a \in (f(\mathfrak{U}_7) - \mathfrak{F} - f(J) - S') \cap N_1$ ,  $f^{-1}(a) \cap \mathfrak{U}_6 = f^{-1}(a) \cap \mathfrak{U}_7 = a_1, \dots, a_p$ . Then for every point  $a'$  near  $a$ ,  $f^{-1}(a') \cap \mathfrak{U}_6 = f^{-1}(a') \cap \mathfrak{U}_7 = a'_1, \dots, a'_p$ . Let  $A(\beta)$  be the subset of  $(f(\mathfrak{U}_7) - \mathfrak{F} - f(J) - S') \cap N_1$  such that for  $a \in A(\beta)$ ,  $f^{-1}(a) \cap \mathfrak{U}_6 = f^{-1}(a) \cap \mathfrak{U}_7$  consists of just  $\beta$  points. The interior of  $A(\beta)$ ,  $\text{Int } A(\beta)$ , is not empty. Let  $b \in (f(\mathfrak{U}_7) - \mathfrak{F} - f(J) - S') \cap N_1 \cap (\text{Int } A(\beta))^c$ . Let  $b_1, \dots, b_k = f^{-1}(b) \cap \mathfrak{U}_6$ . Since  $b \notin f(J)$ , there exist neighborhoods  $B_1, \dots, B_k$  of

for each  $x \in f(W)$ , we say  $f$  is a light analytic map; in this case, it follows immediately from Theorems 5 and 6, exposé XIV of [14], that for each  $x^* \in W$ , there exists a neighborhood  $\mathfrak{U}$  of  $x^*$  such that  $f(\mathfrak{U})$  is an analytic variety of pure dimension  $k$  at  $f(x^*)$ . Moreover, if  $f$  is light analytic and if  $W^0$  is the set of regular points of  $W$ , it is easy to show by a simple dimension argument that the subset  $J$  of  $W^0$ , on which the Jacobian matrix of the mapping  $f$  has rank  $< k$ , is a proper subvariety of  $W^0$ .

$b_1, \dots, b_k$  which are pair-wise disjoint and such that  $f$  is one-to-one on each  $B_i$ . Since, moreover,  $b \notin S'$ ,  $f(B_i)$  contains a neighborhood of  $b$  on  $f(U_7)$ ,  $i = 1, \dots, k$ . Finally,  $b \in (\text{Int } A(\beta))^c$ . Therefore  $k \leq \beta$ . Suppose  $k < \beta$ . Let  $a^{(j)} \rightarrow b$ ,  $f^{-1}(a^{(j)}) = a_1^{(j)}, \dots, a_\beta^{(j)}$ . Then for some  $i$ , there exists a subsequence of  $a_i^{(j)}$  having for limit  $\alpha \neq b_1, \dots, b_k$ ,  $\alpha \in U_6$ , and, by continuity,  $f(\alpha) = b$ , which is impossible. Therefore  $k = \beta$ . Hence  $(\text{Int } A(\beta))^c$  is open in the connected set  $(f(U_7) - \mathcal{F} - f(J) - S') \cap N_1$ . Therefore  $A(\beta) = (f(U_7) - \mathcal{F} - f(J) - S') \cap N_1$ . We let  $\mathcal{S} = f(J) \cup S'$ . This completes the proof of LEMMA III.<sup>(2)</sup>

LEMMA IV. For suitable choice of  $f_{p+1}, \dots, f_s$  ( $s$  sufficiently large),  $\beta$  (of LEMMA III) = 1.

*Proof.* Suppose  $f_{p+1}, \dots, f_s$  are such that  $\beta$  is minimal. Let  $y \in f(U_7) - \mathcal{F} - f(J) - S$ , and suppose that  $U_7 \cap f^{-1}(y) = y_1, \dots, y_\beta$ ,  $\beta > 1$ ,  $y_1 \neq y_2$ . Let  $g$  be an  $\mathcal{A}$ -function on  $U_3$  such that  $g(y_1) \neq g(y_2)$ . If  $\xi \in f(U_7) - \mathcal{F} - f(J) - S$ , let  $f^{-1}(\xi) = \xi_1, \dots, \xi_\beta$ ; let  $\mathcal{N}$  be a neighborhood of  $\xi$  and  $\mathcal{N}_i$  a neighborhood of  $\xi_i$ ,  $i = 1, \dots, \beta$ , such that  $f$  is a biregular mapping of  $\mathcal{N}_i$  onto  $\mathcal{N}$ . Let  $g_{\xi i}$  be defined on  $\mathcal{N}$  by  $g_{\xi i}(\xi) = g((f|_{\mathcal{N}_i})^{-1}(\xi))$  for  $\xi \in \mathcal{N}$ . Let  $I(\mathcal{N}) = \{\xi \mid \xi \in \mathcal{N}, g_{\xi 1}(\xi) = \dots = g_{\xi \beta}(\xi)\}$ .  $I(\mathcal{N})$  is the subset of  $\mathcal{N}$  consisting of those  $\xi \in \mathcal{N}$  such that  $g$  has the same value at each point of  $f^{-1}(\xi)$ , and  $I(\mathcal{N})$  is a subvariety of  $\mathcal{N}$ . Therefore if  $\{\mathcal{N}^{(\alpha)}\}$  is a covering of  $f(U_7) - \mathcal{F} - f(J) - S$  by neighborhoods of the above type,  $I = \bigcup_{\alpha} I(\mathcal{N}^{(\alpha)})$  is a subvariety of  $f(U_7) - \mathcal{F} - f(J) - S$ . It is a proper subvariety since  $y \notin I$ . Therefore  $f^{-1}(I) = I^*$  is a proper subvariety of  $U_7 - V' - J - f^{-1}(S)$ , and the latter is an everywhere dense open subset of  $U_7$ . Define  $f': U_7 \rightarrow C^{s+1}$  by  $f'(y) = (f_1(y), \dots, f_s(y), g(y))$ . By LEMMA III, there exists a neighborhood  $U'_7$  of  $x$  such that  $f'(U'_7)$  is an irreducible variety at  $f'(x)$  and such that  $f'(U'_7)$  is an irreducible variety at  $f'(x)$  and such that if  $\xi$  belongs to an everywhere dense open subset of  $f'(U'_7)$ , then  $f'^{-1}(\xi) = \xi_1, \dots, \xi_\beta$ , and by assumption,  $\beta_1 \geq \beta$ . On the other hand, if  $\xi$  belongs to the everywhere dense open subset  $f'(U_7 - V' - J - f^{-1}(S) - I^*)$ ,  $f'^{-1}(\xi)$  consists of at most  $\beta - 1$  points, which is a contradiction. This establishes the lemma. (Note that since  $f'^{-1}$  of any point is discrete,

$$\dim f'(I^*) = \dim I^* < \dim V = \dim f'(U'_7).$$

<sup>(2)</sup> It is easy to see that at this point, having proved LEMMA III, we could apply Theorem 3 of [4] to obtain immediately that  $V$  is a normal general analytic space. However, we give here, for the sake of completeness, a relatively simple, elementary proof, independent of [4].

Now let  $f_{p+1}, \dots, f_s$  be chosen such that  $\beta$  (of LEMMA III)  $= 1$ . Let  $f(\mathfrak{U}_7) \cap N_1 = W$  and let  $W^\#$  be the canonical normal model<sup>(3)</sup> of  $W$ . We define a mapping  $f^\#$  of  $f^{-1}(V) \cap \mathfrak{U}_7$  into  $W^\#$ . Let  $b \in W$  and let  $f^{-1}(b) = b_1, \dots, b_\gamma$ . By LEMMA III, there exists a neighborhood  $B_i$  of  $b_i$ ,  $i = 1, \dots, \gamma$ , such that  $B_i \subset \mathfrak{U}_7$  and such that  $f(B_i)$  defines an irreducible germ of a variety  $f(B_i)_b$  at  $b$ . Since  $f$  is one-to-one on an everywhere dense subset of  $\mathfrak{U}_7$ ,  $i \neq j$  implies that  $f(B_i)$  and  $f(B_j)$  define distinct irreducible germs at  $b$ . The points of  $W^\#$  are pairs consisting of a point of  $W$  together with an irreducible branch of  $W$  at that point. We define  $f^\#(b_i)$  to be  $(b, f(B_i)_b)$ . It is clear that  $f^\#$  is one-to-one and continuous. If  $p$  is the canonical mapping of  $W^\#$  onto  $W$ ,  $p$  is continuous and  $pf^\# = f$ . If  $K$  is a compact subset of  $W^\#$ ,  $f^{\#-1}(K) = f^{-1}p(K)$ . Since  $f$  is proper, it follows that  $f^\#$  is, and therefore  $f^\#$  is a homeomorphism. Let  $\mathcal{E}$  be the subvariety of  $W$  defined by:  $\mathcal{E} = \{x \mid x \in W, W \text{ not normal at } x \text{ or } x \in f(V')\}$ .  $\dim \mathcal{E} < \dim W$ , and since  $p$  conserves dimensions,  $p^{-1}(\mathcal{E})$  is a proper subvariety  $\mathcal{E}^\#$  of  $W^\#$ . If  $g$  is an  $\mathfrak{A}$ -function on an open subset  $\mathfrak{U}$  of  $f^{-1}(W) \cap \mathfrak{U}_7$ ,  $g \circ f^{\#-1}$  is continuous on  $f^\#(\mathfrak{U})$ , and holomorphic outside  $\mathcal{E}^\#$ ; for if  $a \in V - V'$ ,  $f^\#$  is an analytic isomorphism of a neighborhood  $A$  of  $a$  with the normal model of  $f(A)$ , which is just  $p^{-1}(f(A))$ . Therefore  $g \circ f^{\#-1}$  is holomorphic on all of  $f^\#(\mathfrak{U})$ . Conversely, if  $g'$  is holomorphic on an open subset  $\mathfrak{U}^\#$  of  $W^\#$ ,  $g' \circ f^\#$  is an  $\mathfrak{A}$ -function on  $f^{\#-1}(\mathfrak{U}^\#)$  outside  $V'$  for just the same reason. Therefore, by (3) of Theorem 1,  $g' \circ f^\#$  is an  $\mathfrak{A}$ -function on all of  $f^{\#-1}(\mathfrak{U}^\#)$ . Consequently,  $f^\#$  is an analytic isomorphism of  $f^{-1}(N_1) \cap \mathfrak{U}_7$  onto  $W^\#$ . Thus there is a neighborhood of  $x$  analytically isomorphic to a normal general analytic space. The proof of Theorem 1 is complete.

**THEOREM 2.** *If  $V_{n-1}^*$  has a local basis at each point of  $V_n^*$ , and if  $V_{n-1}^*$  is a general analytic space, then  $V_n^*$  is a normal general analytic space.*

*Proof.* Let  $V = V_n^*$ ,  $V' = V_{n-1}^*$ , each supplied with the analytic structure described in [13].  $V_{n-1}^*$  is an  $\mathfrak{A}$ -variety, since, by hypothesis,  $V_{n-1}^*$  has a local basis at each  $x \in V_n^*$ . We know that  $V_n^* - V_{n-1}^* = V_n = H_n/M_n$  is a normal general analytic space ([15], exposé XII),  $V_{n-1}^*$  is a general

<sup>(3)</sup> By the canonical normal model of a complex analytic variety  $W$ , we mean the space of parameters  $\mathfrak{F}_W$  as defined in no. 9, exposé XIV of [14], supplied with the analytic structure described in no. 10 (loc. cit.). That  $\mathfrak{F}_W$  is isomorphic to a general analytic space is proved in exposés X and XI of [15], the proof depending on the coherence of the sheaf  $\hat{A}(E)$  (exposé X, [15]). The coherence of  $\hat{A}(E)$  was originally demonstrated in [10].

analytic space by hypothesis, while  $\dim V_{n-1} = \frac{1}{2}n(n-1) < \frac{1}{2}n(n+1) = \dim V_n$ , if  $n > 0$ . (1) is satisfied by virtue of the topology with which  $V_n^*$  is supplied (see [13], pp. 266-273), and (2) and (3) are satisfied by virtue of the definition of the analytic structure for  $V_n^*$  and by the properties of the operator  $\Phi$  described on pp. 278-280 of [13].<sup>(4)</sup> (4i) is satisfied by the remarks following the proof of LEMMA II, while (4ii) is satisfied by LEMMA II itself. This completes the proof.

It now remains to be proved that  $V_{n-1}^*$  has a local basis at each  $x \in V_n^*$ . It is trivial that  $V_{n-1}^*$  has a local basis at each  $x \in V_n$ . We now proceed, by a series of lemmas, to show that  $V_{n-1}^*$  has a local basis in  $V_n^*$  at each  $x \in V_{n-1}^*$ . Let  $x \in V_{n-1}^* = V_{n-1} \cup \dots \cup V_0$  and suppose  $x \in V_\rho$ ,  $1 \leq \rho \leq n-1$ . Let  $x$  be the canonical image in  $V_\rho$  of  $Z_1^0 \in F_\rho$ .  $Z_1^0$  is a  $\rho \times \rho$  matrix, reduced in the sense of Siegel, whose imaginary part  $Y_1^0$  is reduced in the sense of Minkowski. Let  $G_0$  be the finite subgroup of  $M_\rho$  leaving  $Z_1^0$  fixed. Let  $D_0$  be the set of  $\rho \times (n-\rho)$  matrices  $Z_{12}$  such that each of the entries of  $Y_{12}$  is less in absolute value than  $A$ , where  $A$  is a suitable large positive number greater than the  $\rho$ -th diagonal element of  $Y_1^0$ , and such that the real part of each entry of  $Z_{12}$  is between  $-L$  and  $+L$ ,  $L$  being a suitable large positive number  $\geq 1$ . Then  $D_0$  is a bounded and therefore relatively compact subset of  $C^{\rho(n-\rho)}$ . It is clear that if

$$\begin{pmatrix} Z_1^0 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix} \in F_n, \text{ where } Z_2 \text{ is } (n-\rho) \times (n-\rho),$$

then  $Z_{12} \in D_0$ . As in [13] we shall omit  ${}^tZ_{12}$  wherever it is redundant.

If  $T_2$  is a positive definite  $(n-\rho) \times (n-\rho)$  integral matrix and if  $T_{12}$  is a rational multiple of  $T_2$  ([13], p. 275), we let

$$\theta_{T_2, T_{12}}(Z_1, Z_{12}) = \epsilon({}^tT_{12}Z_{12}) \sum_{U_{12}} \epsilon(Z_1(T_2[{}^tU_{12}]) + {}^tU_{12}(Z_1T_{12} + {}^2Z_{12}T_2)),$$

$U_{12}$  running over all  $\rho \times (n-\rho)$  integral matrices, where now, as in the future,  $\epsilon(\ ) = e^{2\pi i S\rho(\ )}$ , and  $\epsilon'(\ ) = e^{-\pi S\rho(\ )}$ . Let  $E$  be the identity matrix (of any dimension). Then if  $T_2 = mE$ , where  $m$  is a positive integer, if  $T_{12}$  is a rational multiple of  $mE$  whose entries are even integers (briefly, an even rational multiple of  $mE$ ), and if the columns of  $T_{12}$  are  $2\tau_1, \dots, 2\tau_{n-\rho}$ , we have

<sup>(4)</sup> In fact, it is evident from Lemma 1 of [13] that if  $x \in V_n^*$ , there is a basis of neighborhoods  $\{\mathfrak{U}_\alpha\}$  of  $x$  such that each of the sets  $\mathfrak{U}_\alpha \cap V_n$  is connected; moreover, it is clear from the properties of the operator  $\Phi$  that if  $f$  is continuous on  $\mathfrak{U}_\alpha$  and analytic on  $\mathfrak{U}_\alpha \cap V_n$ , then  $f$  is analytic on  $\mathfrak{U}_\alpha$  by Satake's definition, as we see from Theorem 3 of [13].

$$\theta_{mE, T_{12}}(Z_1, Z_{12}) = \left( \prod_{i=1}^{n-\rho} \Theta_m[\tau_i](z^i; Z_1) \right) \mathcal{E}(Z_1, Z_{12}),$$

where  $z^i$  is the  $i$ -th column of  $Z_{12}$  and where  $\mathcal{E}$  is a non-vanishing exponential factor. It is known ([2], pp. 158 ff.) that for sufficiently large  $m$ ,  $\Theta_m[\tau_i](z^i; Z_1^0)$  have no common zeros as  $\tau_i$  runs over all integral  $\rho$ -vectors, and since  $\tau_1, \dots, \tau_{n-\rho}$  are independent,  $\theta_{mE, T_{12}}(Z_1^0, Z_{12})$  have no common zeros (in  $Z_{12}$ ) as  $T_{12}$  runs over all even rational multiples of  $mE$ . Fix  $m$  subject to this condition. Then by continuity we can find a connected, open neighborhood  $\mathfrak{U}_0$  of  $Z_1^0$  in  $H_\rho$  such that:

(1)  $\mathfrak{U}_0$  is stable under  $G_0$  and  $g\mathfrak{U}_0 \cap \mathfrak{U}_0$  is empty if  $g \in M_\rho - G_0$ .

(2)  $Z_1 \in \mathfrak{U}_0, \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix} \in F_n \Rightarrow Z_{12} \in D_0$ .

(3)  $Z_1 \in \mathfrak{U}_0 \Rightarrow \theta_{mE, T_{12}}(Z_1, Z_{12})$  have no common zeros as  $T_{12}$  runs over all even rational multiples of  $mE$ .

(4)  $\mathfrak{U}_0$  is bounded.

Let  $N_\rho$  be  $\mathfrak{N}$  as defined on p. 270 of [13] with  $r = \rho$ . Then  $G_0$  is naturally identifiable with a subgroup  $G_0'$  of  $M_n$  ( $G_0' \subset M_\rho^{(n)}$  in Satake's notation),  $\mathfrak{G} = G_0' \cdot N_\rho$  is a subgroup of  $M_n$ , and  $N_\rho$  is a normal subgroup of  $G_0' \cdot N_\rho$ . It is clear that we may assume of  $\mathfrak{U}_0$  by appropriate choice of  $A$  and  $L$  to begin with that

(5) If  $Z_1 \in \mathfrak{U}_0, Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix} \in F_n, g \in G_0'$ , and if  $Z' = gZ$ , then  $Z_{12}' \in D_0$ .

Let  $W' = \mathfrak{U}_0 \times D_0$ . By Lemma 2 of [13], there exists a real positive definite, symmetric,  $(n-\rho) \times (n-\rho)$  matrix  $Y_2^0$  such that if  $Y_2 > Y_2^0$ , and if  $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix}$ , where  $(Z_1, Z_{12}) \in W'$ , then  $\sigma(Z) \in \tilde{V}^{(n)}(\mathfrak{U}_0, K)$  for some  $\sigma \in \mathfrak{G}$ , where  $K$  is a fixed, sufficiently large positive number. Let

$$\mathcal{U}(Y_2^0) = \{Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix} \mid (Z_1, Z_{12}) \in \mathfrak{U}_0 \times D_0, Y_2 > Y_2^0\}.$$

Let

$$\theta_{m, T_{12}}^k(Z_1, Z_{12}) = \sum_{g \in G_0'} \theta_{mE, T_{12}}(Z_1^g, Z_{12}^g)^{k\epsilon} (mk(Z_2^g - Z_2)),$$

where if  $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix}$ , then  $gZ = \begin{pmatrix} Z_1^g & Z_{12}^g \\ & Z_2^g \end{pmatrix}$ , and where  $1 \leq k \leq \text{ord } G_0'$ .

The legitimacy of this definition, as well as the fact that  $\theta_{m, T_{12}}^k$  is bounded on  $W'$  follow from (20) of [13].

LEMMA V. The functions  $\sum_{j=1}^r z_j^k$ ,  $k=1, \dots, r$ , have no common zero other than  $z_1 = \dots = z_r = 0$ .

*Proof.* This is immediate since  $z_1, \dots, z_r$  are integral over the polynomial ring generated by  $\sum_{j=1}^r z_j^k$ ,  $k=1, \dots, r$ .

In what follows, let  $\eta(r)$  be the positive minimum of  $\max_{1 \leq k \leq r} |\sum_{j=1}^r z_j^k|$  on the compact set  $\max |z_j| = 1$ . Then if  $a > 1$ , the positive minimum of  $\max_{1 \leq k \leq r} |\sum_{j=1}^r z_j^k|$  on the compact set  $\max |z_j| = a$  is  $\geq a\eta(r)$ .

The following lemma, while practically trivial in content, serves to point up the main idea of our approach in what follows.

LEMMA VI. Let  $\{f_\mu\}$  be a sequence of complex-valued functions on a point set  $D$  such that  $\sum f_\mu$  converges absolutely at each point of  $D$ , and suppose there exists an integer  $C > 0$  and a finite number of subsets  $D_1, \dots, D_A$  of  $D$  such that

$$(i) \quad D = \bigcup_{a=1}^A D_a.$$

(ii) For each  $a=1, \dots, A$ , there exist indices  $m_1, \dots, m_{p_a} < C$  such that  $V(\{f_{m_k}\}_{k=1, \dots, p_a}) \cap D_a$  is empty, and if  $x \in D_a$  and  $f_{m_k}(x) \neq 0$ , then  $\sum_{\mu > C} |(f_\mu(x)/f_{m_k}(x))^l| \leq \frac{1}{2}\eta(C)$ ,  $l=1, \dots, C$ . Then  $V(\{\sum_{\mu} f_\mu^l\}_{l=1, \dots, C})$  is empty.

*Proof.* Let  $x \in D$ . Then  $x \in D_a$  for some  $a=1, \dots, A$ . Suppose  $f_{m_k}(x) \neq 0$ . Choose  $l$ ,  $1 \leq l \leq C$ , such that  $|\sum_{\mu \leq C} (f_\mu(x)/f_{m_k}(x))^l| \geq \eta(C)$ . It is clear that  $|\sum_{\mu} (f_\mu(x)/f_{m_k}(x))^l| \geq \eta(C) - \frac{1}{2}\eta(C) = \frac{1}{2}\eta(C) \neq 0$ . q.e.d.

LEMMA VII. There exist a finite number of  $r \times r$  unimodular matrices  $U_1, \dots, U_{A_r}$  such that for any reduced (semi-definite or positive definite)  $r \times r$  matrix  $Y$ , there is an integer  $a$ ,  $1 \leq a \leq A_r$ , such that  $\text{Sp } Y[U_a] \leq \text{Sp } Y[U]$  for all  $r \times r$  unimodular  $U$ .

*Proof.* If  $U_a$  is any unimodular matrix, the set

$$D_{U_a} = \{Y \mid \text{Sp } Y[U_a] \leq \text{Sp } Y[U], \text{ all unimodular } U\}$$

is evidently closed (and convex). Hence it suffices to prove the lemma for reduced  $Y > 0$ . We recall ([16], p. 629 and pp. 642-643) the following

facts which hold for any reduced  $r \times r$  matrix  $Y$  and any semidefinite  $r \times r$  matrix  $T$ :

(1)  $\text{Sp}(YT) = \text{Sp}(TY) \geq \lambda(r) \sum t_{ii} y_{ii}$ ,  $\lambda(r)$  being a positive number depending only on  $r$ , and  $t_{ii}$ ,  $y_{ii}$  being respectively the diagonal elements of  $T$ ,  $Y$ .

(2)  $\text{Sp } Y \leq r y_{rr}$ .

(3)  $y_{ii} \leq y_{i+1, i+1}$ ,  $|y_{ij}| = |y_{ji}| \leq \frac{1}{2} y_{ii}$ .

If  $U_a$  is any unimodular matrix, denote by  $U_a^{(k)}$  the  $k \times r$  submatrix of  $U_a$  consisting of the last  $k$  rows of  $U_a$ . We say that  $U_a$  is minimizing for  $Y$  if  $Y \in D_{U_a}$ . It is clear that there exist minimizing  $U_a$  for any positive semidefinite  $Y$ . Let  $L_k$  be the assertion that there exist at most finitely many  $U_a^{(k)}$  such that  $U_a$  is minimizing for some positive definite reduced  $Y$ . We shall prove

(a)  $L_1$ .

(b)  $L_k$  implies  $L_{k+1}$ .

Since  $L_r$  implies LEMMA VII, this will complete the proof.

*Proof of (a).* If  $U = (u_{ij})$ , it is clear from (1) that

$$\text{Sp } Y[U] \geq \lambda(r) \sum_i y_{ii} (\sum_j u_{ij}^2) \geq \lambda(r) y_{rr} \sum_j u_{rj}^2.$$

Since by (2),  $\text{Sp } Y \leq r y_{rr}$ , it follows that  $U$  can be minimizing for  $Y$  only if  $\lambda(r) \sum_j u_{rj}^2 \leq r$ , from which  $L_1$  follows.

*Proof of (b).* Suppose  $L_k$  to be true but  $L_{k+1}$  false. Then there exists a sequence of  $r \times r$  unimodular matrices  $\{U_a\}$  and a sequence  $\{Y_a\}$  of reduced  $Y_a > 0$  such that  $U_a$  is minimizing for  $Y_a$  and such that  $\sum_j (u_{r-k, j}^a)^2 \rightarrow +\infty$ , where  $U_a = (u_{ij}^a)$ . Since there are only a finite number of possibilities for  $U_a^{(k)}$ , we may assume  $U_a^{(k)}$  to be the same for each  $U_a$ . Let  $U^*$  be an integral  $(r-k) \times r$  matrix such that  $U_o = \begin{pmatrix} U^* \\ U_a^{(k)} \end{pmatrix}$  is unimodular. We let  $U_a^*$  denote the  $(r-k) \times r$  matrix composed of the first  $r-k$  rows of  $U_a$ .  $Y_a = (y_{ij}^a)$ , and write  $Y_a = \begin{pmatrix} Y_1^a & Y_{12}^a \\ {}^t Y_{12}^a & Y_2^a \end{pmatrix}$ ,  $Y_2^a$  being  $k \times k$ . Then

$$\text{Sp } Y_a[U_a] = \text{Sp } Y_1^a[U_a^*] + 2\text{Sp } {}^t U_a^* Y_{12}^a U_a^{(k)} + \text{Sp } Y_2^a[U_a^{(k)}].$$

Using (1), (3), and the fact that  $U_a^{(k)}$  is fixed, we easily obtain

$$\begin{aligned}\mathrm{Sp} Y_a[U_a] &\geq c \cdot \mathrm{Sp} Y_1^a[U_a^*] + \mathrm{Sp} Y_2^a[U_a^{(k)}] \\ &\geq c \cdot \sum_j (u_{r-k,j}^a)^2 \cdot y_{r-k,r-k}^a + \mathrm{Sp} Y_2^a[U_a^{(k)}],\end{aligned}$$

$c > 0$  being a constant independent of  $a$ . On the other hand, using (3), it is clear that

$$\mathrm{Sp} Y_a[U_0] \leq C y_{r-k,r-k}^a + \mathrm{Sp} Y_2^a[U_a^{(k)}],$$

$C > 0$  being a constant independent of  $a$ . Since  $U_a$  is minimizing for  $Y_a$ , we must therefore have

$$C y_{r-k,r-k}^a \geq c \sum_j (u_{r-k,j}^a)^2 \cdot y_{r-k,r-k}^a,$$

and since  $Y_a > 0$ , we must have  $y_{r-k,r-k}^a > 0$ ; but then the above equation is in evident contradiction to  $\sum_j (u_{r-k,j}^a)^2 \rightarrow +\infty$ . Hence (b) and therefore

LEMMA VII is proved.

LEMMA VIII. *There is a finite set  $S(n)$  of unimodular matrices, a positive definite matrix  $Y_0$ , and a positive constant  $m$  such that if  $Y$  is any reduced  $n \times n$  matrix,  $Y > kY_0$  ( $k \geq 1$ ), then there exists  $U_a$ ,  $1 \leq a \leq A_n$ , such that  $\mathrm{Sp}(Y[U] - Y[U_a] - mk^t U U) > 0$ , for any unimodular  $U \notin S$ .*

*Proof.* By the reduction theory of Minkowski [9], pp. 68-70, the reduced forms constitute a convex cone with vertex at the origin, bounded by a finite number of plane faces, and therefore having a finite number of 1-dimensional edges which span the interior of the cone. Thus there exist a finite number of reduced,  $n \times n$  matrices  $Y_1, \dots, Y_s$  such that if  $Y$  is a reduced  $n \times n$  matrix,  $Y$  can be written in the (not generally unique) form  $Y = \sum_{i=1}^s a_i Y_i$ ,  $a_i \geq 0$ . Let  $Y_1, \dots, Y_p$  be positive definite and  $Y_{p+1}, \dots, Y_s$  be semi-definite, but not positive definite. Let  $Y' = \sum_{i>p} a_i Y_i$ . Then  $Y'$  is reduced. Choose  $a$ ,  $1 \leq a \leq A_n$ , such that  $U_a$  is minimizing for  $Y'$ . Then  $\mathrm{Sp}(Y'[U] - Y'[U_a]) \geq 0$  for all unimodular  $U$ . It is evident that there is a finite set  $S$  of  $n \times n$  matrices such that if  $U$  is unimodular,  $U \notin S$ , then

$$\mathrm{Sp}(Y_i[U] - Y_i[U_a]) > \alpha \cdot \mathrm{Sp}(^t U U), \quad i = 1, \dots, p, \quad a = 1, \dots, A_n,$$

$\alpha$  being a fixed positive number. Let  $Y$  be a positive reduced matrix,  $Y > cE$ ,  $c > 0$ . Then  $Y = \sum_{i=1}^p a_i Y_i + Y'$ ,  $a_i \geq 0$ ,  $Y' \geq 0$ ,  $Y'$  not positive definite. Let

$\delta$  be the biggest diagonal entry of any  $Y_i$ ,  $i=1, \dots, p$ . Since  $Y > cE$ , we have  $a_{i'} > c/p\delta$  for some  $i'$ ,  $1 \leq i' \leq p$ . Then if  $U \notin S$ ,

$$\begin{aligned} \text{Sp}\{Y[U] - Y[U_a]\} &= \sum_{i=1}^p a_i \text{Sp}\{Y_i[U] - Y_i[U_a]\} + \text{Sp}\{Y'[U] - Y'[U_a]\} \\ &\geq \sum_{i=1}^p a_i \text{Sp}\{Y_i[U] - Y_i[U_a]\} \geq \sum_{i=1}^p a_i \alpha \cdot \text{Sp}({}^t U U) \\ &\geq (\alpha c/p\delta) \text{Sp}({}^t U U). \text{ So we can take } Y_0 = E, m = \alpha/p\delta, c = k. \end{aligned}$$

q. e. d.

Let  $\kappa$  be the minimum for  $(Z_1, Z_{12}) \in \text{clos}(\mathcal{U}_0 \times D_0)$  and  $a = 1, \dots, A_{n-p}$  of the quantity

$$\max_{k, T_{12}} |\theta_{m, T_{12}}^k(Z_1, Z_{12}U_a)|,$$

where  $k = 1, \dots, \text{ord } G'_0$ , and where  $T_{12}$  runs over all even rational multiples of  $mE$ .  $\kappa > 0$  by Lemma V applied to

$$\theta_{m, T_{12}}^{-1}(Z_1, Z_{12}U_a), \dots, \theta_{m, T_{12}}^{\text{ord } G'_0}(Z_1, Z_{12}U_a)$$

for each  $a$ . Let

$$\Phi_{m, T_{12}}^k(Z) = \sum_{U_2} \theta_{m, T_{12}}^k(Z_1, Z_{12}U_2) \epsilon(mkU_2 {}^t U_2 Z_2)$$

and

$$\Phi_{m, T_{12}}^{kl}(Z) = \sum_{U_2} \theta_{m, T_{12}}^k(Z_1, Z_{12}U_2) {}^l \epsilon(mklU_2 {}^t U_2 Z_2),$$

where  $U_2$  runs over all non-associate, unimodular  $(n-\rho) \times (n-\rho)$  matrices, and  $1 \leq l \leq C$ ,  $C$  being the number of elements in the set  $S(n-\rho)$  of LEMMA VIII. In legitimatizing the above constructions, we let  $G_{12}$  be the subgroup of  $N_\rho$  such that  $U_{12} = S_{12} = 0$ , and  $G_2$ , the subgroup of  $N_\rho$  such that  $U_2 = E$ , and then note that  $G_2$  is a normal subgroup of  $G'_0 \cdot N_\rho$ ,  $G_{12}$  is stable under the inner automorphisms of  $G'_0 \cdot N_\rho$  induced by elements of  $G'_0$ , and, as previously stated,  $N_\rho$  is a normal subgroup of  $G'_0 \cdot N_\rho$ ; then each  $\Phi_{m, T_{12}}^{kl}$  is defined in the form

$$\Phi(Z) = \sum_{\sigma \in G_{12} \bmod S_2} \left( \sum_{g \in G'_0} f(g\sigma Z) \right) {}^l,$$

where  $f$  is invariant under  $G_2$ , and  $\sigma \in G_{12} \bmod S_2$  means that for each coset of the subgroup of  $G_{12}$  for which  $S_1 = S_{12} = 0$ , and  $U = E$ , we choose a coset representative  $\sigma$ ; from this, it is evident that  $\Phi$  is invariant under  $G'_0 \cdot N$ . That the above series converge uniformly for  $Y_2 \gg 0$  follows easily from LEMMA VIII and the fact that  $\mathcal{U}_0 \times D_0$  is a bounded set.

Denote by  $D_{U_a}'$  the subset of  $F_{n-\rho}$  consisting of those  $Z_2 = X_2 + iY_2$

such that the conclusions of LEMMA VIII are satisfied for  $Y_2$  by the  $(n-\rho) \times (n-\rho)$  unimodular matrix  $U_a$ . Then if  $Y_2^0 \gg 0$ , we see that the series

$$\sum_{U_2 \neq S} \theta_{m, T_{12}}^k(Z_1, Z_{12}U_2) {}^t\epsilon(klm(Z_2[U_2] - Z_2[U_a]))$$

converges uniformly for  $Z \in \mathcal{U}(Y_2^0)$ ,  $Z_2 \in D_{U_a}'$ . Denote this series by  $S_{m, T_{12}}^{kla}$ . Then choose  $Y_2^0$  such that

$$|S_{m, T_{12}}^{kla}(Z)| < \min(\frac{1}{2}\eta(C)\kappa^C, \frac{1}{2}\eta(C))$$

for fixed large  $m$  and all  $k, l, a$ , and  $T_{12}$  an even rational multiple of  $mE$ , provided  $Z \in \mathcal{U}(Y_2^0)$  and  $Z_2 \in D_{U_a}'$ . Since  $\Phi_{m, T_{12}}^{kl}$  are invariant under  $G_0'N_\rho$  and approach 0 as  $Z \rightarrow V_{n-1}^*$ , they induce holomorphic functions  $^{(4)}\Omega_{T_{12}kl}$  on a neighborhood  $\mathfrak{U}$  of  $x$ , and it is clear that  $V_{n-1}^* \cap \mathfrak{U} \subset V(\{\Omega_{T_{12}kl}\})$ , where we may assume  $\Phi_n^{-1}(V_n \cap \mathfrak{U}) \cap F_n \subset \mathcal{U}(Y_2^0)$ . We now show that  $V(\{\Omega_{T_{12}kl}\}) \cap V_n \cap \mathfrak{U}$  is empty. For this, it suffices to show that  $V(\{\Phi_{m, T_{12}}^{kl}\}) \cap \mathcal{U}(Y_2^0)$  is empty for some given fixed  $m$ . Let  $Z \in \mathcal{U}(Y_2^0)$ , assume  $Z_2 \in D_{U_a}'$ , and let  $k, T_{12}$  be such that  $|\theta_{m, T_{12}}^k(Z_1, Z_{12}U_a)| \geq \kappa$ . Then

$$|S_{m, T_{12}}^{kl}(Z)/\theta_{m, T_{12}}^k(Z_1, Z_{12}U_a)^l| \leq \min(\frac{1}{2}\eta(C)\kappa^{C-l}, \frac{1}{2}\eta(C)\kappa^{-l}) \leq \frac{1}{2}\eta(C).$$

Therefore, by LEMMA VI,  $V(\{\Phi_{m, T_{12}}^{kl}\}) \cap \mathcal{U}(Y_2^0)$  is empty, and therefore  $V(\{\Omega_{T_{12}kl}\}) \cap \mathfrak{U} = V_{n-1}^* \cap \mathfrak{U}$ . Thus,  $V_{n-1}^*$  has a local basis at  $x$ , and we have therefore established

**THEOREM 3.**  $V_{n-1}^*$  has a local basis at each  $x \in V_n^*$ .

Combining this with Theorem 2 and the fact that  $V_0^*$  and  $V_1^*$  are general analytic spaces we obtain by an obvious induction

**THEOREM 4.**  $V_n^*$  is a general analytic space.

We shall now obtain

**THEOREM 5.** The general analytic space  $V_n^*$  is isomorphic, analytically, to a (analytically) normal projective variety.

*Proof.* Let  $m_n$  be as in LEMMA I, let  $p > 0$  be such that  $m_n | p$ , and let  $\varphi_1, \dots, \varphi_{s(p)}$  be a basis of the automorphic forms of weight  $p$  on  $V_n^*$ . We define a mapping  $\varphi^p: V_n^* \rightarrow CP^{s(p)}$ , where  $CP^s$  is the  $s$ -dimensional complex projective space, by  $\varphi^p(x) = (\varphi_1(x), \dots, \varphi_{s(p)}(x))$ , which is well defined since  $\varphi_1, \dots, \varphi_{s(p)}$  are simply cross-sections of the analytic sheaf of germs of automorphic forms of weight  $p$  on  $V_n^*$ , and by LEMMA I  $\varphi_1, \dots, \varphi_{s(p)}$  have no common zeros on  $V_n^*$ . Let  $\theta_p: V_n^* \times V_n^* \rightarrow CP^{s(p)} \times CP^{s(p)}$  be defined by

$\theta_p = \varphi^p \times \varphi^p$ . Let  $\Delta_p$  be the diagonal of  $CP^{s(p)} \times CP^{s(p)}$  and let  $D_p = \theta_p^{-1}(\Delta_p)$ .  $D_p$  is a subvariety of  $V_n^* \times V_n^*$ . If  $p_1, p_2, \dots$  is a strictly increasing sequence of positive integers such that  $m_n | p_1 | p_2 | \dots$ , it is clear that  $D_{p_1} \supseteq D_{p_2} \supseteq \dots$ , so that for sufficiently large  $p_k = N$ , we have  $D_{p_k} = D_{p_{k+1}} = \dots$ . Let  $\mathcal{D}$  be the diagonal of  $V_n^* \times V_n^*$ . It is evident by LEMMA II that  $D_N \cap (V_n \times V_n) = \mathcal{D} \cap (V_n \times V_n)$ . Hence if  $x \in V_n^*$ ,  $(\varphi^N)^{-1}\varphi^N(x) \cap V_n$  consists of at most one point. If  $n=1$ , it is obvious that  $(\varphi^N)^{-1}\varphi^N(x) \cap V_{n-1}^*$  is also discrete. If we assume, by induction, that  $(\varphi^N | V_{n-1}^*)^{-1}(y)$  is discrete for each  $y \in CP^{s(N)}$ , it follows that  $(\varphi^N)^{-1}(y)$  is discrete (i.e., 0-dimensional or empty) for all  $y \in CP^{s(N)}$ . Therefore, by a well-established property of analytic mappings,<sup>(1)</sup> we see that if  $x \in V_n^*$  and if  $\mathfrak{U}$  is a small neighborhood of  $x$ , then  $\varphi^N(\mathfrak{U})$  defines an irreducible<sup>(4)</sup> germ of a variety at  $\varphi^N(x)$ , and hence  $\varphi^N(V_n^*) = \mathcal{V}$  is a complex analytic variety in  $CP^{s(N)}$ . Moreover,  $\varphi^N$  is one-to-one on an everywhere dense set of regular points, namely, the regular points of  $V_n$ . Let  $\mathcal{V}^\#$  be the (canonical) normal model of  $\mathcal{V}$ ,  $\mathcal{V}^\# \subset CP^r$ . According to Zariski [17], algebraic normality implies analytic normality. Then to each pair consisting of a point  $y$  of  $\mathcal{V}$  and an irreducible branch  $\mathcal{V}_y$  of  $\mathcal{V}$  at  $y$ , there corresponds precisely one point of  $\mathcal{V}^\#$ . So, just as in the proof of THEOREM 1, we see that  $\varphi^N$  can be naturally lifted to an analytic isomorphism  $\varphi^\#$  of  $V_n^*$  onto  $\mathcal{V}^\#$  such that we have the following commutative diagram:

$$\begin{array}{ccc} V_n^* & \xrightarrow{\varphi^\#} & \mathcal{V}^\# \\ & \searrow \varphi & \downarrow \pi \\ & & \mathcal{V} \end{array}$$

where  $\varphi = \varphi^N$ ,  $\pi$  being the natural map. This completes the proof.

Now let  $f$  be meromorphic on  $H_n$ , let  $f$  be invariant under  $M_n$ , and let  $n \geq 2$ . Let  $x \in H_n$  and let  $M_{nx}$  be the finite subgroup of  $M_n$  leaving  $x$  fixed. Then in a small neighborhood  $\mathfrak{U}$  of  $x$ ,  $f = h_1/h_2$ , where  $h_1$  and  $h_2$  are holomorphic in  $\mathfrak{U}$ .  $h_1$  and  $h_2$  may not be invariant under  $M_{nx}$ , but in this case, we let  $h_2^*$  be the product of the translates of  $h_2$  under all elements of  $M_{nx}$  except the identity; then if we put  $g_1 = h_1 h_2^*$ ,  $g_2 = h_2 h_2^*$ ,  $g_1$  and  $g_2$  are invariant under  $M_{nx}$ , though not relatively prime at  $x$ , and  $f = g_1/g_2$  in a small neighborhood of  $x$ . Thus  $f$  naturally induces a meromorphic function  $f^*$  on  $V_n \subset V_n^*$ . Since  $n \geq 2$ ,  $\dim V_n^* - \dim V_{n-1}^* = n > 1$ . Therefore, by an easy generalization<sup>(5)</sup> of Levi's theorem on removable singularities of

<sup>(5)</sup> Levi's theorem on removable singularities of meromorphic functions, as proved

meromorphic functions [5],  $f^*$  is meromorphic on all of  $V_n^*$  and hence on  $\mathcal{V}^\#$ . Since the correspondence  $\pi$  is birational, this means [1] that  $f^*$  is a rational function of the coordinates in  $\mathcal{V}$ . From this, it follows that  $f$  is a quotient of homogeneous polynomials in  $\varphi_1, \dots, \varphi_{s(N)}$ . We have therefore,

**THEOREM 5.** *If  $f$  is a meromorphic function in  $H_n$  invariant under  $M_n$ ,  $f$  is the quotient of two automorphic forms of like weight (though not, in general, everywhere relatively prime), provided  $n \geq 2$ .*

It would be of interest to learn something of the Betti numbers, etc. of the space  $V_n^*$ , but nothing of this nature is known to the author. It would also be of interest to know the relationship between  $V_2^*$  and  $\bar{V}_2$  (see [12]). It seems quite clear that  $V_2^*$  and  $\bar{V}_2$  are birationally equivalent to each other and, by a result of Lapin [7], and others to  $CP^3$ . It is possible that  $\bar{V}_2$  is obtained from  $V_2^*$  by a monoidal transformation along the irreducible subvariety  $V_1^*$ , but the author has no proof.

It will be shown in a later paper that if  $M^\#$  is any subgroup of  $M_n$  of finite index (e.g., one of its congruence subgroups), Satake's methods yield a compactification  $V^\#$  of the quotient space  $H_n/M^\#$  such that there is a natural analytic map  $\nu: V^\# \rightarrow V_n^*$  having the following properties:

- (1)  $\nu$  is a proper map and if  $x \in V_n^*$ ,  $\nu^{-1}(x)$  is a finite set.
- (2) There is a subvariety  $A$  of  $V_n^*$  of codimension  $\geq 1$  such that  $\nu$  is a locally topological map of  $V^\# - \nu^{-1}(A)$  onto  $V_n^* - A$ .
- (3) If  $x \in A$  and  $x' \in \nu^{-1}(x)$ ,  $x'$  has a basis of neighborhoods  $\{U_\alpha\}$  such that  $U_\alpha \cap (V^\# - \nu^{-1}(A))$  is non-empty and connected.

Then combining this with the results we have already obtained, or, alternatively, using THEOREM 3 of this paper in conjunction with Theorem 3 of [4] and the remark following Theorem 1 of [3] we shall see that  $V^\#$  is (analytically isomorphic to) a normal projective variety. We shall also obtain the analog of Theorem 5.

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in [5], applies only to meromorphic functions in an open domain of  $C^n$ . However, it is not hard to prove that if  $f$  is a meromorphic function on a normal general analytic space  $V$  of pure dimension  $n$  except possibly on an imbedded subvariety of (complex) dimension  $\leq n - 2$ , then  $f$  is meromorphic on all of  $V$ . We omit the proof here.

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# EIGENFUNCTION EXPANSIONS FOR NON-SYMMETRIC PARTIAL DIFFERENTIAL OPERATORS, I.\*

By FELIX E. BROWDER.

**Introduction.** Let  $L$  and  $B$  be two differential operators defined on a domain  $G$  of Euclidean  $n$ -space  $E^n$  ( $n \geq 1$ ). By an eigenfunction expansion for  $L$  with respect to  $B$  on  $G$  is meant, roughly speaking, a sequence of measures  $\{m_j\}$  on the complex plane and a sequence of functions  $\{u_j(x, \xi)\}$  defined for  $x$  in  $G$  and  $\xi$  complex, such that:

$$(a) \quad (L - \bar{\xi}B)u_j(x, \xi) = 0,$$

$$(b) \quad \text{For a suitably smooth function } v \text{ in a restricted class, } U_j(v)(\xi) = \int_G (Bv)(x) \overline{u_j(x, \xi)} dx \text{ is defined except on a set of } m_j\text{-measure zero and lies in } L^2(m_j) \text{ while } (Bv, v) = \sum_j \int |U_j(v)(\xi)|^2 dm_j(\xi),$$

$$(c) \quad v(x) = \sum_j \int U_j(v)(\xi) u_j(x, \xi) dm_j(\xi).$$

The general theory of eigenfunction expansions for singular self-adjoint ordinary differential operators was initiated by Weyl in 1910 with his treatment of the second-order case, and has been intensively developed since that time.<sup>1</sup> Two simplifying circumstances are present in the case of ordinary differential operators as contrasted with partial differential operators proper ( $n \geq 2$ ). The first and most important is that the family of solutions of a linear ordinary differential equation is of finite linear dimension, so that many questions about it can be reduced to problems in finite-dimensional linear algebra. Second, from the point of view of the theory of partial differential operators, an ordinary differential equation is both elliptic, so that all its solutions are smooth functions, and hyperbolic, so that the Cauchy problem has a stable solution. It is not surprising, therefore, that it has been possible to obtain eigenfunction expansions for general classes of

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<sup>1</sup> An inclusive account of the theory for ordinary differential operators is given by Coddington and Levinson [12].

ordinary differential operators by the methods of classical analysis, while in the case of partial differential operators proper, it has been necessary to have recourse to the tools furnished by functional analysis.

Except for the work of Carleman in 1934 on the Schroedinger equation [10], the development of the theory of eigenfunction expansions for partial differential operators has taken place only in recent years. In 1953, using an argument already set forth in general terms by Mautner [22] and based upon the Von Neumann-Stone spectral theory for unbounded self-adjoint operators, Gårding ([14], [15]) and the writer [7] independently proved the existence of an eigenfunction expansion corresponding to each self-adjoint realization of a formally self-adjoint elliptic partial differential operator.<sup>2</sup> For differential operators with constant coefficients, an extension to hypoelliptic operators was given in 1955 by Hormander [20]. In 1955, Gelfand and Kostyucenko announced in [17] the existence of an eigenfunction expansion for each self-adjoint realization of a formally self-adjoint partial differential operator without any requirement of ellipticity. Basing their proof upon a Banach space differentiation theorem due to Gelfand [16], they obtained eigenfunctions which were in the general case distributions of some prescribed order. Using the same differentiation theorem in Hilbert space (where it was first proved by G. Birkhoff [4]), the present writer in [8] extended their result to obtain eigenfunction expansions for formally self-adjoint partial differential operators of arbitrary type without any assumption on the existence of self-adjoint realizations.<sup>3</sup>

In the present paper, we establish more generally an eigenfunction expansion theorem for any subnormal realization  $A$  of a partial differential operator  $L$  in  $L^2(G)$ . The class of (possibly unbounded) subnormal operators which includes all symmetric operators is defined by one of the two equivalent conditions:  $A$  is subnormal if it can be extended to a normal operator in a larger Hilbert space, or if it can be written in the form  $A = \int \xi dF(\xi)$ , where  $F$  is a positive operator-valued measure on the Borel sets of the complex plane  $C^1$  with  $\|Au\|^2 = \int |\xi|^2 d[F_\xi u, u]$ . The proof for subnormal operators is based upon the Birkhoff-Gelfand differentiation theorem, and the eigenfunctions obtained are distribution solutions of the equation  $(L - \xi)u$

<sup>2</sup> See also the paper of Povsner [26] on the Schroedinger equation where related methods are applied. Another proof for self-adjoint realizations of elliptic operators was later given by Bade and Schwartz [2].

<sup>3</sup> The proof of the result announced in [17] was published by Gelfand and Silov [18]. Another proof has been given by Berezanski [3]. At the same time as the writer's paper [8], results in the symmetric case for ordinary differential operators were obtained by Coddington [11].

$= 0$ . In particular, by applying results on the smoothness of solutions of hypoelliptic equations, one obtains for such operators eigenfunction expansions in the classical sense.

If  $B$  is a positive partial differential operator on  $G$ , we also obtain a proof of an expansion theorem in solutions of  $(L - \bar{\zeta}B)u = 0$  under the assumption that  $L$  has a subnormal realization in a Hilbert space  $H_B$  defined by the operator  $B$ . In this case (as in a corresponding argument indicated for self-adjoint operators on  $E^n$  by Gelfand and Kostyucenko [17], [18]), the eigenfunctions obtained are no longer distributions in general, but, rather, linear functionals of a more general type, and it becomes impossible to rely upon results on the smoothness of distribution solutions of hypoelliptic equations. We shall treat this difficulty in the following paper of the present series.

Section 1 discusses subnormal operators, their generalized spectral resolutions, and various operator-theoretical results applied in the subsequent discussion. In Section 2, the Hilbert spaces which we shall use are defined and their properties established. In Section 3, the proof is given of the existence of an eigenfunction expansion corresponding to each generalized spectral resolution of a subnormal operator. Section 4 treats of hypoelliptic operators.

**Section 1.** Let  $H$  be a Hilbert space with inner product  $[u, v]$ ,  $T$  a (possibly unbounded) linear operator with domain  $D_T$  dense in  $H$  and range  $R_T$  in  $H$ ,  $T^*$  the adjoint of  $T$ . Since we shall assume that  $T$  has a closure in  $H$ ,  $T^*$  is a closed, densely defined linear operator in  $H$ .  $T$  is said to be normal if  $TT^* = T^*T$ , with the usual convention for the multiplication of unbounded operators. By the spectral theorem, every normal operator  $T$  in  $H$  has a resolution of the identity  $\{E_\zeta\}$ , i. e., a function  $E$  from the  $\sigma$ -algebra  $\Omega$  of Borel sets of  $C^1$ , the complex plane, to the projections of  $H$  such that: (1)  $E(C^1) = I$ , the identity operator; for each  $u$  in  $H$ ,  $[E(S)u, u]$  is a positive measure on  $\Omega$ ; (2)  $u \in D_T$  if and only if  $\int |\zeta|^2 d[E_\zeta u, u] < \infty$ ; If  $u \in D_T, v \in H$ ,  $[Tu, v] = \int \zeta d[E_\zeta u, v]$ ; (3)  $E(S_1 \cap S_2) = E(S_1)E(S_2)$  for  $S_1, S_2$  in  $\Omega$ .

**Definition 1.**  $T$  is subnormal if there exists a Hilbert space  $H_1$  containing  $H$  as a closed subspace and a normal operator  $T_1$  in  $H_1$  such that  $T \subseteq T_1$ .

For bounded operators, Definition 1 coincides with that given by Halmos

in [19]. Halmos has given a characterization of bounded subnormal operators which in the simplified expression given to it by Bram [5] is the following: the bounded operator  $T$  is subnormal if for every finite family of elements  $u_1, \dots, u_r$  of  $H$ ,

$$(1) \quad \sum_{j,k=1}^r [T^j u_k, T^k u_j] \geq 0.$$

In the more general case of unbounded operators, every operator  $T$  admitting a normal extension in  $H$  itself is trivially subnormal. By a theorem of Neumark ([1]), every symmetric operator has a self-adjoint extension in a larger Hilbert space and therefore is subnormal.

*Remark.* We shall call  $T$  formally normal if  $D_T \subset D_{T^*}$  and  $\|Tu\| = \|T^*u\|$  for  $u$  in  $D_T$ . One might conjecture that every formally normal operator  $T$  is subnormal. The truth or falsity of this conjecture does not seem easy to establish, especially in view of the meagreness of studies as to conditions for the extendability of a formally normal operator to a normal operator in  $H$  itself. (See [21], [24], [25]). If it were true that every formally normal operator were subnormal, then it would follow from the argument employed in the case of bounded operators that if  $R_T \subset D_T$  and (1) holds for every sequence chosen from  $D_T$ , then the unbounded operator  $T$  would be subnormal.

*Definition 2.* Let  $T$  be a subnormal operator in  $H$ ,  $T_1$  a normal extension of  $T$  to a Hilbert space  $H_1$  containing  $H$  as a closed subspace,  $P$  the projection of  $H_1$  on  $H$ ,  $\{E_\zeta\}$  the resolution of the identity of  $T_1$  in  $H_1$ . Then the generalized spectral resolution of  $T$  corresponding to  $T_1$  is the function  $F$  from the  $\sigma$ -algebra  $\Omega$  of Borel subsets of  $C^1$  to  $L(H)$  (the set of bounded linear operators on  $H$ ) defined by  $F(S) = PE(S)$ .

We note some of the properties of such a generalized spectral resolution:

- (a)  $F(C^1) = I$ ; for  $u \in H$ ,  $[F(S)u, u]$  is a positive measure on  $\Omega$ .
- (b) If  $u \in D_T$ ,  $\int_{C^1} |\zeta|^2 d[F_\zeta u, u] = \|Tu\|^2$ .
- (c) For  $u \in D_T$ ,  $v \in H$ ,  $[Tu, v] = \int_{C^1} \zeta d[F_\zeta u, v]$ .

**LEMMA 1.** Let  $T$  be a linear operator in  $H$ ,  $F$  a function from  $\Omega$  to  $L(H)$  for which (a), (b), and (c) are true. Then  $T$  is subnormal and  $\{F_\zeta\}$  is the generalized spectral resolution of  $T$  corresponding to some normal extension  $T_1$  of  $T$ .

*Proof.* It follows from (a) that for each  $S$  in  $\Omega$ ,  $F(S)$  is self-adjoint. By a theorem of Neumark (cf. [23], p. 28), there exists a Hilbert space  $H_1$  containing  $H$  as a closed subspace and a resolution of the identity  $\{E_\zeta\}$  in  $H_1$  such that, if  $P$  is the projection of  $H_1$  on  $H$ , then  $PE(S) = F(S)$  for all  $S$  in  $\Omega$ . Corresponding to this resolution of the identity, there is a normal operator  $T_1$  in  $H_1$  with  $D_{T_1} = \{w \mid \int_{C^1} |\xi|^2 d[E_\zeta w, w] < \infty\}$  while for  $w \in D_{T_1}$ ,  $v \in H_1$

$$[T_1 w, v] = \int_{C^1} \xi d[E_\zeta w, v].$$

For  $u \in H$ ,  $[E_\zeta u, u] = [F_\zeta u, u]$ , and (b) implies that  $D_T$  is contained in  $D_{T_1}$ . For  $u \in D_T$ ,  $v \in H$  the chain of equalities

$$[E_\zeta u, v] = [Pu, E_\zeta v] = [u, F_\zeta v] = [F_\zeta u, v]$$

implies that  $[Tu, v] = \int_{C^1} \xi d[F_\zeta u, v] = \int_{C^1} \xi d[E_\zeta u, v] = [T_1 u, v]$ . Thus  $T \subseteq PT$ , while (b) implies  $T \subseteq T_1$ .

Let  $T$  be a subnormal operator in  $H$ ,  $\{F_\zeta\}$  a generalized spectral resolution corresponding to  $T$ . Let  $T_1$  be a normal extension of  $T$  in a Hilbert space  $H_1$  with projection  $P$  on its closed subspace  $H$ ,  $\{E_\zeta\}$  the spectral resolution of  $T_1$  with the property that  $PE(S) = F(S)$ . If  $g$  is an arbitrary element of  $H_1$ , we define  $H_1(g)$ , the cyclic subspace generated by  $g$  with respect to  $\{E_\zeta\}$ , to be the subspace of  $H_1$  spanned by  $E(S)g$  with  $S$  ranging over the Borel sets of  $C^1$ . It follows by a familiar argument that if  $h$  lies in the orthogonal complement of  $H_1(g)$ , so does  $H_1(h)$ , and we may select an indexed set  $\{g_\beta; \beta \in Q\}$  in  $H_1 = \sum_{\beta \in Q} \oplus H_1(g_\beta)$ .

LEMMA 2. If  $H$  is separable,  $H_1$  and  $T_1$  may be chosen so that  $Q$  is countable.

*Proof.* Let  $Q'$  be any subset of  $Q$ ,  $H' = \sum_{\beta \in Q'} \oplus H_1(g_\beta)$ ,  $T'$  the restriction of  $T_1$  to  $H' \cap D(T_1)$ . Then the range of  $T'$  is contained in  $H'$ ,  $T'$  is a normal operator with dense domain in  $H'$ , and if  $\{E'_\zeta\}$  is its spectral resolution,  $E'_\zeta$  is the restriction of  $E_\zeta$  to  $H'$ .

Let  $\{h_k\}$  be a dense sequence in the separable Hilbert space  $H$ . For each  $k$ ,  $h_k$  has a non-zero projection on  $H_1(g_\beta)$  only for  $\beta$  in a countable set  $Q_k$ . If we set  $Q' = \bigcup_k Q_k$ ,  $Q'$  is countable, and  $H \subset \sum_{\beta \in Q'} \oplus H_1(g_\beta) = H'$ . Further, for  $\beta \in Q'$ , the cyclic subspace generated by  $g_\beta$  with respect to  $\{E'_\zeta\}$  is equal to  $H_1(g_\beta)$  while, if  $P'$  is the projection of  $H'$  on  $H$ ,  $F(S) = P'E'(S)$ .  $H'$  and  $T'$  are the sought extensions. If  $H_1 = H$ , then  $H' = H$ .

To conclude the present section, we establish the following lemma which is applied in the proof of our main theorem.

**LEMMA 3.** *Let  $m$  be a positive finite measure on the  $\sigma$ -algebra  $\Omega$  of Borel set of  $C^1$ ,  $h$  a vector-valued measure on  $\Omega$  with values in a Hilbert space  $H_0$ . Suppose that  $h$  is weakly absolutely continuous with respect to  $m$  and that there exists a constant  $K$  such that for any finite pairwise disjoint family  $S_k$  of sets of  $\Omega$ ,  $\sum_k \|h(S_k)\| \leq K$ . Then there exists an  $m$ -integrable function  $u$  from  $C^1$  to  $H_0$  such that, for  $S \in \Omega$ ,  $\psi \in H_0$ ,*

$$[h(S), \psi] = \int_S [u(\xi), \psi] dm(\xi).$$

*Proof.* To reduce the lemma to the theorem of Birkhoff [4] and Gelfand ([16], pp. 260-266), of which it is a variant, we apply the method of Riesz-Sz. Nagy ([28], Sec. 60, pp. 130-131) to obtain a mapping  $f$  of a Borel set  $Z$  of the unit interval  $J$  onto a Borel set  $C'$  of  $C^1$  such that, if  $m_L$  denotes Lebesgue linear measure,  $m_L(Z) = 1$ ,  $m(C') = m(C^1)$ , while  $m_L(f^{-1}(S)) = m(S)$  for each Borel subset  $S$  of  $C'$ . The mapping  $f$ , as well as  $f^{-1}$ , carries Borel sets into Borel sets, and  $f$  is one-to-one on the complement of a countable, pairwise disjoint family of intervals  $\{Z_j\}$ , each of which is mapped by  $f$  on a point  $x_j$  of  $C^1$  having a non-zero measure with respect to  $m$ .

Let  $Z_1 = Z - \bigcup_j Z_j$ ,  $S$  a Borel set of  $J$ . Let  $h_1$  be defined by

$$h_1(S) = h(f(S \cap Z_1)) + \sum_j h(x_j)m(S \cap Z_j).$$

It is easy to verify that  $h_1$  is a vector-valued measure on the Borel sets of  $J$  which is weakly absolutely continuous with respect to  $m_L$  and satisfies the bounded-variation condition of the hypothesis for  $h$ . By the theorem of Birkhoff-Gelfand,  $h_1$  has a strong derivative a.e. on  $J$ , and if we set  $dh_1/dt = u(t)$ ,  $u$  is strongly  $m_L$ -integrable and  $h_1(S) = \int_S u(t) dm_L(t)$ . On each  $Z_j$ ,  $u(t)$  is the constant  $h(x_j)$ . Thus, if  $f(t) = x$ ,  $u(t)$  is uniquely defined by  $x$  on  $C'$ . The function  $u(x)$  thus defined (and set equal to zero outside of  $C'$ ) is  $m$ -measurable, and by the transformation formulae for integrals,

$$[h(S), \psi] = \int_S [u(\xi), \psi] dm(\xi)$$

for each  $\psi \in H_0$ ,  $S$  in  $\Omega$ .

**Section 2.** Let  $G$  be an open set in the  $n$ -dimensional Euclidean space

$E^n$  with points  $x = (x_1, \dots, x_n)$ ,  $C_c^\infty(G)$  the family of complex-valued infinitely differentiable functions with compact support in  $G$ ,  $L^2(G)$  the Hilbert space of equivalence classes of complex-valued square-summable functions with inner product  $(u, v) = \int u(x) \bar{v}(x) dx$  ( $\bar{v}$  the complex conjugate of  $v$ , integration with respect to Lebesgue  $n$ -measure). If  $\alpha$  is a multi-index  $(\alpha_1, \dots, \alpha_r)$  with  $1 \leq \alpha_j \leq n$ , we shall let  $|\alpha| = r$  while  $D^\alpha = \partial^r / \partial x_{\alpha_1} \cdots \partial x_{\alpha_r}$ . (For the empty set of indices  $\phi$ , we let  $D^\phi = I$ .)

Let  $C^j(G) = \{f \mid f \text{ is defined in } G, D^\alpha f \text{ exists and is continuous in } G \text{ for } |\alpha| \leq j\}$ .

Let  $B$  be a linear operator with  $C_c^\infty(G)$  as its domain and its range contained in the linear space of locally square-summable functions on  $G$ . We shall assume that there exist  $p \in C^0(G)$ ,  $N \in C^r(G)$  for some  $r \geq 0$  with  $p(x) > 0$  in  $G$ ,  $N(x) > 0$  for  $x$  in  $G$ , such that:

$$(2) \quad \text{For } \phi, \psi \in C_c^\infty(G), (B\phi, \psi) = (\phi, B\psi), (B\phi, \phi) \geq (p\phi, \phi);$$

$$(3) \quad (B\phi, \phi) \leq \sum_{|\alpha| \leq r} (ND^\alpha \phi, D^\alpha \phi), \quad \phi \in C_c^\infty(G);$$

$$(4) \quad (p^{-1}B\phi, B\phi) < \infty \text{ for } \phi \in C_c^\infty(G).$$

(Note that if  $B$  is a differential operator, (4) is automatically satisfied.)

On  $C_c^\infty(G)$ , we define the  $B$ -inner product by,

$$(5) \quad [\phi, \psi] = (B\phi, \psi); \phi, \psi \in C_c^\infty(G).$$

The family of functions  $C_c^\infty(G)$  with its ordinary linear structure and the inner product  $[\ , \ ]$  is a pre-Hilbert space.

**Definition 3.**  $H$  is the Hilbert space which is the completion of  $C_c^\infty(G)$  with respect to the norm  $\|u\|_H = [u, u]^{1/2}$ .

Let  $L^2(p)$  be the Hilbert space of equivalence classes of locally square-summable functions  $u$  for which  $(pu, u) < \infty$  with inner product  $(pu, v)$  and norm  $(pu, u)^{1/2}$ . By (2), every Cauchy sequence from  $C_c^\infty(G)$  in  $H$ -norm is also a Cauchy sequence in  $L^2(p)$ . Thus, there exists a continuous mapping  $J$  of  $H$  into  $L^2(p)$  which is the identity mapping on  $C_c^\infty(G)$ . We assert that  $J$  is one-to-one; for if  $\{\phi_s\}$  is a sequence from  $C_c^\infty(G)$ , converging to  $h$  in  $H$  with  $(p\phi_s, \phi_s) \rightarrow \infty$ , and if  $\psi \in C_c^\infty(G)$ , we have  $[h, \psi] = \lim_s [\phi_s, \psi] = \lim_s (\phi_s, B\psi)$ . Now  $|(\phi_s, B\psi)|^2 \leq (p\phi_s, \phi_s)(p^{-1}B\psi, B\psi) \rightarrow 0$  as  $s \rightarrow \infty$  by (4).

But then  $[h, \psi] = 0$  for all  $\psi$  in  $C_c^\infty(G)$  and  $h = 0$  since  $C_c^\infty(G)$  is dense in  $H$ . Thus  $H$  may be identified with a linear subset of  $L^2(p)$ . If  $p_1$  is any other positive continuous function on  $G$  with  $p_1(x) \leq p(x)$  for  $x$  in  $G$ , then

obviously  $L^2(p)$  is a linear subset of  $L^2(p_1)$  and  $H$  may be identified with a subset of  $L^2(p_1)$  and  $H$  may be identified with a subset of  $L^2(p_1)$ .

Given  $N \in C^r(G)$ ,  $N(x) > 0$  in  $G$ ,  $r \geq 0$ , we define the  $(N, r)$ -inner product on  $C_c^\infty(G)$  by  $[u, v]_{N,r} = \sum_{|\alpha| \leq r} (ND^\alpha u, D^\alpha v)$ . If the differential operator  $B_1$  is defined by  $B_1 u = \sum_{|\alpha| \leq r} (-1)^{|\alpha|} D^\alpha (N(x) D^\alpha u)$ , then  $[u, v]_{N,r} = (B_1 u, v)$  for  $u, v \in C_c^\infty(G)$ , and  $B_1$  satisfies the conditions (2), (3), and (4) on  $B$  with the pair of functions  $(p, N)$  replaced by  $(N, N)$ . If we complete  $C_c^\infty(G)$  with respect to this inner product to a Hilbert space  $H_{N,r}$ , there will be a continuous one-to-one mapping  $J_1$  of  $H_{N,r}$  into  $L^2(p_1)$ , where  $p_1(x) = \min\{p(x), N(x)\}$ , which is the identity mapping on  $C_c^\infty(G)$ . Furthermore, (3) implies that there exists a continuous mapping  $J_2$  of  $H_{N,r}$  into  $H$  which is the identity mapping on  $C_c^\infty(G)$ . Let  $J_3$  be the imbedding mapping of  $H$  into  $L^2(p_1)$ . Then  $J_1 = J_3 J_2$ , and since  $J_1$  and  $J_3$  are one-to-one,  $J_2$  must be one-to-one and  $H_{N,r}$  may be identified with a linear subset of  $H$ .

As a special case of the spaces  $H_{N,r}$ , we may choose  $N(x) \equiv 1$ , and designate the corresponding Hilbert spaces and inner products by  $H_r$  and  $[u, v]_r$ . If  $G$  is bounded,  $r_1 > r$ , by a well-known lemma of Rellich (e.g., [6], p. 32), it follows that the imbedding map  $J_{r,r_1}$  of  $H_{r_1}$  into  $H_r$  is compact. For bounded  $G$ , the inner product  $[u, v]'_r = \sum_{|\alpha|=r} (D^\alpha u, D^\alpha v)$  yields a norm on  $H_r$  equivalent to the  $r$ -norm. We designate the Hilbert space with the inner product  $[ , ]'_r$  on  $H_r$  by  $H'_r$ .

LEMMA 4. Suppose  $f \in C^0(G)$  with  $f(x) > 0$  for  $x$  in  $G$ ,  $r \geq 0$ . Then there exists  $g \in C(G)$  with  $g(x) > 0$  in  $G$ , such that

$$|D^\alpha g(x)| \leq f(x)$$

for all  $x$  in  $G$  and all  $|\alpha| \leq r$ .

The proof of Lemma 4 follows from a standard argument using an infinitely differentiable partition of unity corresponding to a suitably fine locally finite covering of  $G$ .

LEMMA 5. Let  $G$  be a bounded open set in  $E^n$ ,  $r \geq 0$ ,  $r_1 = r + n + 1$ . Then there exists a unique compact self-adjoint positive linear operator  $C_r$  on  $H'_{r_1}$ , such that

$$[u, v]_r = [C_r u, v]'_{r_1}$$

for  $u, v \in H'_{r_1}$ . Further, if  $\{\lambda_{k,r}\}$  is the sequence of eigenvalues of  $C_r$  in non-

increasing order corresponding to a complete orthonormal set in  $H'_{r_1}$ , we have  $\sum_{k=1}^{\infty} \{\lambda_{k,r}\}^{\frac{1}{3}} < \infty$ .

*Proof.* For fixed  $u$  in  $H'_{r_1}$ ,  $[u, v]'_r$  is a continuous conjugate-linear functional of  $v$  on  $H'_{r_1}$ . Hence, there exists a unique element  $w$  in  $H'_{r_1}$  such that  $[u, v]'_r = [w, v]'_{r_1}$  for all  $v$  in  $H'_{r_1}$ . If we set  $C_r u = w$ , it follows that  $\|C_r u\|'_{r_1} \leq c_r \|u\|'_r$ , and by the lemma of Rellich mentioned above,  $C_r$  is a compact linear operator on  $H'_{r_1}$ . Further,  $C_r$  is self-adjoint and positive on  $H'_{r_1}$  since  $[C_r u, u]'_{r_1} = [u, u]'_r > 0$  for  $u \neq 0$  in  $H'_{r_1}$ .

If  $r = 0$ , it follows from results on the asymptotic distribution of eigenvalues for the Dirichlet problem for a polyharmonic equation [13] that

$$\lambda_{k,0} \sim C(n, G) k^{-2-2/n}, \text{ and therefore } \sum_{k=1}^{\infty} \lambda_{k,0}^{\frac{1}{3}} < \infty.$$

For  $r > 0$ , we define  $W = \sum_{|\alpha|=r} \oplus H^{\alpha}_{n+1}$ , where, for each  $\alpha$ ,  $H^{\alpha}_{n+1}$  is a copy of  $H'_{n+1}$ . If  $v = \{v_{\alpha}\}$  is an element of  $W$ , let  $\|v\|_0^2 = \sum_{|\alpha|=r} \|v_{\alpha}\|_0^2$ . We define a mapping  $T$  of  $H'_{r_1}$  into  $W$  by setting  $Tu = \{u_{\alpha}\}$ , where the  $\alpha$ -th coordinate  $u_{\alpha} = D^{\alpha}u$ . Then  $T$  is a one-to-one mapping of  $H'_{r_1}$  into  $W$ , and there exists a constant  $c_1 > 0$  such that  $\|Tu\|_W^2 \leq c_1 [\|u\|'_r]^2$ . Moreover,  $\|Tu\|_0^2 = [\|u\|'_r]^2$ . By the minimax principle for eigenvalues of compact self-adjoint operators on a Hilbert space ([27], p. 235),

$$\lambda_{k,r} = \max_{S_k \subset H'_{r_1}} \{\min\{[\|u\|'_r]^2 / [\|u\|'_{r_1}]^2; u \in S_k\}\},$$

where the maximum is taken over all subspaces  $S_k$  of linear dimension  $k$  in  $H'_{r_1}$ . For each  $u$  in  $H'_{r_1}$ ,  $c_1 [\|u\|'_r]^2 / [\|u\|'_{r_1}]^2 \leq \|Tu\|_0^2 / \|Tu\|_W^2$ . Since  $T$  is one-to-one, it follows that  $c_1 \lambda_{k,r} \leq \xi_{k,r}$ , where

$$\xi_{k,r} = \max_{S'_k \subset W} \{\min\{\|v\|_0^2 / \|v\|_W^2; v \in S'_k\}\},$$

where the  $S'_k$  range over the  $k$ -dimensional subspaces of  $W$ .

If  $v = \{v_{\alpha}\}$ ,  $\|v\|_0^2 = \sum_{|\alpha|=r} [C_0 v_{\alpha}, v_{\alpha}]'_{n+1}$ . Thus  $\{\xi_{k,r}\}$  is the sequence of eigenvalues of the operator  $C'v = \{C_0 v_{\alpha}\}$ . The only admissible values for the  $\xi_{k,r}$  are the values taken on by various  $\lambda_{j,0}$  and if  $\xi_{k,r} = \lambda_{j,0}$ , the multiplicity of  $\xi_{k,r}$  as an eigenvalue of  $C'$  is  $2^r(r!)^{-1}$  times the corresponding multiplicity for  $\lambda_{j,0}$ . Thus  $\xi_{2^r(r!)^{-1}k,r} = \lambda_{k,0}$ , and as a consequence,

$$\xi_{k,r} \sim c'(n, r, G) k^{-2-2/n}.$$

Finally,  $\sum_{k=1}^{\infty} \lambda_{k,r}^{\frac{1}{3}} \leq c_1^{-\frac{1}{3}} \sum_{k=1}^{\infty} \xi_{k,r}^{\frac{1}{3}} < \infty$ .

**THEOREM 1.** *Let  $G$  be an open set of  $E^n$ ,  $r \geq 0$ ,  $r_1 = r + n + 1$ ,  $N(x) > 0$  with  $N \in C^r(G)$ . Then there exist  $M \in C^\infty(G)$  with  $M(x) \geq N(x)$  for  $x$  in  $G$  and a complete orthonormal sequence  $\{v_k\}$  in  $H_{M,r_1}$  such that*

$$\sum_{k=1}^{\infty} \|u_k\|_{N,r} < \infty.$$

*Proof.* We remark first that if the theorem is true for a given domain  $G$ , it is also true for any domain  $G_1$  which is mapped by a  $C^\infty$ -homeomorphism onto  $G$ . Since there exists a  $C^\infty$ -homeomorphism of  $E^n$  onto its open unit ball, i.e., the set  $|x| < 1$ , we may assume without loss of generality that  $G$  is bounded.

Since  $G$  is assumed bounded, the lemma of Rellich cited above is valid for  $G$ . On the other hand, by Lemma 4, there exists  $g \in C^\infty(G)$  with  $g(x) > 0$  for all  $x$  in  $G$ , such that  $|D^\alpha g(x)| \leq N(x)^{-1}$  for  $|\alpha| \leq r$ ,  $x \in G$ . Let  $\phi \in C_c^\infty(G)$ . If we set  $\psi = g^{-1}\phi$ , we have,

$$\|\phi\|_{N,r}^2 = \sum_{|\alpha| \leq r} (N^{\frac{1}{2}} D^\alpha(g\psi), N^{\frac{1}{2}} D^\alpha(g\psi)).$$

By the Leibniz formula for the differentiation of products,

$$D^\alpha(g\psi) = \sum_{\pi+\beta=\alpha} c_{\pi\beta} D^\pi(g) D^\beta(\psi),$$

with  $c_{\pi\beta}$  positive constants. It follows that  $|N^{\frac{1}{2}} D^\alpha(g\psi)| \leq \sum_{\pi,\beta} c_{\pi\beta} |D^\beta\psi|$ . Thus there exists  $c > 0$  such that

$$(6) \quad \|\phi\|_{N,r}^2 \leq c \|\psi\|_{M,r}^2.$$

In addition, we can obviously find a function  $M$  in  $C^\infty(G)$  with  $M(x) > 0$  for  $x \in G$ , such that

$$(7) \quad \|\psi\|_{M,r_1}^2 = \sum_{|\beta| \leq r_1} (D^\beta(g^{-1}\phi), D^\beta(g^{-1}\phi)) \leq \sum_{|\beta| \leq r_1} (MD^\beta\phi, D^\beta\phi) = \|\phi\|_{M,r}^2$$

for all  $\phi \in C^\infty(G)$ .

From (7), it follows that if a sequence  $\{\phi_k\}$  from  $C_c^\infty(G)$  is bounded in  $(M, r_1)$ -norm, the corresponding sequence  $\{\psi_k\}$  with  $\psi_k = g^{-1}\phi_k$ , is bounded in  $r_1$ -norm. By the Rellich lemma, there exists a subsequence  $\{\psi_{k_j}\}$  which is a Cauchy sequence in  $r$ -norm. By (6), the corresponding subsequence  $\{\phi_{k_j}\}$  is a Cauchy sequence in  $(N, r)$ -norm. Thus  $[u, v]_{N,r}$  is a completely continuous form on  $H_{M,r_1}$ , and there exists a compact linear self-adjoint operator  $C$  on  $H_{M,r_1}$  such that  $[u, v]_{N,r} = [Cu, v]_{M,r_1}$  for all  $v$  in  $H_{M,r_1}$ .

We choose the sequence  $\{v_k\}$  in  $H_{M,r_1}$  to be a complete orthonormal sequence of eigenfunctions of the compact positive operator  $C$  with a corresponding sequence of eigenvalues  $\{\lambda_k\}$ , which we may assume to be arranged

in a non-increasing order. If  $Cv_k = \lambda_k v_k$ , we have

$$\|v_k\|_{N,r}^2 = [v_k, v_k]_{N,r} = [Cv_k, v_k]_{M,r_1} = \lambda_k.$$

To complete the proof of the theorem, we must show that  $\sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2}} < \infty$ .

If we apply the minimax principle, as in the proof of Lemma 3, we see that the  $k$ -th eigenvalue will not decrease if we increase the norm  $\|\phi\|_{N,r}^2$  to the larger norm  $c\|\psi\|_r^2$ , with  $\psi = g^{-1}\phi$ , or to the even larger norm  $c_1\{\|\psi\|_r'\}^2$ . Similarly, the  $k$ -th eigenvalue will not decrease if we replace the norm  $\|\phi\|_{M,r}^2$  in the denominator of the quotient by the smaller norm  $\{\|\psi\|_r'\}^2$ . It follows directly that  $\lambda_k \leq c_1 \lambda_{k,r}$  where  $\lambda_{k,r}$  is the  $k$ -th eigenvalue of the operator  $C_r$  defined in Lemma 5. Since by Lemma 5,  $\sum_{k=1}^{\infty} \lambda_{k,r}^{\frac{1}{2}} < \infty$ , the conclusion of the theorem follows.

By the argument of the paragraph preceding Lemma 4, there is a continuous injective linear mapping  $J_4$  of  $H_{M,r_1}$  into  $H$  which is the identity on  $C_c^\infty(G)$ . Since  $H$  may be identified with its own conjugate space by the assignment  $u \rightarrow [v, u]$ , with the latter considered as an element of  $H^*$ , the adjoint mapping  $J_4^*$  may be considered as a continuous linear mapping of  $H$  into  $H_{M,r_1}^*$ . Since  $J_4$  is one-to-one, the image of  $J_4^*$  is dense. Since the image of  $J_4$  in  $H_{M,r_1}$  contains the dense subset  $C_c^\infty(G)$  and is itself dense, it follows that  $J_4^*$  is one-to-one. Thus  $J_4^*$  identifies  $H$  with a dense subset of  $H_{M,r_1}^*$ . If we define a  $(M, -r_1, B)$ -norm on  $H$  by the prescription

$$\|u\|_{M,-r_1,B} = \sup \{ |[\psi, u]| \cdot (\|\psi\|_{M,r_1})^{-1} \mid \psi \in H_{M,r_1} \},$$

we see that  $\|u\|_{M,-r_1,B} = \|J_4^*u\|_{H_{M,r_1}^*}$ . It follows that the completion of  $H$  with respect to the  $(M, -r_1, B)$ -norm is a Hilbert space  $H_{M,-r_1,B}$  which is isomorphic to  $H_{M,r_1}^*$ . If  $B = p(x)I$ , the space  $H_{M,-r_1,p}$  for  $p \in C_c^\infty(G)$  is composed of distributions of order  $r_1$  at most.

### Section 3.

**Definition 4.** Let  $A$  be a subnormal operator in  $H$ ,  $\{F_\zeta\}$  a generalized spectral resolution corresponding to  $A$ . By an eigenfunction expansion corresponding to  $\{F_\zeta\}$ , we shall mean a sequence of positive, finite measures  $\{m_j\}$  defined on the Borel  $\sigma$ -algebra  $\Omega$  of the complex plane  $C^1$  and a sequence of functions  $\{u_j\}$  from  $C^1$  to  $H_{M,-r_1,B}$  having the following properties:

- (1) For  $v \in H_{M,r_1}$ ,  $c_j(\zeta) = [v, u_j(\zeta)]$  lies in  $L^2(m_j)$  and we have  $\|v\|_H^2 = \sum_{j=1}^{\infty} \{\|c_j\|_{L^2(m_j)}\}^2$ . Thus the mapping  $U$  defined by  $Uv = \{c_j\}$  is an

isometric mapping of the dense subset  $H_{M,r_1}$  of  $H$  into  $\sum_j \oplus L^2(m_j)$  and can be extended by continuity to an isometric mapping of  $H$  into  $\sum_j \oplus L^2(m_j)$ .

(2) Let  $\{c_j\} \in \sum_j \oplus L^2(m_j)$ ,  $S$  a Borel set in  $C^1$ . Then  $c_j(\xi)u_j(\xi)$  is a strongly integrable function from  $C^1$  to  $H_{M,-r_1,B}$  with respect to the measure  $m_j$ . If  $h_{j,S} = \int_S c_j(\xi)u_j(\xi)dm_j(\xi)$ , the element  $h_{j,S}$  of  $H_{M,-r_1,B}$  always lies in  $H$ , and the sum  $\sum_j h_{j,S}$  converges strongly in  $H$  to an element  $h_S$  with

$$\|h_S\|_H^2 \leq \sum_j \|c_j\|_{L^2(m_j)}^2.$$

(3) If  $\{c_j\} = U(f)$  for  $f \in H$ ,  $S$  a Borel set of  $C^1$ , then

$$F(S)f = \sum_j \int_S c_j(\xi)u_j(\xi)dm_j(\xi).$$

For  $S = C^1$  and such an element  $\{c_j\}$ , equality holds in the inequality of (2).

(4) If  $v \in D_A \cap H_{M,r_1}$ ,  $Av \in H_{M,r^1}$ , then for every  $\xi \in C^1$ ,

$$[u_j(\xi), Av] = \bar{\xi}[u_j(\xi), v].$$

*Definition 5.* The eigenfunction expansion of Definition 4 is said to be full if the mapping  $U$  described in (1) maps  $H$  onto  $\sum_j \oplus L^2(m_j)$ . Then, for  $S = C^1$ , equality holds in the inequality of (2).

The basic result of the present paper is the following:

**THEOREM 2.** Let  $A$  be a subnormal operator defined in the space  $H$ ,  $\{F_\zeta\}$  a generalized resolution of the identity corresponding to  $A$ . Then there exists an eigenfunction expansion corresponding to  $\{F_\zeta\}$  and this expansion is full if  $F_\zeta$  is an ordinary spectral resolution.

*Proof.* There exists a Hilbert space  $H_1$  containing  $H$  as a closed subspace and a normal operator  $T_1$  in  $H_1$  such that  $T \subseteq T_1$  and  $F(S) = PE(S)$ , where  $\{E_\zeta\}$  is the spectral resolution of  $T_1$ , and  $P$  is the projection of  $H_1$  on  $H$ . If  $\{F_\zeta\}$  is an ordinary spectral resolution, we choose  $H_1$  to be equal to  $H$  itself.

By Lemma 2, we may assume that there exists a sequence  $\{g_j\}$  in  $H_1$  such that  $H_1 = \sum_j \oplus H_1(g_j)$ . Let  $m_j(S) = [E(S)g_j, g_j]$ . If we denote  $H_1(g_j)$  by  $H_j$ , the mapping  $V_j$  assigning to a function  $c_j \in L^2(m_j)$  the element  $h_j = \int_{C^1} c_j(\xi)dE_\zeta g_j$  is a unitary mapping of  $L^2(m_j)$  onto  $H_j$ . Let  $f \in H_1$ . Then  $f = \sum_j h_j$  with  $h_j = V_j c_j \in H_j$ . If  $f \in H$ ,  $Pf = f$  implies that

$f = \sum_j f_j$  with  $f_j = Ph_j = \int_{C^1} c_j(\xi) dF_\xi g_j$ . Moreover,  $F(S)f = PE(S)f = \sum_j f_{j,S}$ , where  $f_{j,S} = PE(S)h_j = \int_S c_j(\xi) dF_\xi g_j$ . In particular, the mapping  $U$  defined by  $U(f) = \{c_j\}$  is an isometric mapping of  $H$  into  $\sum_j \oplus L^2(m_j)$ .

If, conversely, we are given an element  $\{c_j\}$  of  $\sum_j \oplus L^2(m_j)$ ,  $S$  a Borel set of  $C^1$ , we define  $h_{j,S} = \int_S c_j(\xi) dF_\xi g_j$ ,  $h_S = \sum_j h_{j,S}$ , the sum converging strongly in  $H$ . Since  $h_{j,S} = P(\int_S c_j(\xi) dE_\xi g_j)$ , it follows that

$$\|h_S\|^2 \leq \sum_j \|c_j\|_{L^2(m_j)}^2.$$

We must show the existence of a strongly  $m_j$ -measurable function  $u_j$  from  $C^1$  to  $H_{M,r_1,B}$  such that, for all  $\psi$  in  $H_{M,r_1}$  and all Borel sets  $S$  of  $C^1$ ,

$$(6) \quad [h_{j,S}, \psi] = \int_S c_j(\xi) [u_j(\xi), \psi] dm_j(\xi).$$

Since  $[h_{j,S}, \psi] = \int_S c_j(\xi) d[F_\xi g_j, \psi]$ , it suffices that

$$(7) \quad [F(S)g_j, \psi] = \int_S [u_j(\xi), \psi] dm_j(\xi)$$

for all  $\psi \in H_{M,r^1}$ .

We shall verify that the assumptions of Lemma 3 are satisfied if  $H_0 = H_{M,r_1,B}$ ,  $m(s) = m_j(S)$ , and  $h(S) = F(S)g_j$ . (In the following argument we drop the subscript  $j$  from  $m_j$  and  $g_j$ .)

For  $v \in H$ ,

$$|[h(S), v]| = |[PE(S)g, v]| = |[E(S)g, v]| \leq m(S) \|v\|_H.$$

Thus  $h$  is weakly absolutely continuous with respect to  $m$ .

Let  $\{S_k\}$  be a finite, pairwise disjoint family of Borel sets of  $C^1$ ,  $\psi \in H_{M,r_1}$ . Then,

$$(8) \quad \begin{aligned} \sum_k |[h(S_k), \psi]| &= \sum_k |E(S_k)g, E(S_k)\psi| \\ &\leq \left\{ \sum_k \|E(S_k)g\|_{H^2}^2 \right\}^{\frac{1}{2}} \|\psi\|_H \leq \|g\|_H \cdot \|\psi\|_H. \end{aligned}$$

Let  $\{\psi_k\}$  be a sequence of elements from  $H_{M,r_1}$  with  $\|\psi_k\|_{H_{M,r_1}} = 1$ . By Theorem 1, there exists a complete orthonormal sequence  $\{v_j\}$  in  $H_{M,r_1}$  with  $\sum_j \|v_j\|_{N,r} < \infty$ . Since  $\|v\|_H \leq \|v\|_{N,r}$ , it follows that  $\sum_j \|v_j\|_H < \infty$ . Let  $\psi_k = \sum_j a_{kj} v_j$  be the Fourier expansion of  $\psi_k$  with respect to this sequence.

Since  $\|\psi_k\|_{M,r_1}^2 = 1 = \sum_j |a_{kj}|^2$ , we have  $|a_{kj}| \leq 1$ .

We choose the  $\psi_k$  so that  $[h(S_k), \psi_k] = \|h(S_k)\|_{M,-r_1,B}$ . Let

$$K = \|g\|_H \cdot \sum_j \|v_j\|_H < \infty.$$

Then,

$$(9) \quad \sum_k \|h(S_k)\|_{M,-r_1,B} = \sum_k [h(S_k), \psi_k] \leq \sum_{k,j} |a_{kj}| \cdot |[h(S_k), v_j]| \leq K.$$

Thus the hypotheses of Lemma 3 are satisfied, and we obtain a function  $u_j$  from  $C^1$  to  $H_{M,-r_1,B}$  satisfying (7).

To complete the proof of Theorem 2, we must verify properties (1) and (4) of Definition 4. Let  $\psi \in H_{M,r_1}$ ,  $f \in H_1$ ; and let  $c_j(\xi)$ ,  $c'_j(\xi)$  be the functions in  $L^2(m_j)$  corresponding to the projections of  $\psi$  and  $f$  in  $H_j$ . By the spectral theory,  $[f, \psi] = \sum_j \int c'_j(\xi) \overline{c_j(\xi)} dm_j(\xi)$ . On the other hand,

$$\begin{aligned} [f, \psi] &= \sum_j \int c'_j(\xi) d[E_\xi g_j, \psi] \\ &= \sum_j \int c'_j(\xi) d[F_\xi g_j, \psi] = \sum_j \int c'_j(\xi) [u_j(\xi), \psi] dm_j(\xi), \end{aligned}$$

by the preceding part of the proof. Since  $c'_j(\xi)$ , for  $f \in H_1$ , fills out the whole of  $\sum_j \oplus L^2(m_j)$ , we must have  $c_j(\xi) = [\psi, u_j(\xi)]$  almost everywhere in  $m_j$ , and property (1) of Definition 4 is satisfied.

Similarly,

$$[f, Av] = \sum_j \int c'_j(\xi) \overline{\xi c_j(\xi)} dm_j(\xi) = \sum_j \int c'_j(\xi) [u_j(\xi), Av] dm_j(\xi)$$

implies that  $[u_j(\xi), Av] = \overline{\xi} [u_j(\xi), v]$  almost everywhere in  $m_j$ . We may alter the functions  $u_j(\xi)$  for  $\xi$  on sets of  $m_j$ -measure zero without affecting the validity of the other properties of the eigenfunction expansion. Let  $G$  be the graph of  $A$  as a mapping from  $H_{M,r_1}$  to  $H_{M,r_1}$  with its domain restricted to  $D(A) \cap H_{M,r_1} \cap A^{-1}(H_{M,r_1})$ .  $G$  is a closed subspace of the separable Hilbert space  $H_{M,r_1} \times H_{M,r_1}$  and is therefore separable. Let  $(v_r, Av_r)$  be a dense sequence in  $G$ . To each  $r$  there corresponds an  $m_j$ -null exceptional set  $Q_r$  such that for  $\xi \notin Q_r$ ,  $[u_j(\xi), Av_r] = \overline{\xi} [u_j(\xi), v_r]$ . On the complement of  $Q = \bigcup_r Q_r$ , which is a null set in  $m_j$ ,  $[u_j(\xi), v] = \overline{\xi} [u_j(\xi), v]$  for all  $v$ . On  $Q$ , we set  $u_j(\xi) = 0$  and after this change on a  $m_j$ -null set, the last equality holds for all  $\xi$ . Thus the proof of the theorem is complete.

**Section 4.** Let  $L$  be a linear differential operator with coefficients in  $C^\infty$ ,  $L = \sum_{|\beta| \leq m} c_\beta(x) D^\beta$ . The adjoint operator  $L^*$  of  $L$  is defined by  $L^*(u) = \sum_{|\beta| \leq m} (-1)^{|\beta|} D^\beta (\bar{c}_\beta(x) u)$ . Let  $B$  be a positive operator for which conditions (2)-(5) of Section 2 are valid. If  $\phi$  and  $\psi \in C_c^\infty(G)$ ,

$$|(L^*\phi, \psi)| \leq c(\phi) (p\psi, \psi)^{\frac{1}{2}} \leq c'(\phi) \|\psi\|_H.$$

Thus, there exists a unique element  $A\phi$  in  $H$  such that  $[A\phi, \psi] = (L^*\phi, \psi)$  for all  $\psi \in C_c^\infty(G)$ . Since  $(L^*\phi, \psi) = (\phi, L\psi)$ ,  $A$  is a closable operator with domain  $C_c^\infty(G)$  dense in  $H$ . If  $A$  is subnormal, we may apply Theorem 2 and get an eigenfunction expansion for  $A$  which is essentially in terms of eigenfunctions of  $B^{-1}L$ . We shall leave the discussion of the character of the eigenfunctions for general  $B$  to another place, and restrict ourselves for the remainder of this paper to the case when  $B = p(x)I$ , with  $p(x) \in C_c^\infty(G)$ .

**THEOREM 3.** Let  $p \in C_c^\infty(G)$  with  $p(x) > 0$  in  $G$ ,  $B = p(x)I$ ,  $L$  a partial differential operator with infinitely differentiable coefficients,  $A$  the corresponding operator in the Hilbert space  $H$  as defined above. If  $A$  is subnormal in  $H$ , then the  $u_j(\xi)$  of Theorem 2 are distribution solutions of  $(L - \bar{\xi}p)u = 0$ ; i. e.,  $(u_j(\xi), (L^* - \xi p)\phi) = 0$  for all  $\phi$  in  $C_c^\infty(G)$ .

*Proof.* From Theorem 2, we have  $[u_j(\xi), A\phi] = \bar{\xi}[u_j(\xi), \phi]$ . Since convergence in  $H_{M, -r, B}$  for  $B = p(x)I$  implies convergence in the distribution sense, we obtain by a simple continuity argument  $(u_j(\xi), L^*\phi) = \bar{\xi}(u_j(\xi), P\phi)$ .

**Definition 6.** The differential operator  $L$  is said to be hypoelliptic if for each integer  $j$ ,  $-\infty < j < \infty$ , and  $M \in C_c^\infty(G)$  with  $M(x) > 0$  for  $x$  in  $G$ , there exists  $M'_j \in C_c^\infty(G)$  with  $M'_j(x) > 0$ , such that every distribution solution of the equation  $Lu = f$  with  $f \in H_{M, j}(G)$  lies in  $H_{M'_j, j+1}(G)$  and such that  $\|u\|_{M'_j, j+1} \leq \|f\|_{M, j} + \|u\|_{M, j}$ .

**THEOREM 4.** Let  $L$  be a hypoelliptic differential operator,  $B = p(x)I$ , with  $L$  having a subnormal realization  $A$  in  $H$ . Then the eigenfunctions  $u_j(\xi)$  of Theorem 2 are functions in  $C^\infty(G)$ , the integrals of Theorem 2 converge in the pointwise sense for  $\psi$  with compact support in  $G$ , and  $\int_S |u_j(\xi)|^2 dm_j(\xi) < K(S, x)$ , where  $K(S, x)$  is bounded for  $S$  any bounded Borel set in  $C^1$  and  $x$  is a compact subset of  $G$ .

*Proof.* Applying the inequality of Definition 6 to the distribution solution  $u_j(\xi)$  of the equation  $Lu = pu$ , we obtain a sequence of functions  $M_j$  such that  $\|u\|_{M_j, j+1} \leq \|u\|_{M_{j-1}, j}$  and  $u$  lies in  $H_{M_j, j}$  for arbitrarily large  $j$ . By

a theorem of Sobolev (see, for instances, [29]),  $u$  lies in  $C^\infty(G)$ . The convergence and boundedness properties follow from the iteration of these inequalities.

We remark that a general class of differential operators with variable coefficients including both elliptic and parabolic operators are hypoelliptic in the sense of Definition 6 [9].

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## A GENERAL THEORY OF ALGEBRAIC GEOMETRY OVER DEDEKIND DOMAINS, II.\*

### Separably Generated Extensions and Regular Local Rings.

By MASAYOSHI NAGATA.

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The main aim of the present paper (Chapters 3 and 4) is to prove some preliminary results on separably generated extensions and on regular local rings. We want to remark here that in the present paper we need not assume that the ground rings satisfy the finiteness condition for integral extensions (cf. Part I ([17])).

In Chapter 3, we first prove some results on the notion of tensor product over Dedekind domains (§ 1), and then we introduce the notions of local and complete tensor products (§ 2) and derivations of a ring (§ 3). Making use of these results and notions, we study separably generated extensions and regular extensions (§ 4). In § 5, we prove a generalization of a lemma due to Zariski [25] on regular extensions, and in § 6 we treat tensor products of normal rings.

In Chapter 4, we first prove some lemmas on extensions of regular local rings (§ 1) and on quadratic transformations (§ 2). In § 3, we state some results due to Serre [22] on regular local rings. In § 4, we study the notion of unramifiedness of regular local rings and of spots, and in § 5, we prove some results on the unique factorization theorem in local rings.

In Appendix 1, we prove that if  $\mathfrak{p}$  is a prime ideal of a simple spot  $P$  in the restricted case, then  $P_{\mathfrak{p}}$  is a simple spot (which is contained in Serre's results) by a similar method as in Zariski [26]. In Appendix 2, we show that the results in Chapter 2 (in Part I) hold good also in the non-restricted case under the assumption that function fields are separably generated. Furthermore, we prove that the completion of a normal spot (in the non-restricted case) in an integral domain.

#### *Terminology.*

Terms such as rings, local rings, semi-local rings, normal rings, derived

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normal rings, rank of rings or ideals, co-rank of ideals and so on are used in the same sense as in Part I ([17]). An integral domain  $\mathfrak{o}$  is called a *unique factorization ring* if every element of  $\mathfrak{o}$  is expressible, uniquely (up to units), as a product of irreducible elements. In the present paper, spots, affine rings, ground rings and function fields are allowed to be in the *non-restricted case* (see Part I), unless the contrary is explicitly stated. A subring  $I$  of a semi-local ring  $\mathfrak{o}$  is called a *semi-ground ring* of  $\mathfrak{o}$  if 1)  $I$  is a field or a Dedekind domain, 2) elements of  $I$  are not zero-divisors in  $\mathfrak{o}$ , and 3) for every maximal ideal  $\mathfrak{p}$  of  $\mathfrak{o}$ ,  $\mathfrak{o}/\mathfrak{p}$  is a finitely generated field over the field of quotients of  $I/(\mathfrak{p} \cap I)$ . Observe that if a semi-local ring  $\mathfrak{o}$  is a ring of quotients of an affine ring over a ground ring  $I$ , then  $I$  is a semi-ground ring of  $\mathfrak{o}$  and that if  $I$  is a semi-ground ring of a semi-local ring  $\mathfrak{o}$ , then  $I$  is also a semi-ground ring of the completion of  $\mathfrak{o}$ .

*Results assumed to be known.*

Besides results assumed to be known in Part I, we assume that the following results are known.

- 1) On the notion of tensor products: Results contained in Bourbaki [1, §§ 1-3].
- 2) On the theory of fields: Results contained in Weil [24, Chapter I].
- 3) On the theory of local rings:

**LEMMA 0.10** (The structure theorem of complete local rings). *If  $\mathfrak{o}$  is a complete local ring, then  $\mathfrak{o}$  has a coefficient ring  $\mathfrak{v}$ ; that is,  $\mathfrak{v}$  is a homomorphic image of a complete discrete valuation ring which satisfies the following conditions: 1)  $\mathfrak{v}$  is dominated by  $\mathfrak{o}$ , 2) the residue class field of  $\mathfrak{o}$  coincides with that of  $\mathfrak{v}$ , and 3) if  $p$  is the characteristic of the residue class field of  $\mathfrak{o}$ , then the maximal ideal of  $\mathfrak{v}$  is generated by  $p$  (that is,  $p$  times the identity). Then  $\mathfrak{o}$  is isomorphic to a homomorphic image of a formal power series ring over  $\mathfrak{v}$ . (Cohen [6]; for the proof see Narita [20].)<sup>1</sup>*

**LEMMA 0.11.** *If a complete local ring  $\mathfrak{o}$  is dominated by a local ring  $\mathfrak{o}'$ , then  $\mathfrak{o}$  is a subspace of  $\mathfrak{o}'$ . (See Chevalley [3], Cohen [6], Nagata [15].)*

**LEMMA 0.12.** *Let  $\mathfrak{p}$  be a prime ideal of a regular local ring  $\mathfrak{r}$ . Then  $\mathfrak{r}/\mathfrak{p}$  is regular if and only if there exists a regular system of parameters*

<sup>1</sup> Cohen's proof is stated also in Samuel [21]. It seems to me that Narita's proof is simpler than the others. The proof in Nagata [11] contains some errors and it is good only in the cases 1) the residue class field is perfect and 2) it contains a field.

$x_1, \dots, x_n$  ( $n = \text{rank } \mathfrak{r}$ ) such that  $\mathfrak{p} = \sum_1^r x_i \mathfrak{r}$  (for some  $\mathfrak{r}$ , whence  $r = \text{rank } \mathfrak{p}$ ). (See Cohen [6], Nagata [15]; cf. Chevalley [3].)

*Remark 1.* Let  $\mathfrak{o}$  be a local ring of rank  $r$ , and let  $\mathfrak{m}$  be its maximal ideal. Then elements  $u_1, \dots, u_n$  of  $\mathfrak{m}$  generate  $\mathfrak{m}$  if and only if their residue classes modulo  $\mathfrak{m}^2$  generate  $\mathfrak{m}/\mathfrak{m}^2$  over the field  $\mathfrak{o}/\mathfrak{m}$ . (See Cohen [6], Nagata [15]; cf. [16, § 6].) Therefore,  $\mathfrak{o}$  is regular if and only if the dimension of the vector space  $\mathfrak{m}/\mathfrak{m}^2$  over  $\mathfrak{o}/\mathfrak{m}$  is equal to the rank of  $\mathfrak{o}$ .

*Remark 2.* Let  $\mathfrak{o}$  be a local ring, and let  $\mathfrak{p}$  be an ideal of rank  $r$ . If  $\mathfrak{p}$  is generated by  $r$  elements and if  $\mathfrak{o}/\mathfrak{p}$  is regular, then  $\mathfrak{o}$  is a regular local ring, as is easily seen.

**LEMMA 0.13.** *A complete local ring  $\mathfrak{o}$  is a Henselian ring, that is, if a monic polynomial  $f(x)$  over  $\mathfrak{o}$  factors into a product of monic polynomials  $g(x)$  and  $h(x)$  modulo the maximal ideal  $\mathfrak{m}$  of  $\mathfrak{o}$ , and if*

$$g(x)\mathfrak{o}[x] + h(x)\mathfrak{o}[x] + \mathfrak{m}[x] = \mathfrak{o}[x],$$

*then there exist monic polynomials  $g'(x)$  and  $h'(x)$  ( $\in \mathfrak{o}[x]$ ) such that  $f(x) = g'(x)h'(x)$  and both  $g(x) - g'(x)$  and  $h(x) - h'(x)$  are in  $\mathfrak{m}[x]$ . (See Cohen [6], Nagata [11], Samuel [21].)*

**LEMMA 0.14.** *Let  $\mathfrak{o}$  be a complete local ring, not necessarily Noetherian. If the maximal ideal of  $\mathfrak{o}$  has a finite base, then  $\mathfrak{o}$  is Noetherian. (See Cohen [6], Nagata [15], Samuel [21].)*

Further, only in Chapter 4, §§ 3-5, the results on regular local rings contained in Serre [22] are assumed to be known (and they are listed in Chapter 4, § 3.)

#### 4) On Noetherian integral domains:

Some results stated in Nagata [19] will be used freely; in particular, we shall use the following

**LEMMA 0.15.** *Let  $\mathfrak{o}$  be a Noetherian integral domain of rank 1, and let  $K$  be a finite algebraic extension of the field of quotients of  $\mathfrak{o}$ . If a subring  $\mathfrak{o}'$  of  $K$  contains  $\mathfrak{o}$ , then  $\mathfrak{o}'$  is a Noetherian ring (of rank not greater than 1). Further, in this case, if  $\mathfrak{p}'$  is a prime ideal of  $\mathfrak{o}'$  different from 0, then  $\mathfrak{o}'/\mathfrak{p}'$  is a finite algebraic extension of  $\mathfrak{o}/(\mathfrak{p}' \cap \mathfrak{o})$ . (Akizuki; cf. Chevalley [5], Cohen [7].)*

#### 5) On homological algebras:

Some basic notions and results contained in Cartan-Eilenberg [2] are assumed to be known (only in Chapter 4, §§ 3-5).

## Chapter 3. Separably Generated Extensions.

**1. Preliminaries on the notion of tensor products.** Let  $I$  be a ring, and let  $M$  and  $N$  be  $I$ -modules. Then, as is well known, we can define the tensor product of  $M$  and  $N$  over  $I$ , which will be denoted by  $M \otimes_I N$  or by  $M \otimes N$ ; if  $M$  and  $N$  are rings, then  $M \otimes N$  becomes a ring.

**LEMMA 1.** *Let  $I$  be a subring of a ring  $\mathfrak{o}$ . If  $S$  is a multiplicatively closed subset of  $I$  which does not contain 0, then  $\mathfrak{o} \otimes_I I_S$  can be identified with  $\mathfrak{o}_S$ .*

The proof consists in remarking that by the definition of the ring of quotients (see [16, § 2]) and the tensor product, both  $\mathfrak{o}_S$  and  $\mathfrak{o} \otimes_I I_S$  are characterized by the property of being the most "universal" ring in which the elements of  $S$  are mapped on units and which is generated by images of  $\mathfrak{o}$  and inverses of images of elements of  $S$ .

**LEMMA 2.** *If  $\mathfrak{o}$ ,  $\mathfrak{o}'$  and  $I'$  are rings containing a common subring  $I$ , then  $(\mathfrak{o} \otimes_I \mathfrak{o}') \otimes_{I'} I'$  can be identified with  $(\mathfrak{o} \otimes_I I') \otimes_{I'} (\mathfrak{o}' \otimes_I I')$ . If furthermore,  $I'$  is a subring of  $\mathfrak{o}'$ , then  $\mathfrak{o} \otimes_I \mathfrak{o}'$  can be identified with  $(\mathfrak{o} \otimes_I I') \otimes_{I'} \mathfrak{o}'$ .<sup>2</sup>*

The proof is easy; see [1, § 3, Exercise 4) and No. 4].

Let  $I$  be an integral domain. We say that an  $I$ -module  $M$  is torsion-free if  $am = 0$  ( $a \in I$ ,  $m \in M$ ) implies  $a = 0$  or  $m = 0$ . An  $I$ -module which is not torsion-free is said to have torsion. Now we shall state the following well known lemma:

**LEMMA 3.** *Assume that  $I$  is a principal ideal integral domain and that  $M$  is a finite  $I$ -module. If  $M$  is torsion-free, then  $M$  has a linearly independent base.*

**PROPOSITION 1.** *Assume that  $I$  is a Dedekind domain. If  $I$ -modules  $M$  and  $N$  are torsion-free, then  $M \otimes N$  is also torsion free.*

*Proof.*<sup>3</sup> If  $M \otimes N$  has torsion, then there exist submodules  $M'$  and  $N'$  of  $M$  and  $N$  respectively which are finite modules such that  $M' \otimes N'$  has torsion (see [1]). Therefore, we may assume that  $M$  and  $N$  are finite modules. Then, defining multiplication in  $M$  and  $N$  so that  $M^2 = N^2 = 0$ ,

<sup>2</sup> In general,  $I'$  may not be identified with the subring  $1 \otimes I'$  of  $\mathfrak{o} \otimes I'$ . Of course, if  $I$  is a field, the identification is permissible.

<sup>3</sup> Another proof can be given making use of the following fact: If a finite  $I$ -module  $M$  is torsion-free, then  $M$  is isomorphic to the direct sum of a finite number of ideals of  $I$  and  $M$  is a projective module. (For the notion of projective module, see [2].)

we construct rings  $I + M$  and  $I + N$  (direct sums as modules). Then  $M \otimes N$  is imbedded in  $(I + M) \otimes (I + N)$  (see [1]), and this last ring is Noetherian because  $M$  and  $N$  are finitely generated. Let  $p^*_1, \dots, p^*_h$  be the prime divisors of zero in  $(I + M) \otimes (I + N)$  and set  $p_i = p^*_i \cap I$  for each  $i$ . Let  $S$  be the intersection of the complements of the  $p_i$ 's in  $I$ . Then for an element  $a \in S$ ,  $su = 0$  ( $u \in (I + M) \otimes (I + N)$ ) implies  $u = 0$ . Therefore,  $(I + M) \otimes (I + N)$  can be imbedded in  $(I_S + (M \otimes I_S)) \otimes_{I_S} (I_S + (N \otimes I_S))$ . Therefore, in particular,  $M \otimes N$  can be imbedded in  $(M \otimes I_S) \otimes_{I_S} (N \otimes I_S)$ . Thus we may assume that  $I$  is a semi-local Dedekind domain. Then  $I$  is a principal ideal ring (see [16, § 9]), and  $M$  and  $N$  have linearly independent bases by Lemma 3. Therefore  $M \otimes N$  is torsion-free (see [1]).

**COROLLARY.** *Let  $\mathfrak{o}$  and  $\mathfrak{s}$  be rings containing a Dedekind domain  $I$ , and let  $S$  be the set of all non-zero elements of  $I$ . If every element of  $S$  is not a zero-divisor in both  $\mathfrak{o}$  and  $\mathfrak{s}$ , then  $\mathfrak{o} \otimes_I \mathfrak{s}$  can be imbedded in  $\mathfrak{o}_S \otimes_{I_S} \mathfrak{s}_S$ .*

The proof is immediate from Lemma 1 and Proposition 1.

*Remark.* As is well known, we can identify  $\mathfrak{o} \otimes 1$  and  $1 \otimes \mathfrak{s}$  with  $\mathfrak{o}$  and  $\mathfrak{s}$  respectively (in the above case). Then  $\mathfrak{o} \otimes_I \mathfrak{s}$  is the subring of  $\mathfrak{o}_S \otimes_{I_S} \mathfrak{s}_S$  generated by  $\mathfrak{o}$  and  $\mathfrak{s}$ . This identification will be made in every similar case.

**THEOREM 1.** *Let  $\mathfrak{o}$  and  $\mathfrak{s}$  be integral domains which contain a Dedekind domain  $I$ . Let  $K$ ,  $L$  and  $\mathbf{k}$  be the fields of quotients of  $\mathfrak{o}$ ,  $\mathfrak{s}$  and  $I$  respectively. Then  $\mathfrak{o} \otimes_I \mathfrak{s}$  is the subring of  $K \otimes_{\mathbf{k}} L$  generated by  $\mathfrak{o}$  and  $\mathfrak{s}$ . Furthermore,  $\mathfrak{o} \otimes \mathfrak{s}$  and  $K \otimes L$  have the same total quotient ring, and therefore  $K \otimes L$  is an integral domain if and only if  $\mathfrak{o} \otimes \mathfrak{s}$  is.*

*Proof.* By the corollary to Proposition 1,  $\mathfrak{o} \otimes \mathfrak{s}$  is the subring of  $\mathfrak{o}_S \otimes_{\mathbf{k}} \mathfrak{s}_S$  generated by  $\mathfrak{o}$  and  $\mathfrak{s}$ ,  $S$  being the set of non-zero elements of  $I$ . Since  $\mathbf{k}$  is a field,  $\mathfrak{o}_S \otimes_{\mathbf{k}} \mathfrak{s}_S$  is a subring of  $K \otimes_{\mathbf{k}} L$ , which proves the first assertion. From this fact, we see that elements of  $\mathfrak{o}$  and  $\mathfrak{s}$  are not zero-divisors in  $\mathfrak{o} \otimes \mathfrak{s}$ . Therefore the total quotient ring  $A$  of  $\mathfrak{o} \otimes \mathfrak{s}$  contains  $K$  and  $L$  and therefore also  $K \otimes L$ , which proves the last assertion.

**LEMMA 4.** *Let  $\mathfrak{o}$  and  $\mathfrak{s}$  be rings which contain a ring  $I$ . Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be ideals of  $\mathfrak{o}$  and  $\mathfrak{s}$  respectively, and assume that  $\mathfrak{a} \cap I = \mathfrak{b} \cap I$ . Denote by  $I'$  the ring  $I/(\mathfrak{a} \cap I)$  and by  $\mathfrak{d}$  the ideal of  $\mathfrak{o} \otimes \mathfrak{s}$  generated by  $\mathfrak{a}$  and  $\mathfrak{b}$ . Then  $(\mathfrak{o} \otimes_I \mathfrak{s})/\mathfrak{d}$  can be identified with  $(\mathfrak{o}/\mathfrak{a}) \otimes_{I'} (\mathfrak{s}/\mathfrak{b})$ .*

This is well known and we shall omit the proof (see [1]).

**LEMMA 5.** *Let  $K$  be a ring of quotients of an affine ring over a ground*

ring  $I$ , and let  $L$  be a Noetherian integral domain containing  $I$ . Then  $K \otimes_I L$  is a Noetherian ring and its zero ideal has no imbedded prime divisor.

*Proof.* Since  $K \otimes L$  is a ring of quotients of a finitely generated ring over the Noetherian ring  $L$ , we see that  $K \otimes L$  is Noetherian. In order to prove the last assertion, we may assume that  $K$ ,  $L$  and  $I$  are fields by Theorem 1. Let  $x_1, \dots, x_n$  be a transcendence base of  $K$  over  $I$ , and let  $L'$  be the field of quotients of  $I(x_1, \dots, x_n) \otimes_I L$ ; since the  $x_i$ 's are algebraically independent over  $L$ ,  $L' = L(x_1, \dots, x_n)$ . Since  $I$  is a field,  $K \otimes L$  is a subring of  $K \otimes_I L'$ , where  $I'$  denotes the field  $I(x_1, \dots, x_n)$ . Now we see our assertion easily, because  $K$  is a finite algebraic extension of  $I'$  and  $K \otimes_{I'} L'$  has minimum condition.

**2. Local tensor product and complete tensor product.** Let  $\mathfrak{o}$  and  $\mathfrak{o}'$  be semi-local rings and assume that  $I$  is a subring of them. Let  $\mathfrak{p}$  and  $\mathfrak{p}'$  be maximal ideals of  $\mathfrak{o}$  and  $\mathfrak{o}'$  respectively, and set  $\mathfrak{q} = \mathfrak{p} \cap I$ ,  $\mathfrak{q}' = \mathfrak{p}' \cap I$ . We assume that  $I/\mathfrak{q}$  is a field or a Dedekind domain and that  $\mathfrak{o}/\mathfrak{p}$  is a function field over  $I/\mathfrak{q}$  for every maximal ideal  $\mathfrak{p}$  of  $\mathfrak{o}$ . Set  $\mathfrak{t} = \mathfrak{o} \otimes_I \mathfrak{o}'$ . Then we have

**LEMMA 1.**  $\mathfrak{p}\mathfrak{t} + \mathfrak{p}'\mathfrak{t} \neq \mathfrak{t}$  if and only if  $\mathfrak{q} = \mathfrak{q}'$ . In this case, 1)  $\mathfrak{t}/(\mathfrak{p}\mathfrak{t} + \mathfrak{p}'\mathfrak{t})$  is Noetherian, 2) every prime divisor  $\mathfrak{P}$  of  $\mathfrak{p}\mathfrak{t} + \mathfrak{p}'\mathfrak{t}$  is a minimal prime divisor, and 3)  $\mathfrak{P}$  has a finite base.

*Proof.* The first assertion is obvious because  $\mathfrak{p}$  and  $\mathfrak{p}'$  are maximal. 1) and 2) follow from Lemmas 3.1.4 and 3.1.5. Since  $\mathfrak{p}$  and  $\mathfrak{p}'$  have finite bases, 3) follows from 1).

Now, let  $S$  be the intersection of the complements of prime divisors of  $\mathfrak{p}\mathfrak{t} + \mathfrak{p}'\mathfrak{t}$ , where  $\mathfrak{p}$  and  $\mathfrak{p}'$  run over all possible maximal ideals. Then  $\mathfrak{t}_S$  is called the local tensor product of  $\mathfrak{o}$  and  $\mathfrak{o}'$  over  $I$ ; this ring will be denoted by  $\mathfrak{o} \times_I \mathfrak{o}'$  or by  $\mathfrak{o} \times \mathfrak{o}'$ . By this definition,  $\mathfrak{o} \times \mathfrak{o}'$  is a semi-local ring if it is Noetherian.

**LEMMA 2.** If  $I$  is a semi-ground ring of  $\mathfrak{o}$  and if  $\mathfrak{o}'$  is torsion-free as an  $I$ -module, then  $\mathfrak{o} \times \mathfrak{o}'$  is also torsion-free. If, furthermore,  $I$  is a semi-ground ring of  $\mathfrak{o}'$  and if  $\mathfrak{o} \times \mathfrak{o}'$  is a semi-local ring, then  $I$  is also a semi-ground ring of  $\mathfrak{o} \times \mathfrak{o}'$ .

*Proof.* By the Corollary to Proposition 1, we see that  $\mathfrak{o} \otimes \mathfrak{o}'$  is torsion-free. Since  $\mathfrak{o} \times \mathfrak{o}'$  is a ring of quotients of  $\mathfrak{o} \otimes \mathfrak{o}'$ , we see the validity of the first assertion (see [16, § 2]). Now the last assertion is obvious.

Under the same assumptions and notations as in Lemma 2, let  $\mathfrak{m}$  be the  $J$ -radical of  $\mathfrak{o} \times \mathfrak{o}'$  and set  $\mathfrak{n} = \bigcap_n \mathfrak{m}^n$ . Then  $(\mathfrak{o} \times \mathfrak{o}')/\mathfrak{n}$  is a semi-local ring which may not be Noetherian. The completion of this semi-local ring is called the complete tensor product of  $\mathfrak{o}$  and  $\mathfrak{o}'$  (over  $I$ ) and will be denoted by  $\mathfrak{o} \bar{\otimes}_I \mathfrak{o}'$ , or by  $\mathfrak{o} \bar{\otimes} \mathfrak{o}'$ . Observe that  $\mathfrak{o} \bar{\otimes} \mathfrak{o}'$  is the limit space of the inverse system  $\{(\mathfrak{o} \times \mathfrak{o}')/\mathfrak{m}^n; n = 1, 2, \dots\}$ .

LEMMA 3.  $\mathfrak{o} \bar{\otimes} \mathfrak{o}'$  is a (Noetherian) semi-local ring.

*Proof.* By Lemma 1, every maximal ideal of  $\mathfrak{o} \times \mathfrak{o}'$  has a finite base and the same is true of  $\mathfrak{o} \bar{\otimes} \mathfrak{o}'$ . Therefore  $\mathfrak{o} \bar{\otimes} \mathfrak{o}'$  is Noetherian by Lemma 0.14.

LEMMA 4. Let  $\mathfrak{o}^*$  and  $\mathfrak{o}'^*$  be semi-local rings which contain  $\mathfrak{o}$  and  $\mathfrak{o}'$  as dense subspaces respectively. Then  $\mathfrak{o}^* \bar{\otimes} \mathfrak{o}'^* = \mathfrak{o} \bar{\otimes} \mathfrak{o}'$ .

*Proof.* Let  $\mathfrak{m}$  and  $\mathfrak{m}'$  be the  $J$ -radicals of  $\mathfrak{o}$  and  $\mathfrak{o}'$  respectively. Then  $\mathfrak{m}\mathfrak{o}^*$  and  $\mathfrak{m}\mathfrak{o}'^*$  are the  $J$ -radicals of  $\mathfrak{o}^*$  and  $\mathfrak{o}'^*$  respectively. Let  $\alpha(n)$  be the ideal of  $\mathfrak{o} \bar{\otimes} \mathfrak{o}'$  generated by  $\mathfrak{m}^n$  and  $\mathfrak{m}'^n$  and let  $\alpha$  be the  $J$ -radical of  $\mathfrak{o} \bar{\otimes} \mathfrak{o}'$ . Then we see that  $\alpha(1)$  contains a power of  $\alpha$  by Lemma 1. On the other hand, it is obvious that  $\alpha(1)^{2n} \subseteq \alpha(n) \subseteq \alpha(1)^n$ . Therefore  $\mathfrak{o} \bar{\otimes} \mathfrak{o}'$  can be identified with the limit space of the inverse system

$$\{(\mathfrak{o} \times \mathfrak{o}')/(\mathfrak{m}^n(\mathfrak{o} \times \mathfrak{o}') + \mathfrak{m}'^n(\mathfrak{o} \times \mathfrak{o}'))\}.$$

Similarly,  $\mathfrak{o}^* \bar{\otimes} \mathfrak{o}'^*$  can be identified with the limit space of the inverse system

$$\{(\mathfrak{o}^* \times \mathfrak{o}'^*)/(\mathfrak{m}^n(\mathfrak{o}^* \times \mathfrak{o}'^*) + \mathfrak{m}'^n(\mathfrak{o}^* \times \mathfrak{o}'^*))\}.$$

Since

$$\begin{aligned} (\mathfrak{o}^* \times \mathfrak{o}'^*)/(\mathfrak{m}^n(\mathfrak{o}^* \times \mathfrak{o}'^*) + \mathfrak{m}'^n(\mathfrak{o}^* \times \mathfrak{o}'^*)) \\ &= (\mathfrak{o}^*/\mathfrak{m}^n \mathfrak{o}^*) \times (\mathfrak{o}'^*/\mathfrak{m}'^n \mathfrak{o}'^*) = (\mathfrak{o}/\mathfrak{m}^n) \times (\mathfrak{o}'/\mathfrak{m}'^n) \\ &= (\mathfrak{o} \times \mathfrak{o}')/(\mathfrak{m}^n(\mathfrak{o} \times \mathfrak{o}') + \mathfrak{m}'^n(\mathfrak{o} \times \mathfrak{o}')), \end{aligned}$$

we have  $\mathfrak{o} \bar{\otimes} \mathfrak{o}' = \mathfrak{o}^* \bar{\otimes} \mathfrak{o}'^*$ .

*Remark.* The notion of complete tensor product of complete local rings over a field was introduced by Chevalley [4], who called it the *Kronecker product*. Though his formulation is different from ours, if we restrict ourselves to his case, then the two definitions are essentially the same: If  $\mathfrak{o}$  and  $\mathfrak{o}'$  are complete local rings which have basic fields  $\mathbf{k}$  and  $\mathbf{k}'$  respectively (for the definition of basic fields in Chevalley's sense, see [3]), and if  $M$  is a field containing both  $\mathbf{k}$  and  $\mathbf{k}'$ , then the Kronecker product of  $\mathfrak{o}$  and  $\mathfrak{o}'$  over  $M$  is  $(\mathfrak{o}' \bar{\otimes}_{\mathbf{k}'} M) \otimes_M (\mathfrak{o} \bar{\otimes}_{\mathbf{k}} M)$ . (This fact can be proved using some results in [4] and a method similar to the proof of Lemma 4.)

**3. Derivations of a ring.** A derivation  $D$  of a ring  $\mathfrak{o}$  is an additive endomorphism of the total quotient ring  $L$  of  $\mathfrak{o}$  which satisfies the following conditions: 1)  $D(xy) = xDy + yDx$  for  $x, y \in L$  and 2) there exists an element  $d$  of  $\mathfrak{o}$  which is not a zero-divisor such that  $dDx \in \mathfrak{o}$  for  $x \in \mathfrak{o}$ .

Here, if  $d$  can be chosen to be 1, then we call  $D$  an *integral derivation* of  $\mathfrak{o}$ .

Let  $D$  be a derivation of  $\mathfrak{o}$  and let  $\mathfrak{o}'$  be a subring of  $\mathfrak{o}$ . If  $Da = 0$  for every  $a \in \mathfrak{o}'$ , then we say that  $D$  is a derivation *over*  $\mathfrak{o}'$ ; if  $Da = 0$  for every  $a \in \mathfrak{o}$ , then we say that  $D$  is the zero derivation or the trivial derivation of  $\mathfrak{o}$ , which will be denoted by 0.

*Remark 1.* Since  $1^2 = 1$ ,  $D1 = D1 + D1$  and  $D1 = 0$  for every derivation  $D$  of  $\mathfrak{o}$ . Therefore every derivation of  $\mathfrak{o}$  is a derivation over the subring of  $\mathfrak{o}$  generated by the identity.

*Remark 2.* If  $x, y \in \mathfrak{o}$  and if  $y$  is not a zero-divisor, then  $D(x/y) = (yDx - xDy)/y^2$ . (For, since  $x = (x/y)y$ ,  $Dx = yD(x/y) + (x/y)Dy$  and  $D(x/y) = (yDx - xDy)/y^2$ .) Therefore, if  $D$  is a derivation over a subring  $\mathfrak{o}'$  of  $\mathfrak{o}$  and if  $S$  is the set of elements of  $\mathfrak{o}'$  which are not zero-divisors in  $\mathfrak{o}$ , then  $Da = 0$  for every element  $a$  of  $\mathfrak{o}'_S$ .

The set of derivations of  $\mathfrak{o}$  over a subring  $\mathfrak{o}'$  is an  $\mathfrak{o}$ -module. Linear dependence of derivations will always mean dependence in this module, hence over  $\mathfrak{o}$ .

Let  $\mathfrak{o}$  be a ring and let  $X_1, \dots, X_n$  be indeterminates. Then there exist derivations  $D_i$  ( $i = 1, \dots, n$ ) of  $\mathfrak{o}[X_1, \dots, X_n]$  over  $\mathfrak{o}$  such that  $D_i X_i = 1$  and  $D_i X_j = 0$  if  $i \neq j$ . These  $D_i$  are called the partial derivations and will be denoted by  $\partial/\partial X_i$  (if  $f \in \mathfrak{o}[X_1, \dots, X_n]$ , then  $D_i f$  may be denoted by  $\partial f/\partial X_i$ ). When  $f_1, \dots, f_r$  are elements of  $\mathfrak{o}[X_1, \dots, X_n]$ , the matrix  $(\partial f_i/\partial X_j)$  is called the Jacobian matrix of  $f_1, \dots, f_r$  and will be denoted by  $J(f_1, \dots, f_r)$ .

Let  $D$  be a derivation of  $\mathfrak{o}$ . Then there exists a uniquely determined derivation  $D'$  of  $\mathfrak{o}[X_1, \dots, X_n]$  which coincides with  $D$  on  $\mathfrak{o}$  and  $D'X_i = 0$ . If  $f$  is in the total quotient ring of  $\mathfrak{o}[X_1, \dots, X_n]$ ,  $D'f$  is denoted by  $f^D$ .

Let  $\mathfrak{a}$  be an ideal of  $\mathfrak{o}[X_1, \dots, X_n]$ , and let  $S$  be the intersection of the complements of all prime ideals which are prime divisors of  $\mathfrak{a}$  or of zero. Let  $\phi$  be the natural homomorphism from  $\mathfrak{o}[X_1, \dots, X_n]_S$  into the total quotient ring of  $\mathfrak{o}[X_1, \dots, X_n]/\mathfrak{a}$  and set  $x_i = \phi(X_i)$ . When  $f \in \mathfrak{o}[X_1, \dots, X_n]_S$ ,  $\phi(\partial f/\partial X_i)$  will be denoted by  $\partial f/\partial x_i$ . If  $D$  is a derivation of  $\mathfrak{o}$  and there exists an element  $d \in S \cap \mathfrak{o}$  such that  $dD$  is an integral derivation,  $\phi(f^D)$  will be denoted by  $f^D(x_1, \dots, x_n)$  for every element  $f = f(X_1, \dots, X_n)$  of  $\mathfrak{o}[X_1, \dots, X_n]_S$ .

*Remark 1.*  $\partial/\partial X_1, \dots, \partial/\partial X_n$  form a maximal linearly independent set of derivations of  $\mathfrak{o}[X_1, \dots, X_n]$  over  $\mathfrak{o}$ .

*Remark 2.* If  $D$  is a derivation of  $\mathfrak{o}[X_1, \dots, X_n]/\mathfrak{a}$  over  $\phi(\mathfrak{o})$ , then for an element  $g$  of  $\mathfrak{o}[X_1, \dots, X_n]$ ,  $D(\phi(g)) = \sum D x_i \cdot (\partial g / \partial x_i)$ .

**LEMMA 1.** Let  $\mathbf{k}$  be a field with a derivation  $D$  and let  $X_1, \dots, X_n$  be indeterminates. Set  $R = \mathbf{k}[X_1, \dots, X_n]$  and let  $\mathfrak{p}$  be a prime ideal of  $R$  with a base  $f_1, \dots, f_s$  and let  $L$  be the field of quotients of  $R/\mathfrak{p}$ . Then there exists a derivation  $D'$  of  $L$  such that 1)  $D'$  coincides with  $D$  on  $\mathbf{k}$  and 2)  $D'x_i = u_i$  (where  $x_i$  is the class of  $X_i$  modulo  $\mathfrak{p}$  and  $u_i \in L$ ) if and only if  $u_1, \dots, u_n$  satisfy the following set of linear equations:

$$f_i^D(x_1, \dots, x_n) + \sum (\partial f_i / \partial x_j) u_j = 0.$$

In this case,  $D'$  is uniquely determined.

For the proof, see Weil [24].

**COROLLARY.** Let  $r$  be the rank of the Jacobian matrix  $J(f_1, \dots, f_s)$  modulo  $\mathfrak{p}$ . Then every maximal linearly independent set of derivations of  $L$  over  $\mathbf{k}$  consists of  $n - r$  elements. (See Weil [24].)

**LEMMA 2.** Let  $L$  be a function field over a ground field  $\mathbf{k}$  of characteristic  $p$ . 1) If  $\dim L = r$ , then there exists at least  $r$  linearly independent derivations of  $L$  over  $\mathbf{k}$ ;  $L$  has no more than  $r$  linearly independent derivations over  $\mathbf{k}$  if and only if  $L$  has a separating transcendence base over  $\mathbf{k}$ . 2) Assume that  $p \neq 0$  and that  $a$  is an element of  $\mathbf{k}$  such that  $a^{1/p} \notin \mathbf{k}$ . Then there exists a derivation  $D$  of  $\mathbf{k}(a^{1/p})$  over  $\mathbf{k}$  such that  $D(a^{1/p}) = 1$ ; every derivation of  $\mathbf{k}(a^{1/p})$  over  $\mathbf{k}$  is of the form  $xD$  with  $x \in \mathbf{k}(a^{1/p})$ .

For the proof, see Weil [24].

**PROPOSITION 2.** Let  $\mathfrak{o}$  be an integral domain finitely generated over a ring  $I$  and let  $\mathbf{k}$  and  $L$  be the fields of quotients of  $I$  and  $\mathfrak{o}$  respectively. Then an additive endomorphism  $D$  of  $L$  is a derivation of  $\mathfrak{o}$  over  $I$  if and only if  $D$  is a derivation of  $L$  over  $\mathbf{k}$ ; in this case,  $D$  is a derivation of an arbitrary ring of quotients of  $\mathfrak{o}$ .

*Proof.* If  $D$  is a derivation of  $\mathfrak{o}$ , then  $D$  is a derivation of  $L$  by definition; if  $D$  is a derivation over  $I$ , then by Remark 2 after the definition, we see that  $D$  is a derivation over  $\mathbf{k}$ . Conversely, assume that  $D$  is a derivation of  $L$  over  $\mathbf{k}$ . Let  $x_1, \dots, x_n$  be elements of  $\mathfrak{o}$  such that  $\mathfrak{o} = I[x_1, \dots, x_n]$  and let  $S$  be a multiplicatively closed subset of  $\mathfrak{o}$  which does not contain zero.

Let  $d (\neq 0)$  be an element of  $\mathfrak{o}$  such that  $dDx_i \in \mathfrak{o}$  for every  $i$ . Then, since every element  $y$  of  $\mathfrak{o}$  is expressed as a polynomial in  $x_i$ 's, we have  $dDy \in \mathfrak{o}$ . Then for an element  $z = y/s$  ( $y \in \mathfrak{o}, s \in S$ ),  $Dz = (sDx - xDs)/s^2$  and we have  $dDz \in \mathfrak{o}_S$ . Thus the proof is completed.

Let  $\mathfrak{a}$  be an ideal of a ring  $\mathfrak{o}$  and let  $D$  be a derivation of  $\mathfrak{o}$ . Let  $\phi$  be the natural homomorphism from  $\mathfrak{o}$  onto  $\mathfrak{o}/\mathfrak{a}$ . Assume that there exists an element  $d$  of  $\mathfrak{o}$  which is not a zero-divisor modulo  $\mathfrak{a}$  and such that  $dD$  is an integral derivation of  $\mathfrak{o}$  and  $dDa \in \mathfrak{a}$  for every element  $a \in \mathfrak{a}$ . Then we can define an operator  $D'$  to be  $D'(\phi(x)) = \phi(dDx)/\phi(d)$ . Then

LEMMA 3.  $D'$  can be extended uniquely to a derivation of  $\mathfrak{o}/\mathfrak{a}$ .

The derivation obtained above is called the derivation induced on  $\mathfrak{o}/\mathfrak{a}$  by  $D$  (observe that  $D'$  is independent of the choice of  $d$ ).

*Proof.* Let  $x/y$  be an element of the total quotient ring of  $\mathfrak{o}/\mathfrak{a}$  ( $x, y \in \mathfrak{o}/\mathfrak{a}$ ). Then we define  $D^*(x/y) = (yD'x - xD'y)/y^2$ . Then it is seen that  $D^*$  is a derivation  $\mathfrak{o}/\mathfrak{a}$ . The uniqueness is obvious because any extension of  $D'$  must satisfy the defining property of  $D^*$ .

**4. Separably generated extensions and regular extensions.** We say that a field  $L$  is *separably generated* over a subfield  $\mathbf{k}$  if either  $\mathbf{k}$  is of characteristic 0 or  $L \otimes_{\mathbf{k}} \mathbf{k}^{1/p}$  is an integral domain, where  $p$  is the characteristic of  $\mathbf{k}$ . We say that  $L$  is a *regular extension* of  $\mathbf{k}$  if  $L$  is separably generated over  $\mathbf{k}$  and if  $\mathbf{k}$  is algebraically closed in  $L$ . Then the following three lemmas are well known (see Weil [24]).

LEMMA 1.  $L$  is separably generated over  $\mathbf{k}$  if and only if every subfield of  $L$  which is finitely generated over  $\mathbf{k}$  has a separating transcendence base. When  $L$  is finitely generated over  $\mathbf{k}$ ,  $L$  is separably generated if and only if  $L$  has a separating transcendence base.

LEMMA 2.  $L$  is separably generated over  $\mathbf{k}$  if and only if  $L \otimes_{\mathbf{k}} K$  is an integral domain for every field  $K$  which contains  $\mathbf{k}$  and in which  $\mathbf{k}$  is separably algebraically closed.

LEMMA 3. The following three conditions for the field  $L$  are equivalent to each other:

- 1)  $L$  is a regular extension of  $\mathbf{k}$ .
- 2)  $L \otimes_{\mathbf{k}} K$  is an integral domain for every field  $K$  containing  $\mathbf{k}$ .
- 3)  $L \otimes_{\mathbf{k}} K$  is an integral domain for every finite algebraic extension of  $\mathbf{k}$ .

We say that an integral domain  $\mathfrak{o}$  is *separably generated* over a subring  $I$  if the field of quotients of  $\mathfrak{o}$  is separably generated over that of  $I$ ; we say that  $\mathfrak{o}$  is a *regular extension* of  $I$  if the field of quotients of  $\mathfrak{o}$  is a regular extension of that of  $I$ . When a subset  $T$  of  $\mathfrak{o}$  forms a separating transcendence base of the field of quotients of  $\mathfrak{o}$  over that of  $I$ , we say that  $T$  is a *separating transcendence base* of  $\mathfrak{o}$  over  $I$ .

Now we shall treat the case where  $I$  is a field or a Dedekind domain. Let  $\mathbf{k}$  be the field of quotients of  $I$ . We denote by  $p$  either the number 1 or the characteristic of  $I$  according as this characteristic is zero or not. By definition, by Theorem 1 and by Lemmas 1-3, we have immediately the following two theorems.

**THEOREM 2.** *Let  $\mathfrak{o}$  be an integral domain which contains the ring  $I$  (which is a field or a Dedekind domain). Then the following four conditions are equivalent to each other:*

- 1)  $\mathfrak{o}$  is separably generated over  $I$ .
- 2) If  $\mathfrak{s}$  is an affine ring over  $I$  contained in  $\mathfrak{o}$ , then  $\mathfrak{s}$  has a separating transcendence base over  $I$ .
- 3) For a fixed natural number  $m$ ,  $\mathfrak{o} \otimes_I I^{1/p^m}$  is an integral domain. (This condition may be separated into infinitely many conditions with  $m = 1, 2, \dots$ )
- 4) If  $\mathbf{k}$  is separably algebraically closed in a field  $K$ , then  $\mathfrak{o} \otimes_I K$  is an integral domain.

**THEOREM 3.** *With the same  $\mathfrak{o}$  as above, the following four conditions are equivalent to each other:*

- 1)  $\mathfrak{o}$  is a regular extension of  $I$ .
- 2)  $\mathfrak{o} \otimes_I I'$  is an integral domain for every finite integral extension  $I'$  of  $I$ .
- 3) If  $\mathfrak{s}$  is an integral domain containing  $I$ ,  $\mathfrak{o} \otimes_I \mathfrak{s}$  is an integral domain.
- 4)  $\mathfrak{o}$  is separably generated over  $I$  and  $\mathbf{k}$  is algebraically closed in the field of quotients of  $\mathfrak{o}$ .

**COROLLARY 1.** *Assume that an integral domain  $\mathfrak{o}$  is a regular extension of the ring  $I$ . If  $K$  is an integral domain containing  $I$ , then  $\mathfrak{o} \otimes_I K$  is a regular extension of  $K$ .*

*Proof.* By the definition of regularity and by Theorem 1, we may assume that  $K$  is a field. Let  $L$  be an arbitrary field containing  $K$ . Then

$(\mathfrak{o} \otimes_I K) \otimes_K L = \mathfrak{o} \otimes_I L$  by Lemma 3.1.2. Since  $\mathfrak{o}$  is a regular extension of  $I$ ,  $\mathfrak{o} \otimes_I L$  is an integral domain, which shows that  $\mathfrak{o} \otimes K$  is a regular extension of  $K$ .

**COROLLARY 2.** *Let  $\mathfrak{o}$  and  $\mathfrak{o}'$  be integral domains which contain the ring  $I$ . If  $\mathfrak{o}$  and  $\mathfrak{o}'$  are regular extensions of  $I$ , then  $\mathfrak{o} \otimes_I \mathfrak{o}'$  is also a regular extension of  $I$ .*

*Proof.* Let  $K$  be a field containing  $I$ . Then

$$(\mathfrak{o} \otimes \mathfrak{o}') \otimes K = (\mathfrak{o} \otimes K) \otimes_K (\mathfrak{o}' \otimes K)$$

by Lemma 3.1.2. Since  $\mathfrak{o} \otimes K$  and  $\mathfrak{o}' \otimes K$  are regular extensions of  $K$  by Corollary 1, we see that  $(\mathfrak{o} \otimes \mathfrak{o}') \otimes K$  is an integral domain, which proves our assertion.

**PROPOSITION 3.** *Let  $\mathfrak{o}$  be an integral domain finitely generated over a ring  $I$  which is not necessarily a Dedekind domain, and let  $\phi$  be a homomorphism from  $I[x_1, \dots, x_n]$  onto  $\mathfrak{o}$ , where  $x_1, \dots, x_n$  are algebraically independent elements over  $I$ . If  $\mathfrak{p}$  is the kernel of  $\phi$  and if  $f_1, \dots, f_m$  is a base of  $\mathfrak{p}$ , then the rank  $r$  of the Jacobian matrix  $J(f_1, \dots, f_m)$  modulo  $\mathfrak{p}$  is not greater than the rank  $s$  of  $\mathfrak{p}$ ;  $r=s$  if and only if  $\mathfrak{o}$  is separably generated.*

*Proof.* We may assume that  $I$  is a field. Then the assertion follows from the Corollary to Lemma 3.3.1 and Lemma 1.

**PROPOSITION 4.** *Let  $x_1, \dots, x_n$  be algebraically independent elements over an integral domain  $I$  of characteristic  $p \neq 0$ . Let  $\alpha$  be an ideal of  $I[x_1, \dots, x_n]$  with  $\alpha \cap I = 0$  and let  $\mathbf{k}$  be the field of quotients of  $I$ . Assume that  $\alpha \mathbf{k}[x_1, \dots, x_n]$  is of rank  $r$ . Then there exist elements  $y_1, \dots, y_n \in I[x_1, \dots, x_n]$  and an element  $a (\neq 0) \in I$  such that*

- 1)  $I[1/a, x_1, \dots, x_n]$  is integral over  $I[1/a, y_1, \dots, y_n]$ ,
- 2)  $\alpha \cap I[1/a, y_1, \dots, y_n]$  is generated by  $y_1, \dots, y_r$  and
- 3) for every derivation  $D$  of  $I[x_1, \dots, x_n]$ ,  $Dy_j = Dx_j$  for  $j = r+1, \dots, n$ .

*Proof.* By Remark 2 after Lemma 1.1.1, we see that the  $y_i$ 's in Corollary 1 to Proposition 1.1 can be chosen so that  $y_j - x_j$  is in  $\pi[x_1^p, \dots, x_r^p]$  for  $j = r+1, \dots, n$ , where  $\pi$  denotes the prime integral domain. In this case, the  $y_i$  are the required elements.

**THEOREM 4. (NORMALIZATION THEOREM FOR SEPARABLY GENERATED INTEGRAL DOMAINS)** *Let  $\mathfrak{o}$  be an integral domain finitely generated over a subring  $I$ . If  $\mathfrak{o}$  is separably generated over  $I$ , then there exist a separating transcendence base  $z_1, \dots, z_t$  of  $\mathfrak{o}$  over  $I$  and an element  $a$  ( $\neq 0$ ) of  $I$  such that  $\mathfrak{o}[1/a]$  is integral over  $I[1/a, z_1, \dots, z_t]$ .*

*Proof.* When  $I$  is of characteristic zero, our assertion is nothing but the normalization theorem (Corollary 2 to Proposition 1.1). Therefore we assume that  $I$  is of characteristic  $p \neq 0$ . Let  $D_1, \dots, D_t$  be a maximal linearly independent set of derivations of  $\mathfrak{o}$  over  $I$ . Then  $t =$  transcendence degree of  $\mathfrak{o}$  over  $I$  by Lemma 3.3.2. Take elements  $y_1, \dots, y_t$  of  $\mathfrak{o}$  such that  $D_i y_j \neq 0$  if and only if  $i = j$ . Let  $w_1, \dots, w_s$  be elements of  $\mathfrak{o}$  such that  $\mathfrak{o} = I[w_1, \dots, w_s, y_1, \dots, y_t]$  and apply the proof of Proposition 4; we see that the  $z_i$ 's in Corollary 2 to Proposition 1.1 can be selected so that  $y_i - z_i \in \pi[w_1^p, \dots, w_s^p]$  ( $\pi$  being the prime integral domain of  $I$ ). Then for every derivation  $D$  of  $\mathfrak{o}$ ,  $Dy_i = Dz_i$ . Therefore, in particular,  $D_i z_j \neq 0$  if and only if  $i = j$ . Let  $D$  be an arbitrary derivation of  $\mathfrak{o}$  over  $I[z_1, \dots, z_t]$ . Then there are elements  $f$  ( $\neq 0$ ),  $f_1, \dots, f_t \in \mathfrak{o}$  such that  $fD = \sum f_i D_i$  (because  $D$  is a derivation of  $\mathfrak{o}$  over  $I$ ). Then  $0 = fDz_j = f_j D_j z_j$ ,  $f_j = 0$  for every  $j$ , and  $D = 0$ . Thus we see that  $\mathfrak{o}$  is separably algebraic over  $I[z_1, \dots, z_t]$ . The other property of the  $z_i$ 's is asserted in Corollary 2 to Proposition 1.1.

**PROPOSITION 5.** *Let  $\mathfrak{o}$  be a Noetherian integral domain which contains a Dedekind domain  $I$ . Assume that the derived normal ring  $\mathfrak{o}'$  of  $\mathfrak{o}$  is a finite  $\mathfrak{o}$ -module. Then for all but a finite number of prime ideals  $\mathfrak{p}$  of  $I$ ,  $\mathfrak{p}\mathfrak{o}$  has no imbedded prime divisor.*

*Proof.* Let  $\mathfrak{P}$  be the set of prime ideals  $\mathfrak{p}$  of  $I$  such that  $\mathfrak{p}\mathfrak{o}$  has imbedded prime divisors. For each  $\mathfrak{p} \in \mathfrak{P}$ , let  $S(\mathfrak{p})$  be the complement of  $\mathfrak{p}$  in  $I$ . Since every element of  $S$  is a unit modulo  $\mathfrak{p}$ ,  $\mathfrak{p}\mathfrak{o}_{S(\mathfrak{p})}$  has imbedded prime divisors. Let  $x_{\mathfrak{p}}$  be an element of  $\mathfrak{p}$  such that  $x_{\mathfrak{p}} I_{\mathfrak{p}} = \mathfrak{p} I_{\mathfrak{p}}$ . Then there exists an element  $a_{\mathfrak{p}} \in \mathfrak{o}_{S(\mathfrak{p})}$  such that  $a_{\mathfrak{p}}/x_{\mathfrak{p}} \notin \mathfrak{o}_{S(\mathfrak{p})}$  and 2)  $a_{\mathfrak{p}}/x_{\mathfrak{p}}$  is integral over  $\mathfrak{o}_{S(\mathfrak{p})}$ . Then multiplying by a suitable element of  $S(\mathfrak{p})$ , we may assume that  $a_{\mathfrak{p}}/x_{\mathfrak{p}}$  is integral over  $\mathfrak{o}$  (but  $a_{\mathfrak{p}}/x_{\mathfrak{p}} \notin \mathfrak{o}$  because  $a_{\mathfrak{p}}/x_{\mathfrak{p}} \notin \mathfrak{o}_{S(\mathfrak{p})}$ ). Since  $\mathfrak{o}'$  is a finite  $\mathfrak{o}$ -module, there exist a finite number of  $\mathfrak{p}$ 's, say  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  such that all  $a_{\mathfrak{p}}/x_{\mathfrak{p}}$  are in the ring  $\mathfrak{o}^* = \mathfrak{o}[a_{\mathfrak{p}_1}/x_{\mathfrak{p}_1}, \dots, a_{\mathfrak{p}_n}/x_{\mathfrak{p}_n}]$ . Let  $x$  be the product of the  $x_{\mathfrak{p}_i}$ 's. We shall show that if  $\mathfrak{p} \in \mathfrak{P}$ , then  $x \in \mathfrak{p}$ . Indeed, if  $x \notin \mathfrak{p}$ , then  $x$  is in  $S(\mathfrak{p})$  and  $\mathfrak{o}_{S(\mathfrak{p})}$  contains  $\mathfrak{o}^*$  and therefore also  $a_{\mathfrak{p}}/x_{\mathfrak{p}}$ , which is a contradiction. Thus  $\mathfrak{P}$  is a finite set.

*Remark.* From the above proof, we see that if  $\mathfrak{o}$  is a normal ring, then for every prime ideal  $\mathfrak{p}$  of  $I$ ,  $\mathfrak{p}\mathfrak{o}$  has no imbedded prime divisor.

**PROPOSITION 6.** *Let  $\mathfrak{o}$  and  $I$  be the same as in Proposition 5. If  $d$  is an element of  $\mathfrak{o}$  which is different from zero, then for all but a finite number of prime ideals  $\mathfrak{p}$  of  $I$ ,  $d$  is not a zero-divisor modulo  $\mathfrak{p}\mathfrak{o}$ .*

*Proof.* Since there exist only a finite number of prime ideals  $\mathfrak{p}$  of  $I$  such that  $\mathfrak{p}\mathfrak{o}$  has imbedded prime divisors by Proposition 5, we may omit such prime ideals from consideration. Then  $d$  being a zero-divisor modulo  $\mathfrak{p}\mathfrak{o}$  implies that  $d$  is in a minimal prime divisor of  $\mathfrak{p}\mathfrak{o}$ . Let  $S$  be the complement of  $\mathfrak{p}$  in  $I$ . Then every prime divisor of  $\mathfrak{p}\mathfrak{o}$  does not meet  $S$  and every (minimal) prime divisor of  $\mathfrak{p}\mathfrak{o}_S$  is of rank 1 (because  $\mathfrak{p}\mathfrak{o}_S$  is principal). Therefore, every prime divisor of  $\mathfrak{p}\mathfrak{o}$  is of rank 1. Therefore, a prime divisor  $\mathfrak{q}$  of  $\mathfrak{p}\mathfrak{o}$  contains  $d$  if and only if  $\mathfrak{q}$  is a minimal prime divisor of  $d\mathfrak{o}$ . Consequently,  $d$  is a zero-divisor modulo  $\mathfrak{p}\mathfrak{o}$  only when  $\mathfrak{p}$  is the intersection of a minimal prime divisor  $\mathfrak{q}$  of  $d\mathfrak{o}$  with  $I$  (except for  $\mathfrak{p}$ 's such that  $\mathfrak{p}\mathfrak{o}$  has imbedded prime divisors). Thus we see our assertion.

**THEOREM 5.** *Let  $\mathfrak{o}$  be an affine ring over a ground ring  $I$ . If  $\mathfrak{o}$  is separably generated over  $I$ , then for all but a finite number of prime ideals  $\mathfrak{p}$  of  $I$ ,  $\mathfrak{p}\mathfrak{o}$  is a semi-prime ideal and  $\mathfrak{o}/\mathfrak{p}'$  is separably generated over  $I/\mathfrak{p}$  for every prime divisor  $\mathfrak{p}'$  of  $\mathfrak{p}\mathfrak{o}$ .*

*Proof.* We take elements  $a, z_1, \dots, z_t$  as in Theorem 4 and let  $L$  be the field of quotients of  $\mathfrak{o}$ . Since only a finite number of prime ideals of  $I$  contain  $a$  and  $a$  is unit modulo the remaining primes, we may consider  $\mathfrak{o}[1/a]$  and  $I[1/a]$  instead of  $\mathfrak{o}$  and  $I$  respectively. Thus we may assume that  $\mathfrak{o}$  is integral over  $I[z_1, \dots, z_t]$ . Let  $b$  be an element of  $\mathfrak{o}$  such that  $I[z_1, \dots, z_t, b]$  is an affine ring of  $L$  and let  $d$  be the discriminant of the irreducible monic polynomial over  $I[z_1, \dots, z_t]$  which has  $b$  as a root. Since the derived normal ring of  $\mathfrak{o}$  is a finite  $\mathfrak{o}$ -module (because  $b$  is separable), there exist only a finite number of prime ideals  $\mathfrak{p}$  of  $I$  such that  $d$  is a zero-divisor modulo  $\mathfrak{p}\mathfrak{o}$  by Proposition 6 and we may omit such prime ideals from consideration. Then  $b$  modulo  $\mathfrak{p}\mathfrak{o}$  ( $\mathfrak{p}$  being a prime ideal of  $I$ ) is separable over  $I[z_1, \dots, z_t]/\mathfrak{p}I[z_1, \dots, z_t]$  and  $\mathfrak{p}I[z_1, \dots, z_t]$  is a semi-prime ideal. Since  $\mathfrak{o}$  is contained in  $I[z_1, \dots, z_t, b, 1/d]$ , and since  $d$  is not a zero-divisor modulo  $\mathfrak{p}\mathfrak{o}$ , we see that  $\mathfrak{p}\mathfrak{o}$  is also a semi-prime ideal and that if  $\mathfrak{p}'$  is a prime divisor of  $\mathfrak{p}\mathfrak{o}$ , then  $\mathfrak{o}/\mathfrak{p}'$  is separably generated over  $I/\mathfrak{p}$ .

We say that a polynomial (in several indeterminates) with coefficients in an integral domain  $I$  is *absolutely irreducible* if it is irreducible even

when it is regarded as a polynomial over the integral closure of  $I$  in the algebraic closure of the field of quotients of  $I$ .

**PROPOSITION 7.** *Let  $f$  be an absolutely irreducible polynomial in indeterminates  $x_1, \dots, x_n$  over a Noetherian integral domain  $I$ . Then for all but a finite number of prime ideals  $\mathfrak{p}$  of rank 1 in  $I$ , the polynomial  $f' = (f \text{ modulo } \mathfrak{p})$  is also absolutely irreducible over  $I/\mathfrak{p}$ .*

*Proof.*<sup>4</sup> Let  $d$  be the degree of  $f$  and  $e$  an arbitrary natural number less than  $d$ . Let  $\{R_1, \dots, R_r\}$ ,  $\{S_1, \dots, S_s\}$ ,  $\{T_1, \dots, T_t\}$  be the set of all monomials in  $x_1, \dots, x_n$  whose degrees are not greater than  $d$ ,  $e$ ,  $d-e$  respectively, and let  $y_1, \dots, y_s, z_1, \dots, z_t$  be indeterminates. Set  $G = \sum y_i S_i$ ,  $H = \sum z_j T_j$  and  $F = GH$ . Further we express  $F$  in the form  $F = \sum w_i R_i$  or  $\sum w'_i R_i$ ,  $w_i$  and  $w'_i$  being polynomials in  $y_1, \dots, y_s, z_1, \dots, z_t$  with coefficients in the prime integral domain of  $I$  or of  $I/\mathfrak{p}$  respectively. Let  $X_1, \dots, X_r$  be indeterminates, and let  $\sigma$  and  $\sigma'$  be the homomorphisms from  $I[X_1, \dots, X_r]$  onto  $I[w_1, \dots, w_r]$  and from  $(I/\mathfrak{p})[X_1, \dots, X_r]$  onto  $(I/\mathfrak{p})[w'_1, \dots, w'_r]$  respectively such that  $\sigma(X_i) = w_i$  and  $\sigma'(X_i) = w'_i$ . Further, let  $\mathfrak{q}$  and  $\mathfrak{q}'$  be the kernels of  $\sigma$  and  $\sigma'$  respectively. Then 1)  $\mathfrak{q}'$  contains  $\mathfrak{q}$  modulo  $\mathfrak{p}$ , 2) if  $a_1, \dots, a_r$  are elements of  $I$  such that  $g(a_1, \dots, a_r) = 0$  for every element  $g(X_1, \dots, X_r)$  of  $\mathfrak{q}$ , then  $\sum a_i R_i$  has a factor of degree  $e$  in some algebraic extension of  $I$  (and conversely), and 3) if  $a'_1, \dots, a'_r$  are elements of  $I/\mathfrak{p}$  such that  $\sum a'_i R_i$  has a factor of degree  $e$  in some algebraic extension of  $I/\mathfrak{p}$ , then  $g'(a'_1, \dots, a'_r) = 0$  for every element  $g'(X_1, \dots, X_r)$  of  $\mathfrak{q}'$  (and conversely). Now let  $c_1, \dots, c_r$  be elements of  $I$  such that  $f = \sum c_i R_i$  and let  $c'_i$  be the residue class of  $c_i$  modulo  $\mathfrak{p}$  for each  $i$ . Since  $f$  is absolutely irreducible, there exists an element  $g(X_1, \dots, X_r)$  of  $\mathfrak{q}$  such that  $g(c_1, \dots, c_r) \neq 0$ . Let  $g'$  be  $g$  modulo  $\mathfrak{p}$ . Then  $g' \in \mathfrak{q}'$ , and  $f'$  has a factor of degree  $d$  in an algebraic extension of  $I/\mathfrak{p}$  only when  $g'(c'_1, \dots, c'_r) = 0$ , i.e.,  $g(c_1, \dots, c_r) \in \mathfrak{p}$ . Thus we see that  $f'$  has no factor of degree  $e$  in any algebraic extension of  $I/\mathfrak{p}$  for all but a finite number of  $\mathfrak{p}$ 's. Applying this for each  $e = 1, \dots, d-1$ , we prove our proposition.

**THEOREM 6.** *Let  $\mathfrak{o}$  be an affine ring over a ground ring  $I$ . If  $\mathfrak{o}$  is a regular extension of  $I$ , then for all but a finite number of prime ideals  $\mathfrak{p}$  of  $I$ ,  $\mathfrak{p}\mathfrak{o}$  is a prime ideal and  $\mathfrak{o}/\mathfrak{p}\mathfrak{o}$  is a regular extension of  $I/\mathfrak{p}$ .*

*Proof.* As in the proof of Theorem 5, we may assume that 1)  $z_1, \dots, z_t$  ( $\in \mathfrak{o}$ ) are algebraically independent over  $I$ , 2)  $\mathfrak{o}$  is integral over  $I[z_1, \dots, z_t]$

<sup>4</sup> The writer owes the present proof to Shimura [23].

and 3)  $b \in \mathfrak{o}$  is separable over  $I[z_1, \dots, z_t]$ , and furthermore, that  $\mathfrak{o}$  and  $I[z_1, \dots, z_t, b]$  have the same field of quotients. Let  $f(X)$  be the irreducible monic polynomial over  $I[z_1, \dots, z_t]$  which has  $b$  as a root and let  $d$  be the discriminant of  $f(X)$ . Since  $\mathfrak{o}$  is a regular extension of  $I$ ,  $f(x)$  is absolutely irreducible as a polynomial in  $z_1, \dots, z_t, X$  over  $I$ . Therefore  $f$  modulo  $\mathfrak{p}$  is absolutely irreducible for all but a finite number of prime ideals  $\mathfrak{p}$  of  $I$  by Proposition 7, that is,  $I[z_1, \dots, z_t, b]/\mathfrak{p}I[z_1, \dots, z_t, b]$  is (an integral domain and) a regular extension of  $I/\mathfrak{p}$ . Since we may neglect those  $\mathfrak{p}$ 's such that  $d$  is a zero-divisor modulo  $\mathfrak{p}\mathfrak{o}$  and since  $\mathfrak{o}$  is contained in  $I[z_1, \dots, z_t, b, 1/d]$ , we see that  $\mathfrak{o}/\mathfrak{p}\mathfrak{o}$  is a regular extension of  $I/\mathfrak{p}$ .

**PROPOSITION 8.** *Let  $\mathfrak{p}$  be a prime ideal of rank  $r$  in an affine ring  $\mathfrak{o}$  over a ground ring  $I$ . If  $\mathfrak{o}$  is separably generated over  $I$ , then there exists a separating transcendence base  $x_1, \dots, x_n$  of  $\mathfrak{o}$  over  $I$  such that 1) when  $\mathfrak{p} \cap I = 0$ ,  $x_1, \dots, x_r$  is a system of parameters of  $\mathfrak{o}_{\mathfrak{p}}$  and the field of quotients of  $I[x_{r+1}, \dots, x_n]$  is a basic field of  $\mathfrak{o}_{\mathfrak{p}}$ ; 2) when  $\mathfrak{p} \cap I \neq 0$ ,  $x_1, \dots, x_{r-1}, x$  ( $x$  being a prime element of  $I(\mathfrak{p} \cap I)$ ) is a system of parameters of  $\mathfrak{o}_{\mathfrak{p}}$  and  $I(\mathfrak{p} \cap I)(x_r, \dots, x_n)$  is a basic ring of  $\mathfrak{o}_{\mathfrak{p}}$ .*

*Proof.* When  $I$  is of characteristic zero or when  $\mathfrak{p} = 0$ , our assertion is obvious and we assume that  $I$  is of characteristic  $p \neq 0$  and that  $\mathfrak{p} \neq 0$ . We set  $P = \mathfrak{o}_{\mathfrak{p}}$ . Let  $D_1, \dots, D_n$  be a maximal linearly independent set of derivations of  $\mathfrak{o}$  over  $I$ ; since  $\mathfrak{o}$  is separably generated,  $n = \text{transcendence degree of } \mathfrak{o} \text{ over } I$  by Lemma 3.3.2 and by Proposition 2. Let  $y_1, \dots, y_n$  be elements of  $\mathfrak{o}$  such that  $D_i y_j \neq 0$  if and only if  $i = j$ ; here we may assume that  $y_j - 1$  is in  $\mathfrak{p}$  for all  $j$  because, otherwise, we may take  $1 + y_i z^p$  with  $z \in \mathfrak{p}$  instead of  $y_i$  (observe that  $D(1 + y_i z^p) = D(y_i z^p) = z^p D y_i$  for every derivation  $D$  of  $\mathfrak{o}$ ). On the other hand, let  $z_1, \dots, z_n$  be elements of  $\mathfrak{o}$  such that 1) when  $\mathfrak{p} \cap I = 0$ ,  $z_1, \dots, z_r$  form a system of parameters of  $P$  and  $z_{r+1}, \dots, z_n$  modulo  $\mathfrak{p}$  are algebraically independent over  $I$ ; 2) when  $\mathfrak{p} \cap I \neq 0$ ,  $x, z_1, \dots, z_{r-1}$  form a system of parameters of  $P$  and  $z_r, \dots, z_n$  modulo  $\mathfrak{p}$  are algebraically independent over  $I/(\mathfrak{p} \cap I)$  (existence follows from Theorem 1.1). Set  $x_i = y_i z_i^p$ . Then  $D_i x_j \neq 0$  if and only if  $i = j$ , which shows that  $x_1, \dots, x_n$  is a separating transcendence base of  $\mathfrak{o}$  over  $I$ . Since  $y_j - 1 \in \mathfrak{p}$ , we see that these  $x_1, \dots, x_n$  are the required elements (see the proof of Proposition 1.3).

**COROLLARY.** *If a spot  $P$  is separably generated over a ground ring  $I$ , then there exists a basic ring  $B$  of  $P$  such that  $P$  is separably generated over  $B$ . Therefore, if  $P$  is a normal spot which is separably generated over a*

ground ring  $I$ , then there exists a basic ring  $B$  of  $P$  such that  $P$  is a regular extension of  $B$ .

*Proof.* The first assertion is immediate from Proposition 8, while the last assertion follows from the first one and from the definition of regularity (take the integral closure in  $P$  of the  $B$  given by the first assertion).

*Remark.* In the above corollary, if  $I$  satisfies the finiteness condition for integral extensions, then we can select  $B$  to satisfy the same condition, as is easily seen from our proof by virtue of Proposition 1.2.

### 5. A lemma due to Zariski.

LEMMA 1. *Let  $L$  be a finite algebraic extension of a field  $K$ . If there exists an element  $a$  of  $L$  such that  $L$  is separable over  $K(a)$ , then  $L$  is a simple extension of  $K$ , that is, there exists an element  $c$  of  $L$  such that  $L = K(c)$ .*

*Proof.* Let  $L'$  be the maximal separable subfield of  $L$  over  $K$ . Then there exists an element  $d$  of  $L'$  such that  $L' = K(d)$ . Let  $b$  be an element of  $L$  such that  $L = K(a, b)$ . Then  $b$  is purely inseparable over  $K(a, d)$  and separable over  $K(a, d)$ . Thus  $b$  is in  $K(a, d)$  and  $L = K(a, d)$ . Since  $d$  is separable over  $K$ ,  $L = K(a, d)$  is a simple extension of  $K$ .

LEMMA 2. *Let  $L$  be a simple algebraic extension of a field  $K$ . Then the number of fields  $L'$  such that  $K \subseteq L' \subseteq L$  is at most  $2^{n-1}$ , where  $n$  is the degree of  $L$  over  $K$ .*

*Proof.* Let  $b$  be an element of  $L$  such that  $L = K(b)$ , and let  $b = b_1, b_2, \dots, b_n$  be the conjugates of  $b$  (if  $L$  is inseparable over  $K$ , each  $b_i$  appears just  $[L:K]_i$ -times). If  $K \subseteq L' \subseteq L$ , then there are  $i_1, \dots, i_r$  (each  $i_j > 1$ ) such that  $(x-b)(x-b_{i_1}) \cdots (x-b_{i_r})$  is the irreducible polynomial over  $L'$  which has  $b$  as a root. Then the field generated over  $K$  by the coefficients of this polynomial coincides with  $L'$  because  $L = L'(b)$ . Therefore, there exists a one-to-one correspondence between  $L'$  and a family of subsets of  $2, \dots, n$ , which proves our assertion.

The following lemma is obvious.

LEMMA 3. *Assume that a function field  $L$  over a field  $k$  is separably generated over  $k$ . Then for an element  $x$  of  $L$ , the following three conditions are equivalent to each other:*

- 1) *There exists a derivation  $D$  of  $L$  over  $k$  such that  $Dx \neq 0$ .*

2) There exists a separating transcendence base of  $L$  over  $\mathbf{k}$  which has  $x$  as a member.

3) When  $\mathbf{k}$  is of characteristic zero,  $x$  is transcendental over  $\mathbf{k}$ ; when  $\mathbf{k}$  is of characteristic  $p \neq 0$ ,  $x$  is not in  $\mathbf{k}(L^p)$ .

*Remark.* Let  $L$  be a function field over a field  $\mathbf{k}$  of characteristic  $p \neq 0$ , and let  $x_1, \dots, x_n$  be elements of  $L$  such that  $L = \mathbf{k}(L^p)(x_1, \dots, x_n)$  with  $n$  such that  $[L : \mathbf{k}(L^p)] = p^n$ . Then the derivations  $D_1, \dots, D_n$ , such that  $D_i x_j$  is 1 or 0 according as  $i = j$  or  $i \neq j$ , form a maximal linearly independent set of derivations of  $L$  over  $\mathbf{k}$ .

Now, making use of the above lemmas, we shall prove a generalization of a lemma due to Zariski [25] by the methods of Zariski [25] and Matusaka [10]:

**THEOREM 7.** Let  $L$  be a function field over a ground field  $\mathbf{k}$ .<sup>5</sup> Assume that  $x$  and  $y$  are elements of  $L$  which are algebraically independent over  $\mathbf{k}$  and that there exists a derivation  $D$  of  $L$  over  $\mathbf{k}$  such that  $Dx \neq 0$ . Let  $L'$  be the algebraic closure of  $\mathbf{k}(x, y)$  in  $L$ . If  $L$  is a regular extension of  $\mathbf{k}$ , then  $L$  is a regular extension of  $\mathbf{k}(x + cy)$  for all except possibly  $2^{n-1} + 1$  elements  $c$  of  $\mathbf{k}$ , where  $n = [L' : \mathbf{k}(x, y)]$ .

*Proof.* For every element  $c$  of  $\mathbf{k}$ , we denote by  $\mathbf{k}_c$  the algebraic closure of  $\mathbf{k}(x + cy)$  in  $L'$  (hence in  $L$ ). Consider the following conditions for an element  $c$  of  $\mathbf{k}$ : 1)  $D(x + cy) \neq 0$  and 2) there exists an element  $d (\neq c)$  of  $\mathbf{k}$  such that  $\mathbf{k}_c(x + dy) = \mathbf{k}_d(x + cy)$ . Since  $D(x + cy) = Dx + cDy$  and since  $Dx \neq 0$ , there exists at most one  $c$  which does not satisfy the condition 1). On the other hand, since  $x \notin \mathbf{k}(L^p)$ , there exists an element  $z$  of  $L'$  such that  $x$  and  $z$  form a separating transcendence base of  $L'$  over  $\mathbf{k}$ , and if  $w$  is an element of  $L'$  such that  $L' = \mathbf{k}(x, z, w)$ , then  $L' = \mathbf{k}(x, y)(z, w)$ , and  $w$  is separable over  $\mathbf{k}(x, y, z)$  and  $L'$  is a simple extension of  $\mathbf{k}(x, y)$  by Lemma 1. Since  $\mathbf{k}_c(x + dy) = \mathbf{k}(x, y)(\mathbf{k}_c)$ , the number of  $c$ 's which do not satisfy the condition 2) is not greater than  $2^{n-1}$  by Lemma 2. Therefore, we have only to show that if an element  $c \in \mathbf{k}$  satisfies the conditions 1) and 2), then  $L$  is a regular extension of  $\mathbf{k}(x + cy)$ . Since  $\mathbf{k}_d$  is a regular extension of  $\mathbf{k}$  by Theorem 3 and since  $x + cy$  is transcendental over  $\mathbf{k}_d$ ,  $\mathbf{k}_d(x + cy) = \mathbf{k}_c(x + dy)$  is a regular extension of  $\mathbf{k}(x + cy)$  by Corollary 1 to Theorem 3. Since  $\mathbf{k}_c$  is algebraic over  $\mathbf{k}(x + cy)$ , we have  $\mathbf{k}_c = \mathbf{k}(x + cy)$ , which shows that  $\mathbf{k}(x + cy)$  is algebraically closed in  $L$ . Since  $D(x + cy) \neq 0$ , there

<sup>5</sup> By the definition of regularity, the assumption that  $\mathbf{k}$  is a field is not essential.

exists a separating transcendence base of  $L$  over  $\mathbf{k}$  which has  $x + cy$  as a member, which shows that  $L$  is a regular extension of  $\mathbf{k}(x + cy)$  and the proof is completed.

**COROLLARY 1.** *With the same  $L$ ,  $\mathbf{k}$ ,  $x$  and  $y$  as above, if  $u$  is transcendental over  $L$ , then  $L(u)$  is a regular extension of  $\mathbf{k}(u)(x + uy)$ .*

*Proof.* Let  $u_1, \dots, u_N$  ( $N > 2^{n-1} + 1$  with the same  $n$  as above) be algebraically independent elements over  $L$ . Let  $L'$  be the same as above.  $L'(u_1, \dots, u_N)$  is the algebraic closure of  $\mathbf{k}(u_1, \dots, u_N, x, y)$  in  $L(u_1, \dots, u_N)$  and  $[L'(u_1, \dots, u_N) : \mathbf{k}(u_1, \dots, u_N, x, y)] = n$ . Therefore, there exists one  $i$  such that  $L(u_1, \dots, u_N)$  is a regular extension of  $\mathbf{k}(u_i)(x + u_i y)$ . It follows that  $L(u_i)$  is a regular extension of  $\mathbf{k}(u_i)(x + u_i y)$  because  $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N$  are algebraically independent over  $\mathbf{k}(u_i)(x + u_i y)$ . Since  $u_i$  is transcendental over  $L$ , we may replace  $u_i$  by  $u$ , which proves our assertion.

**COROLLARY 2.** *Assume that an integral domain  $\mathfrak{o}$  is generated by  $x_1, \dots, x_N$  over a subring  $I$ . If  $\mathfrak{o}$  is a regular extension of  $I$ , if  $I$  contains infinitely many elements and if the transcendence degree of  $\mathfrak{o}$  over  $I$  is greater than 1, then there exists a linear combination  $\sum a_i x_i$  ( $a_i \in I$ ) such that  $\mathfrak{o}$  is a regular extension of  $I[\sum a_i x_i]$ .*

## 6. Tensor products of normal rings.

**LEMMA 1.** *Let  $\mathfrak{o}$  be a normal ring which contains a field  $\mathbf{k}$ . If a field  $K$  is separably generated over  $\mathbf{k}$  and if  $K \otimes_{\mathbf{k}} \mathfrak{o}$  is an integral domain, then  $K \otimes_{\mathbf{k}} \mathfrak{o}$  is a normal ring. (Y. Nakai)*

*Proof.* We may assume that  $K$  is finitely generated over  $\mathbf{k}$ . Let  $x_1, \dots, x_n$  be a separating transcendence base of  $K$  over  $\mathbf{k}$  and set  $\mathfrak{s} = \mathfrak{o}[x_1, \dots, x_n]$ ,  $\mathbf{k}' = \mathbf{k}(x_1, \dots, x_n)$  (with regard to the fact that  $K$  and  $\mathfrak{o}$  are subrings of  $K \otimes \mathfrak{o}$ ). Since  $x_1, \dots, x_n$  are algebraically independent over  $\mathfrak{o}$  and since  $\mathfrak{o}$  is a normal ring, we see that  $\mathfrak{s}$  is a normal ring. Since  $\mathbf{k}'[\mathfrak{o}]$  is a ring of quotients of  $\mathfrak{s}$ ,  $\mathbf{k}'[\mathfrak{o}]$  is a normal ring. Now since every element of  $K$  is a root of a monic polynomial over  $\mathbf{k}'$  (hence over  $\mathbf{k}'[\mathfrak{o}]$ ) whose discriminant is a unit in  $\mathbf{k}'$  (hence in  $\mathbf{k}'[\mathfrak{o}]$ ), we see that  $K[\mathfrak{o}]$  ( $= K \otimes \mathfrak{o}$ ) is a normal ring.

**THEOREM 8.** *Let  $\mathfrak{o}$  and  $\mathfrak{o}'$  be normal rings which contain a field  $\mathbf{k}$ . If  $\mathfrak{o}$  and  $\mathfrak{o}'$  are separably generated over  $\mathbf{k}$  and if  $\mathfrak{o} \otimes_{\mathbf{k}} \mathfrak{o}'$  is an integral domain, then  $\mathfrak{o} \otimes_{\mathbf{k}} \mathfrak{o}'$  is a normal ring. (Y. Nakai)*

*Proof.* Let  $L$  and  $L'$  be the fields of quotients of  $\mathfrak{o}$  and  $\mathfrak{o}'$  respectively. By Lemma 1,  $L \otimes \mathfrak{o}'$  and  $\mathfrak{o} \otimes L'$  are normal rings. Therefore, we have only to show that  $(L \otimes \mathfrak{o}') \cap (\mathfrak{o} \otimes L') = \mathfrak{o} \otimes \mathfrak{o}'$ . Let  $\{u_\lambda\}$  and  $\{u'_\lambda\}$  be linearly independent bases of  $\mathfrak{o}$  and  $\mathfrak{o}'$  over  $k$ , and let  $\{v_\mu\}$  and  $\{v'_\mu\}$  be linearly independent bases of  $L$  and  $L'$  over  $k$  which contain  $\{u_\lambda\}$  and  $\{u'_\lambda\}$  respectively. Then every element  $b$  of  $L \otimes L'$  is expressed uniquely in the form  $\sum v_\mu v'_\mu a_{\mu\mu'}$  ( $a_{\mu\mu'} \in k$ ). If  $b$  is in  $L \otimes \mathfrak{o}'$ , then in this expression  $v'_\mu$  is in  $\{u'_\lambda\}$  for all  $\mu'$  such that  $a_{\mu\mu'} \neq 0$ ; if  $b$  is in  $\mathfrak{o} \otimes L'$ , then  $v_\mu$  is in  $\{u_\lambda\}$  for all  $\mu$  such that  $a_{\mu\mu'} \neq 0$ . Therefore, if  $b$  is in  $(L \otimes \mathfrak{o}') \cap (\mathfrak{o} \otimes L')$ , then the expression must be of the form  $\sum u_\lambda u'_\lambda a_{\lambda\lambda'}$  ( $a_{\lambda\lambda'} \in k$ ), that is  $b \in \mathfrak{o} \otimes \mathfrak{o}'$ . Thus we see that  $(L \otimes \mathfrak{o}') \cap (\mathfrak{o} \otimes L') = \mathfrak{o} \otimes \mathfrak{o}'$  and the proof is completed.

The analogue of Theorem 8 does not hold in general for tensor products over a ring which is not a field, even if the ring is the case over a ground place. Though we shall discuss below under what condition such an extended result does hold, we shall first give a counter example:

*Example.* Let  $I$  be the ring of rational integers and let  $x$  and  $y$  be algebraically independent elements over  $I$ . Set  $I^* = I_{2I}$  and  $I' = I^*(x, y)$ . Then  $I'$  is a discrete valuation ring with a prime element 2 and is a ground place. Let  $z, z', w, w'$  be algebraically independent elements over  $I'$  and set  $\mathfrak{o} = I'[z, w, \sqrt{x+2z}, \sqrt{y+2w}]$ ,  $\mathfrak{o}' = I'[z', w', \sqrt{x+2z'}, \sqrt{y+2w'}]$ . Then  $\mathfrak{o}$  and  $\mathfrak{o}'$  are regular extensions of  $I'$ , and  $2\mathfrak{o}$  and  $2\mathfrak{o}'$  are prime ideals. Set  $P = \mathfrak{o}_{2\mathfrak{o}}$ ,  $P' = \mathfrak{o}'_{2\mathfrak{o}'}$ . Then  $P$  and  $P'$  are the valuation rings, hence they are normal spots over  $I'$  (and they are regular extensions of  $I'$ ). Since  $(P \otimes_{I'} P')/(2) = (P/2P) \otimes_{I'/2I'} (P'/2P')$ , we see that there exists only one minimal prime divisor  $\mathfrak{p}$  of  $2(P \otimes P')$  and that  $\mathfrak{p}(P \otimes P')_{\mathfrak{p}}/(2)$  is not principal. Therefore  $\mathfrak{p}(P \otimes P')_{\mathfrak{p}}$  is not principal. Since  $\mathfrak{p}$  is of rank 1, it follows that  $(P \otimes P')_{\mathfrak{p}}$  is not normal and  $P \otimes P'$  is also not normal.

**PROPOSITION 9.** Let  $\mathfrak{o}$  and  $\mathfrak{o}'$  be normal rings which contain a discrete valuation ring  $I$  with a prime element  $x$ . Assume that  $\mathfrak{o}^* = \mathfrak{o} \otimes_I \mathfrak{o}'$  is a Noetherian integral domain and that both  $\mathfrak{o}$  and  $\mathfrak{o}'$  are separably generated over  $I$ . Then  $\mathfrak{o}^*$  is a normal ring if and only if  $\mathfrak{o}^*_{\mathfrak{p}^*}$  is a normal ring for every prime divisor  $\mathfrak{p}^*$  of  $x\mathfrak{o}^*$ .

*Proof.* Let  $k$  be the field of quotients of  $I$ . Then  $k = I[1/x]$  and we have  $\mathfrak{o}^*[1/x] = \mathfrak{o}[1/x] \otimes \mathfrak{o}'[1/x]$  by Theorem 1. Therefore  $\mathfrak{o}^*[1/x]$  is a normal ring by Theorem 8. Now we prove our assertion easily (see [16, § 9]).

**COROLLARY.** Let  $\mathfrak{o}$  and  $\mathfrak{o}'$  be normal affine rings or spots over the same ground ring  $I$ . Assume that  $I$  is a valuation ring with a prime element  $x$ .

If the following four conditions are satisfied, then  $\mathfrak{o} \otimes_I \mathfrak{o}'$  is a normal ring.

- 1)  $\mathfrak{o}$  and  $\mathfrak{o}'$  are separably generated over  $I$ .
- 2)  $x\mathfrak{o}$  and  $x\mathfrak{o}'$  are semi-prime ideals.
- 3)  $\mathfrak{o} \otimes \mathfrak{o}'$  is an integral domain.
- 4) If  $\mathfrak{p}$  and  $\mathfrak{p}'$  are prime divisors of  $x\mathfrak{o}$  and  $x\mathfrak{o}'$  respectively, then  $(\mathfrak{o}/\mathfrak{p}) \otimes_{I/xI} (\mathfrak{o}'/\mathfrak{p}')$  has no nilpotent elements.

*Proof.* By our assumption,  $x(\mathfrak{o} \otimes \mathfrak{o}')$  is a semi-prime ideal and we see the assertion by Proposition 9.

*Remark 1.* Condition 4) is satisfied if one of  $\mathfrak{o}/\mathfrak{p}$  and  $\mathfrak{o}'/\mathfrak{p}'$  is separably generated over  $I/xI$  by Theorem 1.

*Remark 2.* Since the intersection of normal rings is again a normal ring, we can apply Proposition 9 and its corollary to tensor products over a Dedekind domain by virtue of Theorem 1.

#### Chapter 4. Preliminary Results from the Theory of Regular Local Rings.

##### 1. Ring extensions of regular local rings.

**PROPOSITION 1.** Let  $\mathfrak{o}$  be a local integral domain with maximal ideal  $\mathfrak{m}$  and let  $a$  be an element of an integral extension of  $\mathfrak{o}$ . Assume that 1)  $a$  is not in the field of quotients of  $\mathfrak{o}$ , 2) the characteristic  $p$  of  $\mathfrak{o}$  is different from zero and 3)  $a^p \in \mathfrak{o}$ . Then  $\mathfrak{o}[a]$  is a local ring. Furthermore, the dimension of the vector space  $\mathfrak{m}'/\mathfrak{m}'^2$  over  $\mathfrak{o}'/\mathfrak{m}'$  is not less than that of  $\mathfrak{m}/\mathfrak{m}^2$  over  $\mathfrak{o}/\mathfrak{m}$  ( $\mathfrak{m}'$  being the maximal ideal of  $\mathfrak{o}'$ ); they coincide with each other if and only if either the irreducible polynomial  $X^p - a^p$  over  $\mathfrak{o}$  is irreducible modulo  $\mathfrak{m}$ , or there exists an element  $b \in \mathfrak{o}$  such that  $(a - b)^p \in \mathfrak{m}$ ,  $\notin \mathfrak{m}^2$ .

*Proof.* Since  $a$  is purely inseparable over  $\mathfrak{o}$ ,  $\mathfrak{o}[a]$  is a local ring. Let  $x_1, \dots, x_r$  be a minimal base of  $\mathfrak{m}$ .<sup>6</sup>

1) When  $X^p - a^p$  is irreducible modulo  $\mathfrak{m}$ , it is easy to see that  $\mathfrak{m}'$  is generated by  $\mathfrak{m}$ . Furthermore,  $x_1, \dots, x_r$  is a minimal base of  $\mathfrak{m}'$ . For, otherwise, there exists an element  $x$  of  $\mathfrak{m}$  which is not in  $\mathfrak{m}^2$  such that  $x \in \mathfrak{m}'^2 = \mathfrak{m}^2 \mathfrak{o}[a]$ , which is impossible because  $1, a, \dots, a^{p-1}$  are linearly independent over  $\mathfrak{o}$ . Thus the case is settled.

<sup>6</sup> We call a base of an ideal is minimal if any proper subset of the base cannot generate the ideal. When  $\mathfrak{o}$  is a local ring with maximal ideal  $\mathfrak{m}$ , a subset of an ideal  $\mathfrak{a}$  of  $\mathfrak{o}$  generates  $\mathfrak{a}$  if and only if their residue classes modulo  $\mathfrak{m}\mathfrak{a}$  generates  $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$  over  $\mathfrak{o}/\mathfrak{m}$ . Therefore  $r$  is equal to the dimension of  $\mathfrak{m}/\mathfrak{m}^2$  in the present case.

2) Assume that  $X^p - a^p$  is reducible modulo  $m$ . Then there exists an element  $b \in o$  such that  $a^p - b^p \in m$ . Therefore, considering  $a - b$  instead of  $a$ , we may assume that  $b = 0$ , that is,  $a \in m'$ . Assume that an element  $x \in m$  which is not in  $m^2$  is in  $m'^2$ . Then, since  $m' = mo[a] + ao[a] = m + ao[a]$ , there is a relation of the form:

$$x = \sum c_{ij} x_i x_j + (\sum d_j x_j) a + (\sum_{i=0}^{p-1} e_i a^i) a^2 \quad (c_{ij}, d_j, e_i \in o).$$

Since  $1, a, \dots, a^{p-1}$  are linearly independent over  $o$ , we have

$$x = \sum c_{ij} x_i x_j + e_{p-2} a^p.$$

This shows that the residue class of  $x$  modulo  $m^2$  is of the form unit  $\times (a^p$  modulo  $m^2)$ . Therefore, the dimension of  $m'/m'^2$  is either  $r$  or  $r + 1$  according as  $a^p \notin m^2$  or  $a^p \in m^2$ . Thus the proof is completed.

COROLLARY 1. If furthermore  $o[a]$  is a regular local ring, then so is  $o$ .

COROLLARY 2. Besides the assumptions in Proposition 1, we assume that  $o$  is a regular local ring. Then  $o[a]$  is regular if and only if either  $X^p - a^p$  is irreducible modulo  $m$ , or there exists an element  $b$  of  $o$  such that  $(a - b)^p \in m$ ,  $\notin m^2$ .

PROPOSITION 2. Let  $o$  be a normal local ring and let  $a$  be a root of an irreducible monic polynomial  $f(x)$  over  $o$ . Let  $p$  be a maximal ideal of  $o[a]$ , and set  $o' = o[a]_p$ ,  $m' = po'$ . If the discriminant  $d$  of  $f(x)$  is a unit in  $o$ , then 1)  $m' = mo'$  and 2)  $o'$  is a normal ring.

Proof. Since  $d$  is a unit in  $o$ , we have that  $o[a]$  is a normal ring, and 2) follows immediately. Let  $g(x)$  be a monic polynomial over  $o$  such that i)  $g(x)$  modulo  $m$  is irreducible over  $o/m$  and ii)  $a$  modulo  $p$  is a root of  $g(x)$  modulo  $m$ . Set  $b = g(a)$ . Then  $b \in p$ . Set  $p'' = mo[b] + bo[b]$ , which is a maximal ideal of  $o[b]$  and  $p'' = p \cap o[b]$ . Since  $g(x)$  modulo  $m$  is irreducible over  $o/m = o[b]/p''$ ,  $p = p''o[a]$ . It follows, in particular, that  $p$  is the unique maximal ideal of  $o[a]$  containing  $b$ . Let  $h(x)$  be a monic polynomial over  $o$  such that i)  $f(x) - g(x)h(x) \in m[x]$  and ii) every irreducible factor of  $h(x)$  is irreducible modulo  $m$ . Then applying the same observation to each irreducible factor of  $h(x)$ , we see that  $c = h(a)$  is not in  $p$  (because  $d$  is a unit and  $g(x), h(x)$  have no common factor modulo  $m$ ). Since  $g(a) = b$ ,  $bc \in mo[a]$  and  $b \in m'$ , which shows that  $m' = mo'$  (because  $p = p''o[a]$ ).

COROLLARY. If furthermore  $o$  is a regular local ring, then  $o'$  is also a regular local ring.

**PROPOSITION 3.** *Let  $\mathfrak{o}$  be an integral domain and let  $x_1, \dots, x_n$  be algebraically independent elements over  $\mathfrak{o}$ . Let  $\mathfrak{p}$  be a prime ideal of  $\mathfrak{o}' = \mathfrak{o}[x_1, \dots, x_n]$ . If  $\mathfrak{o}_{(\mathfrak{p} \cap \mathfrak{o})}$  is a regular local ring, then so is  $\mathfrak{o}'_{\mathfrak{p}}$ .*

*Proof.* By Corollary 5 to Proposition 1.1,  $\mathfrak{o}'_{\mathfrak{p}}/(\mathfrak{p} \cap \mathfrak{o})\mathfrak{o}'_{\mathfrak{p}}$  is a regular local ring, from which the assertion follows immediately.

**LEMMA 1.** *Let  $x$  be a transcendental element over a local ring  $\mathfrak{o}$ . Then a minimal base of the maximal ideal  $\mathfrak{m}$  of  $\mathfrak{o}$  is also a minimal base of  $\mathfrak{m}\mathfrak{o}(x)$ . Therefore,  $\mathfrak{o}$  is regular if and only if  $\mathfrak{o}(x)$  is.*

The proof is easy.

**COROLLARY.** *The converse of Proposition 3 holds.*

*Proof.* From the regularity of  $\mathfrak{o}'_{\mathfrak{p}}$  and  $\mathfrak{o}'_{\mathfrak{p}}/(\mathfrak{p} \cap \mathfrak{o})\mathfrak{o}'_{\mathfrak{p}}$ , it follows that  $(\mathfrak{p} \cap \mathfrak{o})\mathfrak{o}'_{\mathfrak{p}}$  is generated by a subset of a regular system of parameters of  $\mathfrak{o}'_{\mathfrak{p}}$  (Lemma 0.12) and  $\mathfrak{o}'_{(\mathfrak{p} \cap \mathfrak{o})\mathfrak{o}'} = \mathfrak{o}(x_1, \dots, x_n)$  is regular, from which it follows that  $\mathfrak{o}_{(\mathfrak{p} \cap \mathfrak{o})}$  is regular.

**LEMMA 2.** *Let  $\mathfrak{o}$  be a regular local ring with maximal ideal  $\mathfrak{m}$ . If a monic polynomial  $f(x)$  over  $\mathfrak{o}$  is irreducible modulo  $\mathfrak{m}$ , then the ring  $\mathfrak{o}' = \mathfrak{o}[x]/f(x)\mathfrak{o}[x]$  is a regular local ring with maximal ideal  $\mathfrak{m}\mathfrak{o}'$ . Further,  $\mathfrak{m}^n\mathfrak{o}' \cap \mathfrak{o} = \mathfrak{m}^n$  for every  $n = 1, 2, \dots$ .*

*Proof.* Since  $f(x)$  itself is also irreducible, the assertion is proved easily.

*Remark.* By the same way as above, we see that Lemma 2 can be generalized as follows: Let  $\mathfrak{o}$  be a normal local ring with maximal ideal  $\mathfrak{m}$ . If a monic polynomial  $f(x)$  over  $\mathfrak{o}$  is irreducible modulo  $\mathfrak{m}$ , then the ring  $\mathfrak{o}' = \mathfrak{o}[x]/f(x)\mathfrak{o}[x]$  is a local ring with maximal ideal  $\mathfrak{m}\mathfrak{o}'$  and  $\mathfrak{m}^n\mathfrak{o} \cap \mathfrak{o} = \mathfrak{m}^n$ . Therefore, if  $\mathfrak{o}'$  is regular, then  $\mathfrak{o}$  is also a regular local ring.

**2. Quadratic transforms of local integral domains.** Let  $\mathfrak{o}$  be a local integral domain with maximal ideal  $\mathfrak{m}$  and let  $x_1, \dots, x_n$  be a base of  $\mathfrak{m}$ . Further, let  $\mathfrak{v}$  be a valuation ring which dominates  $\mathfrak{o}$  and let  $\mathfrak{n}$  be the maximal ideal of  $\mathfrak{v}$ . Let  $v$  be a valuation whose valuation ring is  $\mathfrak{v}$ . We may assume that  $v(x_1) \leq v(x_j)$  for every  $j$ . Set  $\mathfrak{o}' = \mathfrak{o}[x_2/x_1, \dots, x_n/x_1]$ ,  $\mathfrak{m}' = \mathfrak{n} \cap \mathfrak{o}'$  (observe that  $\mathfrak{o}'$  is a subring of  $\mathfrak{v}$ ),  $\mathfrak{o}'' = \mathfrak{o}'_{\mathfrak{m}'}$  and  $\mathfrak{m}'' = \mathfrak{m}'\mathfrak{o}''$ . Then

**PROPOSITION 4.** 1)  $\mathfrak{o}''$  is a local integral domain; 2)  $\mathfrak{o}''$  is determined by  $\mathfrak{v}$  above, that is,  $\mathfrak{o}''$  is independent of the choice of the base  $x_1, \dots, x_n$ .

of  $m$ ; 3) if  $\mathfrak{p}$  is a prime ideal of  $\mathfrak{o}$  which does not contain  $x_1$  and if there exists a prime ideal  $\mathfrak{p}''$  of  $\mathfrak{o}''$  such that  $\mathfrak{o}_{\mathfrak{p}}$  is dominated by  $\mathfrak{o}''_{\mathfrak{p}'}$ , then  $\mathfrak{o}_{\mathfrak{p}} = \mathfrak{o}''_{\mathfrak{p}'}$ ; and 4) if  $\mathfrak{o}$  is a regular local ring, then  $\mathfrak{o}''$  is also a regular local ring.

This  $\mathfrak{o}''$  is called the *quadratic transform* of  $\mathfrak{o}$  with respect to the valuation ring  $\mathfrak{v}$ .

*Proof.* (1) Since  $\mathfrak{o}$  is a Noetherian ring and since  $\mathfrak{o}'$  is finitely generated over  $\mathfrak{o}$ ,  $\mathfrak{o}'$  is Noetherian and  $\mathfrak{o}''$  is a local integral domain.

(2) Let  $y_1, \dots, y_m$  be another base of  $m$ . In order to prove 2), we may assume that  $x_j = y_j$  for  $j \leq n$ . First set  $\mathfrak{o}^* = \mathfrak{o}[y_2/y_1, \dots, y_m/y_1]$ . Then  $\mathfrak{o}' \subseteq \mathfrak{o}^*$ . Since  $y_i$  is a linear combination of  $x_j$ 's with coefficients in  $\mathfrak{o}$ , we have  $\mathfrak{o}' = \mathfrak{o}^*$  and  $\mathfrak{o}'' = \mathfrak{o}^*_{(\pi \cap \mathfrak{o}^*)}$ . Assume that  $v(y_i) = v(y_1)$  ( $= v(x_1)$ ). Set  $\mathfrak{o}^{**} = \mathfrak{o}[y_1/y_i, \dots, y_m/y_i]$ . Then  $y/y_i$  is a unit in  $\mathfrak{v}$  and  $y/y_i$  is not in  $m^{**} = \pi \cap \mathfrak{o}^{**}$ . Therefore  $\mathfrak{o}^{**}_{m^{**}}$  contains  $y_1/y_i$  as a unit, hence  $\mathfrak{o}^{**}_{m^{**}}$  contains  $y_2/y_1, \dots, y_m/y_1$ . Therefore  $\mathfrak{o}^*_{m^*} \subseteq \mathfrak{o}^{**}_{m^{**}}$ . By the same reason, we have the converse inclusion, and  $\mathfrak{o}'' = \mathfrak{o}_{m^*} = \mathfrak{o}^{**}_{m^{**}}$ .

(3) Since  $x_1 \notin \mathfrak{p}$ ,  $\mathfrak{o}'$  is a subring of  $\mathfrak{o}_{\mathfrak{p}}$  and 3) is easy.

(4) By 2), we may assume that  $x_1, \dots, x_n$  is a regular system of parameters of  $\mathfrak{o}$ .  $m$  is contained in  $x_1 \mathfrak{o}'$ , and  $\mathfrak{o}'/x_1 \mathfrak{o}'$  is generated by the residue classes of  $x_2/x_1, \dots, x_n/x_1$  over the residue class field of  $\mathfrak{o}$ ; they are obviously algebraically independent over  $\mathfrak{o}/m$ . Therefore,  $\mathfrak{o}''/x_1 \mathfrak{o}''$  is a regular local ring by Corollary 5 to Proposition 1.1, which shows that  $\mathfrak{o}''$  is also a regular local ring.

*Remark.* Observe that if  $v(x_1) < v(x_j)$  for every  $j > 1$ , then the maximal ideal of  $\mathfrak{o}''$  is generated by  $x_1, x_2/x_1, \dots, x_n/x_1$ .

**3. Results due to Serre [22].** If  $M$  is module over a ring  $\mathfrak{o}$ , the projective dimension of  $M$  in the sense of Cartan-Eilenberg [2] is called the *homological dimension* of  $M$  (according to Serre [22]), which will be denoted by  $\text{dh}_{\mathfrak{o}} M$  or by  $\text{dh } M$ . A series  $a_1, \dots, a_r$  of elements of the maximal ideal  $m$  of  $\mathfrak{o}$  is called an  *$M$ -series* if for every  $i \leq r$  the element  $a_i$  is not a zero-divisor in the module  $M/(\sum_{j=1}^{i-1} a_j M)$ . The number  $r$  is called the length of the series.

**PROPOSITION 5.** *If  $M$  is a finite module over a regular local ring  $\mathfrak{o}$  of rank  $n$ , then every maximal  $M$ -series is of length  $n - \text{dh } M$ . (Auslander-Buchsbaum-Serre)*

For the proof, see Serre [22].

**COROLLARY.** *If  $\mathfrak{a}$  is an ideal of  $\mathfrak{o}$  and if  $n \geq 1$ , then  $\text{dh } \mathfrak{a} \leq n - 1$ ; if  $\mathfrak{p}$  is a prime ideal of  $\mathfrak{o}$  different from  $\mathfrak{m}$  and if  $n \geq 2$ , then  $\text{dh } \mathfrak{p} \leq n - 2$ .*

**THEOREM 1.** *If  $\mathfrak{p}$  is a prime ideal of a regular local ring  $\mathfrak{o}$ , then  $\mathfrak{o}_{\mathfrak{p}}$  is a regular local ring. (Serre [22])*

For the proof, see Serre [22]; the base of his proof is the following

**PROPOSITION 6.** *A local ring  $\mathfrak{o}$  is regular if and only if there exists an integer  $m$  such that  $\text{dh } M \leq m$  for every  $\mathfrak{o}$ -module  $M$  (that is, global homological dimension of  $\mathfrak{o}$  is finite). (Serre [22])*

For the proof, see Serre [22].

In the latter parts of our papers, we shall need only Theorem 1 in the case where  $\mathfrak{o}$  is a spot of the restricted case, among results stated here. Therefore, we shall give another proof for this case in Appendix 1. (A proof for this case is also given by Nagata [18].)

**4. Unramifiedness of regular local rings.** Let  $r$  be a regular local ring with a semi-ground ring  $I$ . We say that  $r$  is *unramified* with respect to  $I$  if either  $r$  contains the field of quotients of  $I$ , or a prime element of  $I_{(\mathfrak{m} \cap I)}$  is not in  $\mathfrak{m}^2$ , where  $\mathfrak{m}$  denotes the maximal ideal of  $r$  (that is, a prime element of  $I_{(\mathfrak{m} \cap I)}$  can be selected to be a member of a regular system of parameters of  $r$ ). If furthermore the residue class field of  $r$  is separably generated over that of  $I_{(\mathfrak{m} \cap I)}$ , then we say that  $r$  is *tamely unramified* (with respect to  $I$ ).

A spot  $P$  over a ground ring  $I$  is called a *simple spot* if it is a regular local ring. It is called an *unramified simple spot* (over  $I$ ) if it is unramified with respect to  $I$ ; it is called a *tamely unramified simple spot* if it is tamely unramified. It is called *ramified* (over  $I$ ) if it is not unramified.

*Remark.* Since an "unramified regular local ring" is a regular local ring which contains a field or in which  $p$  is not in the square of the maximal ideal of the ring ( $p$  being the characteristic of the residue class field of the ring), we must distinguish between unramified simple spots and simple spots which are unramified regular local rings.

**PROPOSITION 7.** *If  $P$  is a tamely unramified regular local ring with respect to a semi-ground ring  $I$ , then the completion  $P^*$  of  $P$  contains a complete discrete valuation ring  $I^*$  (which may be a field) such that 1)  $I$  is a subring of  $I^*$  and 2)  $P^*$  is a formal power series ring over  $I^*$ .*

The proof is easy because  $P^*$  is a Henselian ring (Lemma 0.13).

**THEOREM 2.** *Let  $\mathfrak{p}$  be a prime ideal of a regular local ring  $\mathfrak{r}$ . If  $\mathfrak{r}$  is unramified with respect to a semi-ground ring  $I$ , then  $\mathfrak{r}_{\mathfrak{p}}$  is also unramified with respect to  $I$ .*

*Proof.*<sup>7</sup> Set  $\mathfrak{q} = \mathfrak{p} \cap I$ . If  $\mathfrak{q} = 0$ , then  $\mathfrak{r}_{\mathfrak{p}}$  contains the field of quotients  $k$  of  $I$  and  $\mathfrak{r}_{\mathfrak{p}}$  is unramified with respect to  $I$ . Assume that  $\mathfrak{q} \neq 0$ . Let  $x$  be a prime element of  $I_{\mathfrak{q}}$ . Since  $\mathfrak{r}$  is unramified with respect to  $I$ ,  $\mathfrak{r}/x\mathfrak{r}$  is a regular local ring. Therefore, by Theorem 1,  $(\mathfrak{r}/x\mathfrak{r})_{(\mathfrak{p}/x\mathfrak{r})} = \mathfrak{r}_{\mathfrak{p}}/x\mathfrak{r}_{\mathfrak{p}}$  is a regular local ring, which proves that  $\mathfrak{r}_{\mathfrak{p}}$  is unramified with respect to  $I$ .

More generally, we can prove the following assertion, making use of the notion of quadratic transformation (cf. § 2) or some results in Nagata [18]:

*Let  $\mathfrak{o}$  be a regular local ring with maximal ideal  $\mathfrak{m}$  and let  $\mathfrak{p}$  be a prime ideal of  $\mathfrak{o}$ . If an element  $f$  of  $\mathfrak{p}$  is not in  $\mathfrak{m}^n$ , then  $f$  is not in  $\mathfrak{p}^n\mathfrak{o}_{\mathfrak{p}}$ .*

We shall here only sketch the proofs. Let  $\mathfrak{o}^*$  be the completion of  $\mathfrak{o}$  and let  $\mathfrak{p}^*$  be a prime divisor of  $\mathfrak{p}\mathfrak{o}^*$ . Then  $\mathfrak{p}^* \cap \mathfrak{o} = \mathfrak{p}$  by the corollary to Lemma 0.4,  $\mathfrak{p}^{*n}\mathfrak{o}^*_{\mathfrak{p}^*}$  contains  $\mathfrak{p}^n\mathfrak{o}_{\mathfrak{p}}$ , and we have only to show that  $\mathfrak{p}^{*n}\mathfrak{o}^*_{\mathfrak{p}^*}$  does not contain  $f$ . Thus we may assume that  $\mathfrak{o}$  is complete.

i) The proof by the theory of multiplicity (This can be applied only for the case where  $\mathfrak{o}$  contains a field, because, for the proof of the result in [18] which we want to use, we used the present assertion for the case when  $\mathfrak{o}$  does not contain any field): Since  $f \notin \mathfrak{m}^n$ , the multiplicity of the ring  $\mathfrak{o}/f\mathfrak{o}$  is less than  $n$ . Since  $\mathfrak{o}/f\mathfrak{o}$  is complete, it follows that the multiplicity of the ring  $\mathfrak{o}_{\mathfrak{p}}/f\mathfrak{o}_{\mathfrak{p}}$  is less than  $n$  (see [18]), which proves that  $f \notin \mathfrak{p}^n\mathfrak{o}_{\mathfrak{p}}$ .

ii) The proof by quadratic transformation: By Lemma 4.1.2, we may assume that the residue class field of  $\mathfrak{o}$  is algebraically closed. On the other hand, making use of an induction argument on the rank of  $\mathfrak{o}$ , we may assume that  $\text{co-rank } \mathfrak{p} = 1$ . Let  $\mathfrak{v}$  be a valuation ring which dominates  $\mathfrak{o}$  and which has a prime ideal  $\mathfrak{n}$  such that  $\mathfrak{v}/\mathfrak{n}$  is algebraic over  $\mathfrak{o}/\mathfrak{p}$  and, furthermore, the residue class field of  $\mathfrak{v}$  coincides with that of  $\mathfrak{o}$  (existence follows from Proposition 2 in Appendix 1 of the present paper). Let  $\mathfrak{v}$  be a valuation whose valuation ring is  $\mathfrak{v}$ . Then we can select a regular system of parameters  $x_1, \dots, x_r$  of  $\mathfrak{o}$  so that  $v(x_1) < v(x_j)$  for every  $j > 1$ . Let  $\mathfrak{o}'$  be the quadratic transform of  $\mathfrak{o}$  with respect to  $\mathfrak{v}$  and set  $f' = f/x_1^m$  with  $m$  such that  $f \in \mathfrak{m}^m$ ,  $f \notin \mathfrak{m}^{m+1}$ ; we have only to prove that  $f' \notin \mathfrak{p}^n\mathfrak{o}_{\mathfrak{p}}$ . Repeating

<sup>7</sup> The writer owes the present proof to Prof. Serre.

successively quadratic transformations, we reach a case where  $\mathfrak{p}$  is generated by a subset of a regular system of parameters of  $\mathfrak{o}$  (see Appendix 1, §2) and the assertion is proved.

### 5. Unique factorization theorem in local rings.

**LEMMA 1.** *A Noetherian integral domain  $\mathfrak{o}$  is a unique factorization ring if and only if every prime ideal of rank 1 in  $\mathfrak{o}$  is principal. In this case, an ideal ( $\neq 0, \mathfrak{o}$ ) is principal if and only if it is purely of rank 1 (that is, each of its prime divisors is of rank 1).*

*Proof.* 1) Assume that  $\mathfrak{o}$  is a unique factorization ring and let  $\mathfrak{p}$  be a prime ideal of rank 1. Let  $a$  be an irreducible element in  $\mathfrak{p}$  (such exists because  $\mathfrak{o}$  is Noetherian). If  $bc \in \mathfrak{p}$ , then by the uniqueness of factorization, one of  $b$  and  $c$  is divisible by  $a$ , which shows that  $a\mathfrak{o}$  is prime and  $\mathfrak{p} = a\mathfrak{o}$ .

2) Assume that every prime ideal of rank 1 is principal. Then every irreducible element generates a prime ideal (of rank 1), from which we see the uniqueness of factorization by induction on the number of irreducible factors.

3) Let  $\mathfrak{q}$  be a primary ideal of rank 1 and let  $\mathfrak{p}$  be its prime divisor. Then  $\mathfrak{p} = a\mathfrak{o}$  ( $a \in \mathfrak{o}$ ). Let  $n$  be such that  $\mathfrak{q} \subseteq a^n\mathfrak{o}$ ,  $\mathfrak{q} \not\subseteq a^{n+1}\mathfrak{o}$  and set  $\mathfrak{q}' = \mathfrak{q} : a^n\mathfrak{o}$ . Since  $\mathfrak{q} \subseteq a^n\mathfrak{o}$ ,  $\mathfrak{q} = a^n\mathfrak{q}'$ . Since  $\mathfrak{q} \not\subseteq a^{n+1}\mathfrak{o}$ ,  $\mathfrak{q}' \not\subseteq \mathfrak{p}$ . Since by its definition  $\mathfrak{q}'$  is of rank 1 unless  $\mathfrak{q}' = \mathfrak{o}$ , and we have  $\mathfrak{q}' = \mathfrak{o}$  and  $\mathfrak{q} = a^n\mathfrak{o}$ . Now we consider an ideal  $\mathfrak{a}$  which is purely of rank 1. Let  $\mathfrak{q}_1, \dots, \mathfrak{q}_h$  be the primary components of  $\mathfrak{q}$ . Then each  $\mathfrak{q}_i$  is generated by one element  $a_i$  and

$$\mathfrak{a} = a_1(\mathfrak{a} : a_1\mathfrak{o}) = \dots = a_1 \dots a_h(\mathfrak{a} : a_1 \dots a_h\mathfrak{o}) = a_1 \dots a_h\mathfrak{o}.$$

4) Assume that an ideal  $\mathfrak{a}$  is generated by an element  $a$  and let  $\mathfrak{q}_1, \dots, \mathfrak{q}_h$  be the primary components of  $\mathfrak{a}$  belonging to prime divisors of rank 1. By 3)  $\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_h$  is principal: Let  $b$  be a generator. Then  $\mathfrak{a} = a\mathfrak{o} \subseteq b\mathfrak{o}$  and there exists an element  $c \in \mathfrak{o}$  such that  $a = bc$ . If  $c$  is not a unit in  $\mathfrak{o}$ , then there exists a prime ideal  $\mathfrak{p}$  of rank 1 which contains  $c$  and we have a contradiction from  $b\mathfrak{o}_{\mathfrak{p}} = a\mathfrak{o}_{\mathfrak{p}} \subseteq b\mathfrak{p}\mathfrak{o}_{\mathfrak{p}}$  (see [16]). Thus  $a\mathfrak{o} = b\mathfrak{o}$  and the proof is completed.

*Remark 1.* By the proofs 3) and 4) above, we see that when  $\mathfrak{a}$  is an ideal of a Noetherian integral domain  $\mathfrak{o}$  and every minimal prime divisor of  $\mathfrak{a}$  is principal, then  $\mathfrak{a}$  is principal if and only if  $\mathfrak{a}$  is purely of rank 1.

*Remark 2.* A unique factorization ring  $\mathfrak{o}$  is a normal ring. (Gauss' Lemma)

*Proof.* Every principal ideal of  $\mathfrak{o}$  has no imbedded prime divisor. Therefore  $\mathfrak{o} = \bigcap \mathfrak{o}_{\mathfrak{p}}$ , where  $\mathfrak{p}$  runs over all prime ideals of rank 1 in  $\mathfrak{o}$  (see [17]). Further, if  $\mathfrak{p}$  is a prime ideal of rank 1, then  $\mathfrak{p}$  is principal and  $\mathfrak{o}_{\mathfrak{p}}$  is a discrete valuation ring. It follows that  $\mathfrak{o}$  is normal.

*Remark 3.* If an integral domain  $\mathfrak{o}$  is a unique factorization ring, then every ring of quotients of  $\mathfrak{o}$  is also a unique factorization ring.

The proof is easy.

**PROPOSITION 8.** *Let  $\mathfrak{o}$  and  $\mathfrak{o}'$  be local rings such that 1)  $\mathfrak{o}$  is a subring of  $\mathfrak{o}'$  and 2)  $a\mathfrak{o}' \cap \mathfrak{o} = a\mathfrak{o}$  for every element  $a \in \mathfrak{o}$ . Then an ideal  $\alpha$  of  $\mathfrak{o}$  is principal if (and only if)  $\alpha\mathfrak{o}'$  is principal.*

*Proof.* Let  $b'$  be an element of  $\mathfrak{o}'$  such that  $\alpha\mathfrak{o}' = b'\mathfrak{o}'$ . If there is no element  $a$  of  $\alpha$  such that  $a\mathfrak{o}' = b'\mathfrak{o}'$ , then  $\alpha\mathfrak{o}' \subseteq b'm'\mathfrak{o}'$ , where  $m'$  is the maximal ideal of  $\mathfrak{o}'$ . It follows that  $b'\mathfrak{o}' = b'm'\mathfrak{o}'$ , which is impossible unless  $b' = 0$  (see [16]). Therefore, there exists an element  $a$  of  $\alpha$  such that  $a\mathfrak{o}' = \alpha\mathfrak{o}'$ , hence  $a\mathfrak{o} \subseteq \alpha \subseteq a\mathfrak{o}' \cap \mathfrak{o} = a\mathfrak{o}$  and  $\alpha = a\mathfrak{o}$ .

**COROLLARY 1.** *Let  $\mathfrak{o}^*$  be the completion of a local ring  $\mathfrak{o}$ . Then an ideal  $\alpha$  of  $\mathfrak{o}$  is principal if and only if  $\alpha\mathfrak{o}^*$  is principal.*

**COROLLARY 2.** *Assume that a local ring  $\mathfrak{o}'$  dominates a local integral domain  $\mathfrak{o}$ , that no element of  $\mathfrak{o}$  is a zero-divisor in  $\mathfrak{o}'$  and that  $K \cap \mathfrak{o}' = \mathfrak{o}$  with  $K$  the field of quotients of  $\mathfrak{o}$ . Then an ideal  $\alpha$  of  $\mathfrak{o}$  is principal if and only if  $\alpha\mathfrak{o}'$  is principal.*

**THEOREM 3.** *Let  $\mathfrak{o}^*$  be the completion of a local integral domain  $\mathfrak{o}$ . If  $\mathfrak{o}^*$  is a unique factorization ring, then  $\mathfrak{o}$  is also a unique factorization ring.\**

*Proof.* By Lemma 1, we have only to show that every prime ideal  $\mathfrak{p}$  of rank 1 in  $\mathfrak{o}$  is principal. Let  $S$  be the complement of  $\mathfrak{p}$  in  $\mathfrak{o}$ . Since  $\mathfrak{o}^*/\mathfrak{p}\mathfrak{o}^*$  is the completion of  $\mathfrak{o}/\mathfrak{p}$  (Lemma 0.3), no element of  $S$  is a zero-divisor modulo  $\mathfrak{p}\mathfrak{o}^*$  (Corollary to Lemma 0.4). If we know that  $\mathfrak{o}$  is a normal ring, then the proof proceeds as follows:  $\mathfrak{o}_{\mathfrak{p}}$  is a valuation ring and  $\mathfrak{p}\mathfrak{o}^*_{\mathfrak{s}}$  is a principal ideal, which shows that every prime divisor of  $\mathfrak{p}\mathfrak{o}^*_{\mathfrak{s}}$  is of rank 1 (because  $\mathfrak{o}^*_{\mathfrak{s}}$  is normal or because  $\mathfrak{o}^*_{\mathfrak{s}}$  is a unique factorization ring). Since

\* These results were announced by Mr. Y. Mori at the spring meeting of the Mathematical Society of Japan in 1949. The present proofs were found by the writer independently in March of 1950, which were sketched in [15]. Recently the writer saw that the same results were proved by Krull [9].

no element of  $S$  is a zero-divisor modulo  $\mathfrak{p}\mathfrak{o}^*$ , every prime divisor of  $\mathfrak{p}\mathfrak{o}^*$  is of rank 1. Hence  $\mathfrak{p}\mathfrak{o}^*$  is principal by Lemma 1 and  $\mathfrak{p}$  is principal by Corollary 1 to Proposition 8. Now the proof is completed by the following

LEMMA 2. *Let  $\mathfrak{o}^*$  be the completion of a local integral domain  $\mathfrak{o}$ . If  $\mathfrak{o}^*$  is a normal ring, then  $\mathfrak{o}$  is also a normal ring.*

But this follows immediately from the following

LEMMA 3. *Let  $\mathfrak{o}^*$  be the completion of a semi-local integral domain  $\mathfrak{o}$ . If  $K$  is the field of quotients of  $\mathfrak{o}$ , then  $\mathfrak{o} = K \cap \mathfrak{o}^*$ . (See Nagata [14] or [15].)*

*Proof.* It is obvious that  $\mathfrak{o} \subseteq K \cap \mathfrak{o}^*$ . We shall prove the converse inclusion. Let  $a/b$  ( $a, b \in \mathfrak{o}$ ) be an element of  $K \cap \mathfrak{o}^*$ . Then  $a/b \in \mathfrak{o}^*$  shows that  $a \in \mathfrak{b}\mathfrak{o}^* \cap \mathfrak{o} = \mathfrak{b}\mathfrak{o}$  (Lemma 0.3) and  $a/b \in \mathfrak{o}$ .

PROPOSITION 9. *Let  $\mathfrak{a}$  be an ideal of a local ring  $\mathfrak{o}$ . If there exist an ideal  $\mathfrak{b}$  of  $\mathfrak{o}$  contained in  $\mathfrak{a}$  and a non-unit  $x$  of  $\mathfrak{o}$  such that 1)  $\mathfrak{a}:x\mathfrak{o} = \mathfrak{a}$  and 2)  $\mathfrak{a} \subseteq \mathfrak{b} + x\mathfrak{o}$ , then we have  $\mathfrak{a} = \mathfrak{b}$ .<sup>9</sup>*

*Proof.* Since  $\mathfrak{a} \subseteq \mathfrak{b} + x\mathfrak{o}$ , every element  $a$  of  $\mathfrak{a}$  is expressed in the form  $a = b + xz$  with  $b \in \mathfrak{b}$ ,  $z \in \mathfrak{o}$ . Then  $xz \in \mathfrak{a}$  and  $z \in \mathfrak{a}$  because  $\mathfrak{a}:x\mathfrak{o} = \mathfrak{a}$ . Thus  $\mathfrak{a} = \mathfrak{b} + x\mathfrak{a}$ , which shows that  $\mathfrak{a} = \mathfrak{b}$  because  $x$  is in the maximal ideal of  $\mathfrak{o}$  ([16]).

COROLLARY 1. *Let  $\mathfrak{a}$  be an ideal of a local ring  $\mathfrak{o}$ . If there exists a non-unit  $x$  of  $\mathfrak{o}$  such that 1)  $\mathfrak{a}:x\mathfrak{o} = \mathfrak{a}$  and 2)  $\mathfrak{a} \subseteq x\mathfrak{o}$ , then  $\mathfrak{a} = 0$ .*

COROLLARY 2. *Let  $\mathfrak{a}$  be an ideal of a local ring  $\mathfrak{o}$ . If there exists a non-unit  $x$  of  $\mathfrak{o}$  such that 1)  $\mathfrak{a}:x\mathfrak{o} = \mathfrak{a}$  and 2)  $\mathfrak{a}$  is principal modulo  $x\mathfrak{o}$ , then  $\mathfrak{a}$  is also principal.*

THEOREM 4. *Let  $\mathfrak{o}$  be a regular local ring of rank  $r$ . If either  $\mathfrak{o}$  is unramified, or  $r$  is not greater than 2, then  $\mathfrak{o}$  is a unique factorization ring.<sup>8</sup>*

*Proof.* When  $r=1$ ,  $\mathfrak{o}$  is a discrete valuation ring and our assertion is obvious. Assume next that  $r=2$ . Let  $x$  be an element of the maximal ideal  $\mathfrak{m}$  of  $\mathfrak{o}$  which is not in  $\mathfrak{m}^2$ . Let  $\mathfrak{p}$  be a prime ideal of rank 1 in  $\mathfrak{o}$ .

<sup>9</sup> If we want to generalize this assertion to general Noetherian rings, then a similar proof shows the following:

Let  $\mathfrak{a}$  be an ideal of a Noetherian ring  $\mathfrak{o}$  and let  $\mathfrak{b}$  be an ideal contained in  $\mathfrak{a}$ . If there exists an element  $x$  of  $\mathfrak{o}$  such that 1)  $\mathfrak{a}:x\mathfrak{o} = \mathfrak{a}$ , 2)  $\mathfrak{q} + x\mathfrak{o} \neq \mathfrak{o}$  for every prime divisor  $\mathfrak{q}$  of  $\mathfrak{b}$  and 3)  $\mathfrak{a}$  is contained in  $\mathfrak{b} + x\mathfrak{o}$ , then we have  $\mathfrak{a} = \mathfrak{b}$ .

If  $x \in \mathfrak{p}$ , then  $\mathfrak{p} = x\mathfrak{o}$ . If  $x \notin \mathfrak{p}$ , then  $\mathfrak{p}$  is principal modulo  $x\mathfrak{o}$  because  $\mathfrak{o}/x\mathfrak{o}$  is a valuation ring, which shows that  $\mathfrak{p}$  is principal by Corollary 2 to Proposition 9 and the assertion is proved in this case. On the other hand, if  $\mathfrak{o}$  is unramified, then the completion  $\mathfrak{o}^*$  of  $\mathfrak{o}$  is a unique factorization ring (Lemma 0.9) and  $\mathfrak{o}$  is also a unique factorization ring by Theorem 3.

**PROPOSITION 10.** *If a regular local ring  $\mathfrak{o}$  is tamely unramified with respect to a semi-ground ring, then  $\mathfrak{o}$  is a unique factorization ring.*

*Proof.* By Theorem 3, we may assume that  $\mathfrak{o}$  is complete. Then  $\mathfrak{o}$  is isomorphic to a formal power series ring over a ring  $I$ , which is a field or a discrete valuation ring. Then the proposition is well known in this case (see for example Cohen [6]) and can be proved by induction on the rank of  $\mathfrak{o}$  just as in the case of polynomial rings. Therefore, we shall omit the proof. (Cohen's proof can be simplified a little.)

Though it seems to the writer very likely that every regular local ring is a unique factorization ring, the writer cannot prove it yet. But we can prove the following

**PROPOSITION 11.** *If every regular local ring of rank 3 is a unique factorization ring, then so is every regular local ring.<sup>10</sup>*

In order to prove this proposition, we shall prove the following two lemmas:

**LEMMA 4.** *Assume that  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an exact sequence of modules  $A, B, C$  (over a ring  $\mathfrak{o}$ ). Then 1) if  $\text{dh } A > \text{dh } B$ , then  $\text{dh } A = \text{dh } C - 1$ ; 2) if  $\text{dh } A < \text{dh } C - 1$ , or if  $\text{dh } A < \text{dh } B - 1$ , or if  $\text{dh } A < \text{dh } B$  and  $\text{dh } A < \text{dh } C$ , then  $\text{dh } B = \text{dh } C$ .*

*Proof.* From the exactness of the sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , we have the exactness of the sequence

$$\begin{aligned} \cdots \rightarrow \text{Ext}^{n-1}(A, X) \rightarrow \text{Ext}^n(C, X) \rightarrow \text{Ext}^n(B, X) \rightarrow \text{Ext}^n(A, X) \\ \rightarrow \text{Ext}^{n+1}(C, X) \rightarrow \text{Ext}^{n+1}(B, X) \rightarrow \text{Ext}^{n+1}(A, X) \rightarrow \cdots \end{aligned}$$

with an arbitrary module  $X$ , from which the assertion follows immediately.

**COROLLARY.** *Let  $a$  and  $b$  be elements of an integral domain  $\mathfrak{o}$ . If  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o} + b\mathfrak{o}) \neq 0$ , then  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o} : b\mathfrak{o}) = \text{dh}_{\mathfrak{o}}(b\mathfrak{o} + a\mathfrak{o}) - 1$ .*

<sup>10</sup> This result was proved independently by Zariski.

*Proof.* Since  $0 \rightarrow b\mathfrak{o} \rightarrow a\mathfrak{o} + b\mathfrak{o} \rightarrow \mathfrak{o}/(a\mathfrak{o}:b\mathfrak{o}) \rightarrow 0$  is exact and since  $\text{dh}_{\mathfrak{o}} b\mathfrak{o} = 0$ ,  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o} + b\mathfrak{o}) = \text{dh}_{\mathfrak{o}} \mathfrak{o}/(a\mathfrak{o}:b\mathfrak{o})$ . Since

$$0 \rightarrow (a\mathfrak{o}:b\mathfrak{o}) \rightarrow \mathfrak{o} \rightarrow \mathfrak{o}/(a\mathfrak{o}:b\mathfrak{o}) \rightarrow 0$$

is exact,  $\text{dh}_{\mathfrak{o}} \mathfrak{o}/(a\mathfrak{o}:b\mathfrak{o}) = \text{dh}_{\mathfrak{o}}(a\mathfrak{o}:b\mathfrak{o}) + 1$ . Thus the assertion is proved.

**LEMMA 5.** *Let  $\mathfrak{o}$  be a regular local ring of rank  $r \geq 3$ . Let  $\mathfrak{m}$  be the maximal ideal of  $\mathfrak{o}$ . Assume that every regular local ring of rank less than  $r$  is a unique factorization ring. If  $a$  and  $b$  are elements of  $\mathfrak{m}$  such that  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o} + b\mathfrak{o}) \leq r - 2$ , then the ideal  $a\mathfrak{o}:b\mathfrak{o}$  is principal.*

*Proof.* From  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o} + b\mathfrak{o}) \leq r - 2$ , it follows that either  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o} + b\mathfrak{o}) = 0$  or  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o}:b\mathfrak{o}) \leq r - 3$ . If  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o} + b\mathfrak{o}) = 0$ ,  $a\mathfrak{o} + b\mathfrak{o}$  is principal (because  $\mathfrak{o}$  is a local ring), hence either  $a\mathfrak{o} \subseteq b\mathfrak{o}$  or  $b\mathfrak{o} \subseteq a\mathfrak{o}$  (again by the fact that  $\mathfrak{o}$  is a local ring). Therefore, the assertion is easy in this case. In the other case, since  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o}:b\mathfrak{o}) \leq r - 3$ , it follows that  $\text{dh}_{\mathfrak{o}} \mathfrak{o}/(a\mathfrak{o}:b\mathfrak{o}) \leq r - 2$ . This shows that there exists an  $\mathfrak{o}/(a\mathfrak{o}:b\mathfrak{o})$ -series of length 2, that is, there exists an element  $x \in \mathfrak{m}$  which is not a zero-divisor modulo  $a\mathfrak{o}:b\mathfrak{o}$  such that  $((a\mathfrak{o}:b\mathfrak{o}) + x\mathfrak{o}) : \mathfrak{m} = (a\mathfrak{o}:b\mathfrak{o}) + x\mathfrak{o}$ , by Proposition 5. We can select such an  $x$  so that  $x \notin \mathfrak{m}^2$  (see [19]; cf. [16]). Let  $\mathfrak{q}$  be an arbitrary prime divisor of  $(a\mathfrak{o}:b\mathfrak{o}) + x\mathfrak{o}$ . Since  $\mathfrak{o}_{\mathfrak{q}}$  is a regular local ring of rank less than  $r$ ,  $a\mathfrak{o}_{\mathfrak{q}}:b\mathfrak{o}_{\mathfrak{q}}$  is principal and  $((a\mathfrak{o}:b\mathfrak{o}) + x\mathfrak{o})_{\mathfrak{o}_{\mathfrak{q}}}$  is generated by two elements. It follows that  $\mathfrak{q}$  is of rank 2 by the unmixedness theorem (see Cohen [6] or Nagata [18]), which shows that  $((a\mathfrak{o}:b\mathfrak{o}) + x\mathfrak{o})/x\mathfrak{o}$  is purely of rank 1, hence it is principal by Lemma 1. Since  $x$  is not a zero-divisor modulo  $a\mathfrak{o}:b\mathfrak{o}$ , it follows that  $a\mathfrak{o}:b\mathfrak{o}$  is principal by Corollary 2 to Proposition 9.

*Remark.* If  $\mathfrak{o}$  is a unique factorization ring, then  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o} + b\mathfrak{o})$  is not greater than 1 for arbitrary elements  $a$  and  $b$  of  $\mathfrak{o}$ .

*Proof.* If  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o} + b\mathfrak{o}) \neq 0$ , then  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o}:b\mathfrak{o}) = \text{dh}_{\mathfrak{o}}(a\mathfrak{o} + b\mathfrak{o}) - 1$ . But  $a\mathfrak{o}:b\mathfrak{o}$  is purely of rank 1 or is  $\mathfrak{o}$  itself, hence it is free and  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o}:b\mathfrak{o}) = 0$ . It follows that  $\text{dh}_{\mathfrak{o}}(a\mathfrak{o} + b\mathfrak{o}) = 1$  or 0.

Now we shall prove Proposition 11. Let  $\mathfrak{o}$  be a regular local ring of rank  $r$ ; we prove the assertion by induction on  $r$ . By our assumption, we may assume that  $r \geq 4$ . Let  $\mathfrak{p}$  be a prime ideal of rank 1 in  $\mathfrak{o}$ , and let  $a$  and  $b$  be elements of  $\mathfrak{o}$  such that 1)  $a\mathfrak{o}_{\mathfrak{p}} = \mathfrak{p}\mathfrak{o}_{\mathfrak{p}}$ , 2)  $b \notin \mathfrak{p}$ , but  $b$  is in every primary component of  $a\mathfrak{o}$  belonging to a prime divisor other than  $\mathfrak{p}$ . Then  $a\mathfrak{o}:b\mathfrak{o} = \mathfrak{p}$ . Let  $\mathfrak{a}$  be the ideal such that  $\mathfrak{a} = (a\mathfrak{o}:b\mathfrak{o}) : \mathfrak{m}^t$ , with  $t$  an integer, and  $\mathfrak{a}:\mathfrak{m} = \mathfrak{a}$ , where  $\mathfrak{m}$  is the maximal ideal of  $\mathfrak{o}$ . Let  $x$  be an element of  $\mathfrak{m}$

which is not in  $m^2$  such that  $a: x_0 = a$ . By our assumption,  $\mathfrak{o}/x_0$  is a unique factorization ring, and therefore  $\text{dh}_{\mathfrak{o}/x_0}(a_0 + b_0 + x_0)/x_0$  is not greater than 1. Since  $r$  is not less than 4, it follows that  $(a_0 + b_0 + x_0): m = a_0 + b_0 + x_0$  by Proposition 5, which shows that  $a_0 + b_0 + x_0$  contains  $a$ . Since  $a: x_0 = a$ , we have  $a_0 + b_0 = a$  by Proposition 9, which shows that  $(a_0 + b_0): m = (a_0 + b_0)$  and  $\text{dh}_{\mathfrak{o}} \mathfrak{o}/(a_0 + b_0) \leq r - 1$  by Proposition 5, that is,  $\text{dh}_{\mathfrak{o}}(a_0 + b_0) \leq r - 2$ . Therefore  $p = a_0: b_0$  is principal by Lemma 5, which proves our assertion.

## Appendix 1. Rings of Quotients of Regular Local Rings.

### 1. Some preliminary results.

**LEMMA 1.** *If  $\mathfrak{o}$  is a complete local integral domain, then the derived normal ring  $\mathfrak{o}'$  of  $\mathfrak{o}$  is a finite  $\mathfrak{o}$ -module. (Nagata [14]).*

*Proof.* Since  $\mathfrak{o}$  is complete, there exists a complete regular local ring  $r$  such that  $\mathfrak{o}$  is a finite  $r$ -module (Lemma 0.8). Therefore, we have only to prove the following

**LEMMA 2.** *Let  $r$  be a complete regular local ring, and let  $L$  be a finite algebraic extension of the field of quotients  $K$  of  $r$ . Then the integral closure  $\mathfrak{o}$  of  $r$  in  $L$  is a finite  $r$ -module.*

*Proof.* If  $r$  is of characteristic zero, then  $L$  is separable over  $K$  and our assertion is obvious ([16, § 5]). Therefore, we assume that  $L$  is of characteristic  $p \neq 0$ . Then we may assume that  $r$  is the ring of formal power series in indeterminates  $x_1, \dots, x_n$  over a field  $k$  (Lemma 0.10). Let  $L'$  be a purely inseparable finite algebraic extension of  $K$  such that  $L(L')$  is separable over  $L'$ . If we know that the integral closure  $\mathfrak{o}'$  of  $r$  in  $L'$  is a finite  $r$ -module, then the integral closure  $\mathfrak{o}''$  of  $r$  in  $L(L')$  is a finite  $r$ -module because  $L(L')$  is separable over  $L'$ , and  $\mathfrak{o}$  is also finite  $r$ -module. Thus, we may assume that  $L$  is purely inseparable over  $K$ . Set  $e = [L:K]$ ,  $\mathfrak{k} = k^{1/e}$ ,  $y_i = x_i^{1/e}$  ( $i = 1, \dots, n$ ) and  $r^* = \mathfrak{k}\{y_1, \dots, y_n\}$ . Then  $\mathfrak{o}$  is a subring of  $r^*$  because  $r^*$  is regular and hence normal, so that an element  $a$  of  $\mathfrak{o}$  can be expressed as a power series in  $y_1, \dots, y_n$  with coefficients in  $\mathfrak{k}$ . The leading form of the expression will be called the leading form of  $a$  in the present proof.

**LEMMA 3.** *If  $f_1, \dots, f_m$  are leading forms of elements  $a_1, \dots, a_m$  of  $\mathfrak{o}$  respectively, and if  $1, f_1, \dots, f_m$  are linearly independent over  $r$ , then  $1, a_1, \dots, a_m$  are linearly independent over  $r$ .*

*Proof.* We consider a linear combination  $\sum a_i b_i$  ( $b_i \in r$ ). Let  $g_i$  be the leading form of  $b_i$ . Then  $f_i g_i$  is the leading form of  $a_i b_i$ . We may assume that  $\deg(f_1 g_1) = \cdots = \deg(f_r g_r) = d$  and that  $\deg(f_i g_i) > d$  for  $i > r$ . Then the leading form of  $\sum a_i b_i$  is  $\sum_1^r f_i g_i$ , which is not in  $r$  because  $1, f_1, \dots, f_r$  are linearly independent over  $r$ . Therefore  $\sum a_i b_i \notin r$  unless all the  $b_i$ 's are zero, which proves Lemma 4.

Now we proceed with the proof of Lemma 2. Since  $L$  is finite over  $K$ , there are leading forms  $f_1, \dots, f_s$  of elements of  $\mathfrak{o}$  such that  $1, f_1, \dots, f_s$  is a maximal linearly independent set of leading forms of elements of  $\mathfrak{o}$  by Lemma 3. Let  $c_1, \dots, c_t$  be the coefficients of  $f_1, \dots, f_s$  and let  $f$  be the leading form of an element of  $\mathfrak{o}$ . Since  $f$  is linearly dependent on  $1, f_1, \dots, f_s$  over  $r$ , it follows that the coefficients of  $f$  are in  $\mathbf{k}(c_1, \dots, c_t)$ . Now let  $d_0 = 1, d_1, \dots, d_u$  be a linear base of  $\mathbf{k}(c_1, \dots, c_t)$  over  $\mathbf{k}$  and let  $m_0 = 1, m_1, \dots, m_v$  be the set of monomials in  $y_1, \dots, y_n$  of degree less than one. Then  $f$  is contained in the module generated by all the  $d_i m_j$  over  $r$ . Therefore, there exist elements  $a_1, \dots, a_w$  of  $\mathfrak{o}$  such that the leading form of every element of  $\mathfrak{o}$  is contained in the  $r$ -module generated by the leading forms of  $a_1, \dots, a_w$ . Since  $r[a_1, \dots, a_w]$  is a finite  $r$ -module, it is a complete local integral domain (Lemma 0.5). Since  $r^*$  dominates  $r[a_1, \dots, a_w]$ ,  $r[a_1, \dots, a_w]$  is a subspace of  $r^*$  (Lemma 0.11). On the other hand, by our choice of  $a_1, \dots, a_w$ , we see that  $\mathfrak{o}$  is contained in  $r[a_1, \dots, a_w] + \mathfrak{m}^{*i}$  for every natural number  $i$ , where  $\mathfrak{m}^*$  denotes the maximal ideal of  $r^*$ . Since  $r[a_1, \dots, a_w]$  is complete and is a subspace of  $r^*$ , it must be a closed set of  $r^*$ . Therefore, we have  $\mathfrak{o} \subseteq r[a_1, \dots, a_w]$  and  $r[a_1, \dots, a_w] = \mathfrak{o}$ . Thus the proof is completed.

**COROLLARY 1.** *Every complete local integral domain satisfies the finiteness conditions for integral extensions.*

**COROLLARY 2.** *If a complete semi-local ring  $\mathfrak{o}$  has no nilpotent elements, then the integral closure of  $\mathfrak{o}$  in its total quotient ring is a finite  $\mathfrak{o}$ -module.*

**PROPOSITION 1.** *If a semi-local integral domain  $\mathfrak{o}$  is analytically unramified, then the derived normal ring  $\mathfrak{o}'$  of  $\mathfrak{o}$  is a finite  $\mathfrak{o}$ -module. (Nagata [11])*

*Proof.* Let  $\mathfrak{o} = \mathfrak{o}_0 \subseteq \mathfrak{o}_1 \subseteq \cdots \subseteq \mathfrak{o}_n \subseteq \cdots$  be a chain of subrings of  $\mathfrak{o}'$  such that each  $\mathfrak{o}_i$  is a finite  $\mathfrak{o}$ -module. Then the completions  $\mathfrak{o}_i^*$  of the rings  $\mathfrak{o}_i$  can be imbedded in the integral closure  $\mathfrak{o}^{*'}$  of the completion  $\mathfrak{o}^*$  of  $\mathfrak{o}$  (Lemma 0.6). Since  $\mathfrak{o}^*$  has no nilpotent elements,  $\mathfrak{o}^{*'}$  is a finite  $\mathfrak{o}^*$ -module and there exists an  $i$  such that  $\mathfrak{o}_i^* = \mathfrak{o}_j^*$  for every  $j > i$ . Then we have  $\mathfrak{o}_i = \mathfrak{o}_j$  ( $j > i$ ) by Lemma 4.5.3. Thus  $\mathfrak{o}'$  is a finite  $\mathfrak{o}$ -module.

**PROPOSITION 2.** *Let  $\mathfrak{o}$  be a Noetherian integral domain and let  $L$  be a finite algebraic extension of the field of quotients of  $\mathfrak{o}$ . If  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  are prime ideals of  $\mathfrak{o}$  such that  $\mathfrak{p}_1 \supset \dots \supset \mathfrak{p}_n$ , then there exists a valuation ring  $\mathfrak{v}$  of  $L$  which has prime ideals  $\mathfrak{n}_1, \dots, \mathfrak{n}_n$  such that 1)  $\mathfrak{v}_{\mathfrak{n}_i}$  dominates  $\mathfrak{o}_{\mathfrak{p}_i}$  and 2)  $\mathfrak{v}_{\mathfrak{n}_i}/\mathfrak{n}_i$  is a finite algebraic extension of  $\mathfrak{o}_{\mathfrak{p}_i}/\mathfrak{p}_i\mathfrak{o}_{\mathfrak{p}_i}$  (for each  $i$ ).*

*Proof.* We may assume that  $\mathfrak{p}_n$  is of rank 1. Let  $\mathfrak{o}'$  be the integral closure of  $\mathfrak{o}_{\mathfrak{p}_n}$  in  $L$ . Then  $\mathfrak{o}'$  is a Dedekind domain by Lemma 0.15. Let  $\mathfrak{p}'$  be a maximal ideal of  $\mathfrak{o}'$ . Then  $\mathfrak{o}'_{\mathfrak{p}'}$  is a valuation ring and  $\mathfrak{o}'_{\mathfrak{p}'}/\mathfrak{p}'\mathfrak{o}'_{\mathfrak{p}'}$  is a finite algebraic extension of  $\mathfrak{o}_{\mathfrak{p}_n}/\mathfrak{p}_n\mathfrak{o}_{\mathfrak{p}_n}$  by Lemma 0.15. The rest of the proof is like the proof of Proposition 1.5.

*Remark.* If  $\mathfrak{o}$  is a regular local ring, then there exists a valuation ring of the field of quotients of  $\mathfrak{o}$  which dominates  $\mathfrak{o}$  and whose residue class field coincides with that of  $\mathfrak{o}$ .

*Proof.* Let  $\mathfrak{m}$  be the maximal ideal of  $\mathfrak{o}$  and let  $f$  be an element of  $\mathfrak{m}$  which is not in  $\mathfrak{m}^2$ . Then  $\mathfrak{o}/f\mathfrak{o}$  is a regular local ring and  $\mathfrak{o}_{f\mathfrak{o}}$  is a valuation ring. Therefore, our assertion is easily proved by induction on the rank of  $\mathfrak{o}$ .

## 2. A proof of Theorem 1 in some special cases.

**LEMMA 1.** *Let  $\mathfrak{o}$  be a regular local ring of rank  $r$  and let  $\mathfrak{p}$  be a prime ideal of co-rank 1 in  $\mathfrak{o}$ . If the derived normal ring of  $\mathfrak{o}/\mathfrak{p}$  is a finite  $\mathfrak{o}/\mathfrak{p}$ -module, then  $\mathfrak{o}_{\mathfrak{p}}$  is a regular local ring of rank  $r-1$ . (Zariski [26])*

*Proof.* Let  $\mathfrak{v}$  be a valuation ring which dominates  $\mathfrak{o}$  and has a prime ideal  $\mathfrak{n}$  such that  $\mathfrak{n} \cap \mathfrak{o} = \mathfrak{p}$  and  $\mathfrak{v}/\mathfrak{n}$  is algebraic over  $\mathfrak{o}/\mathfrak{p}$  (existence of  $\mathfrak{v}$  follows from Proposition 2). Let  $\mathfrak{o}_1, \dots, \mathfrak{o}_n, \dots$  be successive quadratic transforms of  $\mathfrak{o}$  with respect to  $\mathfrak{v}$  (that is,  $\mathfrak{o}_1$  is the quadratic transform of  $\mathfrak{o}$  with respect to  $\mathfrak{v}$ ,  $\mathfrak{o}_2$  is that of  $\mathfrak{o}_1$  and so on). Then by Proposition 4.4, each  $\mathfrak{o}_i$  is a regular local ring and has a prime ideal  $\mathfrak{p}_i$  such that  $\mathfrak{o}_{\mathfrak{p}} = (\mathfrak{o}_i)_{\mathfrak{p}_i}$  ( $\mathfrak{p}_i = \mathfrak{n} \cap \mathfrak{o}_i$ ). Let  $\mathfrak{o}'$  be the derived normal ring of  $\mathfrak{o}/\mathfrak{p}$  and set  $\mathfrak{m}' = (\mathfrak{m}/\mathfrak{n}) \cap \mathfrak{o}'$ , where  $\mathfrak{m}$  is the maximal ideal of  $\mathfrak{v}$ . Then, as is easily seen,  $\mathfrak{o}_i/\mathfrak{p}_i$  is contained in  $\mathfrak{o}'_{\mathfrak{m}'}$  for every  $i$ . Since  $\mathfrak{o}'$  is finite over  $\mathfrak{o}/\mathfrak{p}$ , there exists an  $n$  such that  $\mathfrak{o}_n/\mathfrak{p}_n = \mathfrak{o}_{n+1}/\mathfrak{p}_{n+1}$ . Since  $\mathfrak{o}_{n+1}/\mathfrak{p}_{n+1}$  is the quadratic transform of  $\mathfrak{o}_n/\mathfrak{p}_n$  with respect to  $\mathfrak{v}/\mathfrak{n}$ , it follows that  $\mathfrak{o}_n/\mathfrak{p}_n$  is a discrete valuation ring. Hence  $\mathfrak{p}_n$  is generated by  $r-1$  elements (Lemma 0.12) and  $\mathfrak{o}_{\mathfrak{p}} = (\mathfrak{o}_n)_{\mathfrak{p}_n}$  is a regular local ring of rank  $r-1$ , which proves the assertion.

Making use of an induction argument on the rank of a regular local ring, we easily see the following fact:

Let  $\mathfrak{o}$  be a regular local ring and let  $\mathfrak{p}$  be a prime ideal of  $\mathfrak{o}$ . If there exist prime ideals  $\mathfrak{p} = \mathfrak{p}_t \subset \mathfrak{p}_{t-1} \subset \cdots \subset \mathfrak{p}_0$  such that each  $\mathfrak{p}_i$  is of co-rank  $i$  and the derived normal ring of  $\mathfrak{o}_{\mathfrak{p}_{i-1}}/\mathfrak{p}_i \mathfrak{o}_{\mathfrak{p}_{i-1}}$  is a finite module over the latter ring for every  $i$ , then  $\mathfrak{o}_{\mathfrak{p}}$  is a regular local ring.

As consequences of the above result, we have

(1) If  $\mathfrak{o}$  is a complete regular local ring, then  $\mathfrak{o}_{\mathfrak{p}}$  is a regular local ring for every prime ideal  $\mathfrak{p}$  of  $\mathfrak{o}$ .

(See Lemma 1 in §1.)

(2) If  $P$  is a simple spot in the restricted case, then  $P_{\mathfrak{p}}$  is a regular local ring for every prime ideal  $\mathfrak{p}$  of  $P$ .

## Appendix 2. Some Remarks on the Non-restricted Case.

PROPOSITION 1. Let  $P$  be a local ring which dominates a discrete valuation ring  $\mathfrak{v}$ , and let  $\mathfrak{v}^*$  be a valuation ring which contains  $\mathfrak{v}$  as a dense subspace. Let  $\mathfrak{m}$  be the maximal ideal of  $P$  and set  $R = \mathfrak{v}^* \otimes_{\mathfrak{v}} P$ ,  $\mathfrak{m}' = \mathfrak{m}R$ . Then  $\mathfrak{m}'$  is a maximal ideal of  $R$ . Set  $P' = R/\mathfrak{m}'$ . If  $P'$  is Noetherian, then  $P'$  is a local ring which contains  $P$  as a dense subspace.

*Proof.* Let  $\mathfrak{p}$  be the maximal ideal of  $\mathfrak{v}$  and set  $\mathfrak{p}^{(i)} = \mathfrak{m}^i \cap \mathfrak{v}$ ,  $\mathfrak{p}^{*(i)} = \mathfrak{p}^{(i)} \mathfrak{v}^*$ . Then

$$R/\mathfrak{m}^i R = (\mathfrak{v}^*/\mathfrak{p}^{*(i)}) \otimes_{\mathfrak{v}/\mathfrak{p}^{(i)}} (P/\mathfrak{m}^i) = (\mathfrak{v}/\mathfrak{p}^{(i)}) \otimes (P/\mathfrak{m}^i) = P/\mathfrak{m}^i$$

for every natural number  $i$ . Therefore, by the case  $i=1$ , we see that  $\mathfrak{m}'$  is a maximal ideal of  $R$ . If  $P'$  is Noetherian, then  $P'$  is a local ring. Therefore,  $P'/\mathfrak{m}'^i P' = P/\mathfrak{m}^i$  shows that  $P'$  contains  $P$  as a dense subspace.

COROLLARY. If  $P$  is a spot over a ground ring  $\mathfrak{v}$  which is a valuation ring, then  $P'$  as defined above is a local ring which contains  $P$  as a dense subspace.

*Proof.* Let  $\mathfrak{o}$  be an affine ring which has a prime ideal  $\mathfrak{p}$  such that  $P = \mathfrak{o}_{\mathfrak{p}}$ . Then  $R$  is a ring of quotients of  $\mathfrak{v}^* \otimes \mathfrak{o}$ , and this last ring is finitely generated over  $\mathfrak{v}^*$ . It follows that  $R$  is Noetherian and  $P'$  is also Noetherian.

PROPOSITION 2. Let  $P$  be a spot over a ground ring  $I$ . If  $P$  is separably generated over  $I$ , then  $P$  is analytically unramified.

*Proof.* Let  $L$  be the field of quotients of  $P$  and let  $I'$  be the integral

closure of  $I$  in  $L$ . Since  $L$  is separably generated over  $I$ ,  $I'$  is separable over  $I$  and  $I'$  is a finite  $I$ -module. Set  $P' = P[I']$ . Then  $P'$  is a semi-local integral domain and  $P$  is a subspace of  $P'$  (Lemma 0.6). Therefore, we have only to show that  $P'$  is analytically unramified. Since the completion of  $P'$  is the direct sum of completions of local rings (spots)  $P'_{\mathfrak{m}'}$ , where  $\mathfrak{m}'$  runs over all maximal ideals of  $P'$  (Lemma 0.2), we have only to show that  $P'_{\mathfrak{m}'}$  is analytically unramified. Thus we may assume that  $P$  contains  $I'$ . Since  $I'$  is also a Dedekind domain, we may assume that  $I = I'$ , and we may assume further that  $I$  is the ground place of  $I$  dominated by  $P$ . If  $I$  is a field, then  $P$  is a spot in the restricted case, and therefore we assume that  $I$  is a discrete valuation ring. Let  $I^*$  be the completion of  $I$  and set  $R = I^* \otimes_I P$ ,  $\mathfrak{n} = \mathfrak{m}R$ ,  $P'' = R_{\mathfrak{n}}$  (see Proposition 1). By the Corollary to Proposition 1,  $P''$  is a local ring which contains  $P$  as a dense subspace. Since  $P$  is a regular extension of  $I$ ,  $R$  is an integral domain by Theorem 3.3, whence  $P''$  is a spot over  $I^*$  (by the proof of the Corollary to Proposition 1). Since  $I^*$  is complete, it satisfies the finiteness condition for integral extensions, hence  $P''$  is a spot in the restricted case and  $P''$  is analytically unramified. Since  $P''$  contains  $P$  as a dense subspace, we see that  $P$  is also analytically unramified.

**COROLLARY.** *If a spot  $P$  is separably generated over a ground ring  $I$ , then the derived normal ring of  $P$  is a finite  $P$ -module.*

This follows from Proposition 1 in Appendix 1 and the above proposition.

**PROPOSITION 3.** *The completion of a normal spot is an integral domain.*

*Proof.* (1) We first prove the assertion under the assumption that the spot  $P$  is analytically unramified. Let  $L$  be the field of quotients of  $P$  and let  $I'$  be the integral closure of  $I$  in  $L$ . Since  $P$  is normal,  $I'$  is contained in  $P$ , and therefore we may assume that  $I = I'$  (because  $I'$  is a Dedekind domain by Lemma 0.15). Thus we assume that the field of quotients  $K$  of  $I$  is algebraically closed in  $L$ . We may also assume that  $I$  is a discrete valuation ring dominated by  $P$ . Let  $I^*$  be the completion of  $I$  and  $K^*$  be the field of quotients of  $I^*$ . Let  $K'$  be a maximal separably generated extension of  $K$  contained in  $K^*$  and set  $I' = I^* \cap K'$ . Then  $I'$  is a valuation ring which contains  $I$  as a dense subspace. Set  $R = I^* \otimes_I P$ ,  $R' = I' \otimes_I P$ ,  $\mathfrak{n} = \mathfrak{m}R$ ,  $\mathfrak{n}' = \mathfrak{m}R'$ ,  $P'' = R_{\mathfrak{n}}$  and  $P' = R'_{\mathfrak{n}'}$ , where  $\mathfrak{m}$  is the maximal ideal of  $P$  (see Proposition 1). Since  $I'$  is separably generated over  $I$  and since  $K$  is algebraically closed in  $L$ ,  $R'$  is an integral domain by Theorem 3.2. Let  $x$  be a prime element of  $I$ . Then by Lemma 3.6.1,  $R'[1/x]$  is a normal

ring.  $R'/xR' = (I'/xI') \otimes_{I/xI} (P/xP) = P/xP$ . Since  $P$  is a normal ring,  $xP$  has no imbedded prime divisor and  $xR'$  has no imbedded prime divisor. Let  $\mathfrak{p}'$  be a (minimal) prime divisor of  $xR'$  and set  $\mathfrak{p} = \mathfrak{p}' \cap P$ . Then the fact that  $R'/xR' = P/xP$  shows that  $\mathfrak{p}$  is a prime divisor of  $xP$ . Let  $S$  be the complement of  $\mathfrak{p}$  in  $P$ . Then  $R'_S = I' \otimes_I P_S$  and  $P_S = P_{\mathfrak{p}}$ . Since  $P$  is normal,  $P_{\mathfrak{p}}$  is a discrete valuation ring. On the other hand,  $\mathfrak{p}R'_S$  is a maximal ideal of  $R'_S$  and  $(R'_S)_{(\mathfrak{p}R'_S)} = R'_{\mathfrak{p}'}$  contains  $P_{\mathfrak{p}}$  as a dense subspace by the Corollary to Proposition 1. Therefore  $R'_{\mathfrak{p}'}$  is a valuation ring. Therefore, we see that  $R'$  is a normal ring ([16]). Now, since  $I^*$  is purely inseparable over  $I'$ ,  $I^*$  is integral over  $I'$  and the zero ideal of  $R$  is a primary ideal. Since  $P''$  contains  $P$  as a dense subspace and since the completion of  $P$  contains no nilpotent elements, we see that  $P''$  is an integral domain and  $P''$  is a spot over  $I^*$ . Since  $I^*$  is purely inseparable over  $I'$  and since  $P'$  is a normal ring, the derived normal ring of  $P''$  has only one maximal ideal. Since  $P''$  is a spot in the restricted case, it follows that the completion of  $P''$  is an integral domain. Since  $P''$  contains  $P$  as a dense subspace, we see the assertion in this case.

(2) Now we have only to prove that  $P$  is analytically unramified. To do this, we shall prove a lemma due to Zariski [27]:

LEMMA 1. *Let  $\mathfrak{o}$  be a normal local ring. If there exists a non-unit  $a$  of  $\mathfrak{o}$  such that every prime divisor of  $a\mathfrak{o}$  is analytically unramified, then  $\mathfrak{o}$  is analytically unramified.*

*Proof.* Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  be the prime divisors of  $a\mathfrak{o}$  (they are of rank 1 because  $\mathfrak{o}$  is normal), and let  $\mathfrak{p}_{i1}^*, \dots, \mathfrak{p}_{i,n(i)}^*$  be the prime divisors of  $\mathfrak{p}_i\mathfrak{o}^*$  for each  $i$ ; here  $\mathfrak{o}^*$  denotes the completion of  $\mathfrak{o}$ . If  $S$  is the intersection of the complements of  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  in  $\mathfrak{o}$ , then  $\mathfrak{o}_S$  is a principal ideal ring ([16]); let  $x_i$  be an element of  $\mathfrak{o}$  such that  $\mathfrak{p}_i\mathfrak{o}_S = x_i\mathfrak{o}_S$  for each  $i$ . Then by assumption,  $x_i\mathfrak{o}^*_S = \bigcap_j \mathfrak{p}_{ij}^*\mathfrak{o}^*_S$ . Since no element of  $S$  is a zero-divisor modulo  $a\mathfrak{o}^*$ , we see that  $a\mathfrak{o}^*$  has no prime divisor other than the  $\mathfrak{p}_{ij}^*$ 's. If  $\mathfrak{P}_{ij}^*$  denotes the kernel of the natural homomorphism from  $\mathfrak{o}^*$  into  $\mathfrak{o}^*_{\mathfrak{p}_{ij}^*}$ , the primary component of  $a\mathfrak{o}^*$  belonging to  $\mathfrak{p}_{ij}^*$  contains  $\mathfrak{P}_{ij}^*$  so that  $a\mathfrak{o}^*$  contains the intersection  $\mathfrak{d}$  of the  $\mathfrak{P}_{ij}^*$ 's. Since the same holds with  $a$  replaced by any power of  $a$ , we see that  $\mathfrak{d} = 0$ . Since  $\mathfrak{P}_{ij}^*$  is a prime ideal by Lemma 1.4.1, we see that  $\mathfrak{d}$  is a semi-prime ideal and  $\mathfrak{o}$  is analytically unramified.

Now we shall return to the proof of Proposition 3. Let  $x$  be the same as in part (1) of the proof. Then for every prime divisor  $\mathfrak{p}$  of  $xP$ ,  $P/\mathfrak{p}$  is a spot over the field  $I/xI$ , is in the restricted case, and is analytically

unramified. Therefore, Lemma 1 shows that  $P$  is analytically unramified, which completes the proof.

**PROPOSITION 4.** *If an affine ring  $\mathfrak{o}$  is separably generated over a ground ring  $I$ , then the derived normal ring of  $\mathfrak{o}$  is a finite  $\mathfrak{o}$ -module.*

*Proof.* Using the Corollary to Proposition 2, we prove the assertion in the same way as in Theorem 1.3.

Once this proposition is proved, we see the existence of the derived normal model of a model in the same way as in Chapter 2 when the function field is separably generated. (For the projective model, observe that the field of quotients of the homogeneous coordinate ring is obtained by adjoining a transcendental element to the function field of the model under the consideration and the separable generation is preserved.)

Now the other assertions in Chapter 2 can be generalized easily for function fields which are separably generated over a ground ring.

Observe here that even when the function field of a model is separably generated over the ground ring, the function field of the induced model of the model defined by a spot may not be separably generated.

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*Added in Proof.* [20] appeared in the *Journal of Mathematical Society of Japan*, vol. 7 (1955) and [18] and [22] appeared in the *Proceedings of International Symposium on Algebraic Number Theory*, Tokyo-Nikko 1955 (1956).

# LOCAL UNIQUENESS, EXISTENCE IN THE LARGE, AND THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS.\*

By FRED BRAUER and SHLOMO STERNBERG.

1. It has been remarked by several authors that there is a close analogy between theorems assuring existence in the large of ordinary differential equations on the one hand and local uniqueness theorems on the other, cf. [2], [5], [8], [9]. For example, the majorization principle of Wintner [9] is the counterpart of Kamke's "allgemeine Eindeutigkeitssatz," and Wintner's criterion [8] corresponds to Osgood's uniqueness criterion. Furthermore, these criteria all have analogues which assure the convergence of successive approximations; cf. [7], where the Osgood criterion is shown to suffice, and [3], [1], where the same is shown for the Kamke condition (the additional monotony assumption being superfluous, as will be shown). This second situation is somewhat unsatisfactory, as we know that uniqueness of solutions and the convergence of the successive approximations are logically independent. In fact, Müller, in [6], gives an example of a differential equation which has a unique solution and yet the successive approximations do not converge, and we shall present below an example, due to Dieudonné, of a differential equation such that the successive approximations converge for an arbitrary initial curve and yet the solution is not unique. In the present paper we shall give a uniqueness criterion which includes Kamke's as a special case and is sufficiently general so as not to imply the convergence of successive approximations. We shall also explain the source of the duality between local uniqueness and global existence by showing that global existence can be viewed as a sort of uniqueness theorem at infinity.

2. We first consider the uniqueness of solutions of

$$(1) \quad x'(t) = f(x, t), \quad x(0) = 0,$$

where  $x$  and  $f$  are  $n$ -dimensional vectors.

Let  $V(x, t)$  be a function, defined for vectors  $x$  and real  $t$ , with non-negative real values, which is continuous in  $(x, t)$ , has one-sided partial

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derivatives with respect to  $t$  and the components of  $x$ , and such that  $V(x, t) = 0$  implies  $x = 0$ . We will use  $V_t$  to denote a partial derivative of  $V$  with respect to  $t$ ,  $V_x$  to denote some gradient vector of  $V$ , and  $\cdot$  to denote the usual scalar product of vectors. Any condition which involves  $V_x$  or  $V_t$  will be understood to be required for all one-sided derivatives.

**THEOREM 1.** *Let  $\omega(r, t)$  be a continuous non-negative function defined on  $r \geq 0$ ,  $0 < t < a$ . Suppose the only solution of*

$$(2) \quad r'(t) = \omega(r, t)$$

*which satisfies*

$$(3) \quad r(0) = r'(0) = 0$$

*on  $0 \leq t \leq \alpha$ , for any  $\alpha$  in  $0 < \alpha < a$ , is the identically zero solution. Let  $f(x, t)$  be continuous on a bounded region  $0 < t < a$ ,  $V(x, t) < b$  and*

$$(4) \quad V_t(x - y, t) + V_x \cdot [f(x, t) - f(y, t)] \leq \omega[V(x - y, t), t]$$

*in this region. Then there is at most one solution of (1) on  $0 \leq t < a$ .*

*Proof.* Suppose  $x_1(t)$  and  $x_2(t)$  are two solutions of (1) on  $0 \leq t < a$ , and let  $y(t) = x_2(t) - x_1(t)$ . Then

$$(5) \quad (d/dt)[y^i(t)] = f^i[x_2(t), t] - f^i[x_1(t), t],$$

where the superscript  $i$  denotes the  $i$ -th component. Let  $m(t) = V[y(t), t]$ , and  $m^*(t) = \limsup_{h \rightarrow 0} [m(t) - m(t-h)]/h$ . Then

$$\begin{aligned} [m(t) - m(t-h)]/h &= \sum_{i=1}^n (V[y^1(t), \dots, y^{i+1}(t-h), \dots, y^n(t-h), t] \\ &\quad - V[y^1(t), \dots, y^i(t-h), \dots, y^n(t-h), t])/h \\ &\quad + [V(x(t-h), t) - V(x(t-h), t-h)]/h. \end{aligned}$$

Since  $V$  has one-sided partial derivatives, this is bounded above by a sum of the form

$$[V_{x^i}(y(t), t) + \epsilon_i][y^i(t) - y^i(t-h)]/h + V_t(y(t), t) + \epsilon_{n+1},$$

where  $V_{x^i}$  stands for a suitable one-sided partial derivative of  $V$  with respect to  $x^i$ ,  $V_t$  stands for a suitable one-sided partial derivative of  $V$  with respect to  $t$ , and the  $\epsilon_i$  tend to zero as  $h \rightarrow 0$ . Letting  $h \rightarrow 0$ , we obtain, using (4), (5), and the continuity of  $\omega$ ,

$$(6) \quad m^*(t) \leq \omega[m(t), t].$$

Suppose there exists  $\sigma, 0 < \sigma \leq a$ , such that  $m(\sigma) > 0$ . There is a minimum solution  $r(t)$  of (2) through the point  $(\sigma, m(\sigma))$ , existing on some interval to the left of  $\sigma$ .

As far to the left of  $\sigma$  as  $r(t)$  exists, it satisfies

$$(7) \quad 0 < r(t) \leq m(t).$$

To prove this, we first note that  $r(t)$  must remain strictly positive, for if  $r(t)$  vanishes at a point it can be continued to the interval  $0 < t \leq \sigma$  as a solution of (2), (3), and therefore  $r(t)$  must vanish identically by hypothesis. Next, we observe that

$$(8) \quad r'(t) = \omega(r, t) + \epsilon, \quad r(\sigma) = m(\sigma)$$

has solutions  $r(t, \epsilon)$  for all sufficiently small  $\epsilon > 0$ , existing as far to the left of  $\sigma$  as  $r(t)$  exists, and  $\lim_{\epsilon \rightarrow 0+} r(t, \epsilon) = r(t)$  ([4], p. 83). Thus, it suffices to prove

$$(9) \quad r(t, \epsilon) \leq m(t)$$

for all  $\epsilon > 0$  and all solutions of (8). If this inequality does not hold, there is a least upper bound  $\xi$  of numbers  $t \leq \sigma$  for which (9) is false. Since  $m(\sigma) = r(\sigma) = r(\sigma, \epsilon)$ , and the functions  $m(t), r(t, \epsilon)$  are continuous,

$$(10) \quad m(\xi) = r(\xi, \epsilon), \quad m^*(\xi) \geq r'(\xi, \epsilon).$$

Then  $\omega(m(\xi), \xi) + \epsilon = \omega(r(\xi, \epsilon), \xi) + \epsilon = r'(\xi, \epsilon) \leq m^*(\xi) \leq \omega(m(\xi), \xi)$ , using (6), (8), (10). This contradiction proves (9), which, as we have remarked, implies (7).

Now  $r(t)$  can be continued to the whole interval  $0 < t \leq \sigma$  as a solution of (2) which satisfies (7). Since  $\lim_{t \rightarrow 0+} m(t) = 0$ ,  $\lim_{t \rightarrow 0+} r(t) = 0$ , and we may define  $r(0) = 0$ . If  $\|x\|$  denotes the Euclidean norm of a vector  $x$ , then by the continuity of  $f$  at the origin, there exists  $\delta > 0$  such that

$$\|x_2(t) - x_1(t)\| = \left\| \int_0^t [f(x_2(s), s) - f(x_1(s), s)] ds \right\| < \eta t,$$

when  $0 \leq t < \delta$ , for any  $\eta > 0$ . By the continuity of  $V$ , this implies that given any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $V[x_2(t) - x_1(t), t] < \epsilon t$  for  $0 \leq t < \delta$ . This implies  $0 < m(t)/t < \epsilon$  for  $0 < t < \delta$ , and  $\lim_{t \rightarrow 0+} m(t)/t = 0$ ,

which proves  $r'(0) = 0$ . Thus  $r(t)$  is a solution of (2) which satisfies (3), and by hypothesis,  $r(t)$  must vanish identically on  $0 \leq t \leq \sigma$ , contradicting the choice of  $r(\sigma)$ . Hence  $x_1(t) = x_2(t)$  on  $0 \leq t < a$ , which is the desired uniqueness result.

Instead of a single function  $V(x, t)$ , we may use a vector-valued function  $v(x, t)$ , and require that (4) be satisfied for each component of  $v(x, t)$ . Instead of requiring  $x=0$  if  $V(x, t)=0$ , we may impose the weaker condition that  $x=0$  if all components of  $v(x, t)$  vanish. Our proof shows that all components of  $v[x_2(t) - x_1(t), t]$  vanish identically on  $0 \leq t < a$ , and the uniqueness follows.

If  $x$  has components  $(x_1, \dots, x_n)$ , a possible choice of  $V$  is  $V(x, t) = \sum_{i=1}^n |x_i|$ , which is easily verified to have the required properties. With this choice of  $V$ , the condition (4) becomes

$$(11) \quad \sum_{i=1}^n |f_i(x, t) - f_i(y, t)| \leq \omega \left[ \sum_{i=1}^n |x_i - y_i|, t \right]$$

and Theorem 1 yields Kamke's general uniqueness theorem ([4], p. 139).

Let  $\omega(r, t)$  be a continuous non-negative function defined on  $r \geq 0$ ,  $0 < t < a$ . Suppose the only solution of (2) which satisfies (3) on  $0 \leq t \leq \alpha$ , for any  $\alpha$  in  $0 < \alpha < a$ , is the identically zero solution. Let  $f(x, t)$  be continuous in a region  $0 < t < a$ ,  $\sum_{i=1}^n x_i(t) < b$  and satisfy (11) there. Then there is at most one solution of (1) on  $0 \leq t < a$ .

If we take  $\omega(r, t) = \phi(r)\lambda(t)$ , we obtain Osgood's uniqueness theorem.

If  $f(x, t)$  is continuous in  $0 < t < a$ ,  $\sum_{i=1}^n |f_i(x, t)| < b$ , and satisfies

$$\sum_{i=1}^n |f_i(x, t) - f_i(y, t)| \leq \phi \left( \sum_{i=1}^n |x_i - y_i| \right) \lambda(t),$$

where

$$\int_0^\infty dr/\phi(r) = \infty, \quad \int_0^a \lambda(t) dt < \infty,$$

then there is at most one solution of (1) on  $0 \leq t < a$ .

**3.** Let  $f(x, t)$  be defined and continuous on all of  $R^n \times R^1$ . As is shown by the example  $f(x, t) = 1 + x^2$  ( $n=1$ ), there may be solutions to (1) which can not be continued for all  $t$ , and thus additional conditions must be imposed on  $f$  to ensure the existence of global solutions. The fundamental fact which underlies all the explicit criteria is the following (cf. Wintner [8]):

**THEOREM 2.** If  $x(t)$  is a solution of (1) which is defined for some time interval bounded on the right by  $t_0$ , then either  $\|x(t)\| \rightarrow \infty$  as  $|t| \rightarrow t_0$  or the solution can be extended beyond  $t_0$ .

The proof consists merely in observing that if  $\|x(t)\|$  did not go to infinity, then there would exist a sequence  $t_k$  such that  $t_k \rightarrow t_0$  and  $x(t_k) \rightarrow x_0$ , where  $x_0$  is some point in  $R^n$ . But the function  $f$  will be bounded in any compact neighbourhood  $N$  of  $(x_0, t_0)$ , so that the local existence theorem assures the existence of a solution curve through any point of  $N$ , and this solution will be defined for a length of time depending only on the distance to the boundary of  $N$ . Thus, applying the local existence theorem to  $(x(t_k), t_k)$  for  $k$  large enough, we obtain a continuation of  $x(t)$  beyond  $t_0$ .

An immediate corollary of this theorem is that if  $T$  is a mapping of  $R^n \times R^1 \rightarrow R^j$  with the property that the inverse image of a compact set is compact, then either  $\|T[x(t), t]\| \rightarrow \infty$  as  $t \rightarrow t_0$  or  $x(t)$  can be continued beyond  $t_0$ . In particular, using arguments similar to those in Section 2, we have the following criterion due to Conti [2] (who gives essentially the same proof).

**THEOREM 3.** *Let  $V(x, t)$  be a positive real-valued function as in Section 2 such that*

$$V_t(x, t) + V_x \cdot f(x, t) \leq \omega[V(x, t), t],$$

where  $\omega$  has the property that any solution of (2) can be continued for all time. Then if the inverse image of a compact set under  $V$  in  $R^n$  is compact for all fixed  $t$ , then the solutions of (1) can be continued for all time.

If we take  $V(x, t)$  to be  $\|x\|$ , we obtain the majorization principle of Wintner [9]. If, in addition, we take  $\omega(r, t) = \phi(r)\lambda(t)$ , we obtain the following well-known criterion due to Wintner [8].

*If  $|f(x, t)| \leq \phi(\|x\|)\lambda(t)$ , where  $\int_0^\infty dr/\phi(r) = \infty$ , then the solutions of (1) can be extended for all time.*

The striking analogy between the theorems of Section 2 and the criteria of the present section are easily understood on the basis of Theorem 2. In fact, Theorem 2 says that the solution at infinity is not unique if continuation for all time is not possible. This notion can be made precise by choosing some smooth homeomorphism  $M$  of  $0 < \|y\| < \infty$  onto itself such that  $M(y) \rightarrow 0$  for  $\|y\| \rightarrow \infty$ . If we choose  $M$  to vanish sufficiently rapidly at infinity, then the right side of the equation

$$(12) \quad z'(s) = -M_y \cdot f \text{ if } z \neq 0, \quad z' = 0 \text{ if } z = 0,$$

where  $z = M(y)$ ,  $s = -t$ , will be continuous for all  $(z, s)$ . The existence

in the large of solutions of (1) is then equivalent to uniqueness at  $z=0$  of solutions of (12). If, for example,  $n=1$  and  $f(x, t) = \phi(|x|)\lambda(t)$ , then the Osgood condition goes over into the Wintner condition by a change of variable in the integral.

4. In this section we show that the criterion of Section 2 does not imply convergence of the successive approximations. The well-known example, due to Müller [6], of an initial value problem (1) which has a unique solution but for which the successive approximations do not converge will do this. In fact, there exist  $V$  and  $\omega$  such that Müller's example satisfies the hypotheses of Theorem 1.

Müller's example is the function  $f$  ( $n=1$ ) defined by

$$\begin{aligned} f(x, t) &= 0, & t=0, -\infty < x < +\infty \\ f(x, t) &= 2t, & 0 < t \leq 1, -\infty < x < 0 \\ f(x, t) &= 2t - 4x/t, & 0 < t \leq 1, 0 \leq x \leq t^2 \\ f(x, t) &= -2t, & 0 < t \leq 1, t^2 < x < +\infty. \end{aligned}$$

We choose  $V(x, t) = |x|^{\frac{1}{2}}$ , and then  $V_x(x, t) = \pm \frac{1}{2} |x|^{-\frac{1}{2}}$ . It is clear that  $|f(x, t) - f(y, t)| \leq 4t$  and that no smaller bound can be used. The condition (4) becomes

$$\pm \frac{1}{2} |x - y|^{-\frac{1}{2}} [f(x, t) - f(y, t)] \leq \omega(|x - y|^{\frac{1}{2}}, t),$$

which is satisfied with  $\omega(r, t) = 2t/r$ . The problem (2), (3), with this choice of  $\omega$ , has only the trivial solution, and thus, by Theorem 1, the problem (1) for this choice of  $f$  has a unique solution.

5. Müller's example, described in Section 4, shows that uniqueness does not imply the convergence of successive approximations. If we start with an equation with non-unique solution, then the successive approximations will trivially converge if we choose a solution as initial curve. The question arises as to whether convergence for an *arbitrary* initial curve is sufficient to imply uniqueness. Here, we provide a counter-example, suggested by Prof. J. Dieudonné and reproduced here with his kind permission after some slight modifications. That is, we shall choose an  $f$  such that for arbitrary integrable initial curves  $x_0(t)$  with  $x_0(0) = 0$ , the sequence

$$(13) \quad x_j(t) = \int_0^t f[x_{j-1}(s), s] ds, \quad (n=1, 2, \dots)$$

converges, and hence to a solution of (1), and yet (1) has more than one solution.

The most popular example of non-uniqueness, namely  $f(x, t) = 2|x|^{\frac{1}{2}}$ , has this property. To prove this, we first remark that the procedure (13) is monotone: if  $|x_0| \geq |x_0^*|$  for all sufficiently small  $t$ , then  $|x_j| \geq |x_j^*|$  for such  $t$ . Next, we observe that the procedure converges for  $x_0(t) = 0$  [ $t < a$ ],  $x_0(t) = k(t-a)^2$  [ $t \geq a$ ]. In fact,

$$x_j(t) = 0 \quad [t < a], \quad x_j(t) = k^{\frac{1}{2}j}(t-a)^2 \quad [t \geq a],$$

so that  $x_j(t)$  converges to the solution  $x(t) = 0$  [ $t < a$ ],  $x(t) = (t-a)^2$  [ $t \geq a$ ].

Now we may assume, iterating several times if necessary, that the initial curve  $x_0$  is non-negative, continuous, and monotone increasing. If  $\tau = \text{g.l.b. } t > 0$ , then  $x_0(t) > 0$ , then

$$(14) \quad k_1(t-\tau-\delta)^2 > x_0(t) > k_2(t-\tau+\delta)^2,$$

for any fixed bounded range of  $t > \tau$ ,  $\delta > 0$ , and suitably large  $k_1$  and small  $k_2$ . Since the procedure (13) is monotone and  $\delta$  can be made arbitrarily small, (14) implies that  $x_j(t)$  converges to the solution  $x(t) = 0$  [ $t < \tau$ ],  $x(t) = (t-\tau)^2$  [ $t \geq \tau$ ]. In fact, it follows that by proper choice of the initial curve  $x_0(t)$ , the approximations  $x_j(t)$  can be made to converge to any solution of (1).

6. In the present section we shall impose an additional condition on  $V$  which will imply the convergence of successive approximations. We assume that  $V$  has the properties required in Section 2, and also obeys

$$(15) \quad V\left(\int_{t-h}^t [f(x(s), s) - f(y(s), s)] ds, t\right) \\ \leq \int_{t-h}^t V_t[x(s) - y(s), s] ds + \int_{t-h}^t V_x \cdot [f(x(s), s) - f(y(s), s)] ds,$$

for any continuous functions  $f, x, y$ .

**THEOREM 4.** Let  $V$  be as in Section 2, with the additional property (15). Suppose the hypotheses of Theorem 1 are satisfied with a function  $\omega(r, t)$  which is monotone non-decreasing in  $r$  for each fixed  $t$ . Then the successive approximations (13) converge to the solution  $x(t)$  of (1), in the sense that  $V[x(t) - x_j(t), t]$  tends uniformly to zero.

*Proof.* It is easy to see that since  $f$  is continuous, and hence bounded, the sequence  $x_j(t)$  of successive approximations is uniformly bounded and equicontinuous in the Euclidean norm on some interval. It follows that

there is a subsequence  $x_{j_k}(t)$  which converges uniformly on this interval (in the Euclidean norm) to a function  $x(t)$ . Since

$$x_{j_{k+1}}(t) = \int_0^t f[x_{j_k}(s), s] ds,$$

$x_{j_{k+1}}(t)$  converges uniformly to a function  $x^*(t)$ . We shall prove

$$(16) \quad V[x_{j+1}(t) - x_j(t), t] \rightarrow 0,$$

on this interval, which will imply  $x(t) = x^*(t)$ , so that  $x(t)$  is a solution of (1). Since this solution is unique by Theorem 1, every convergent subsequence converges to  $x(t)$ , and it follows that the original sequence converges to  $x(t)$ . Since this sequence is uniformly bounded and equicontinuous, the convergence is uniform in the Euclidean norm, which implies, by the continuity of  $V$ , that  $V[x(t) - x_j(t), t]$  converges uniformly to zero on some interval.

To prove (16), let  $w_j(t) = x_{j+1}(t) - x_j(t)$ , and  $m(t) = \limsup V[w_j(t), t]$  as  $j \rightarrow \infty$ . Then  $m(0) = 0$ , and  $m(t)$  is continuous, since it is the upper limit of a uniformly bounded equicontinuous sequence of functions. We have

$$(17) \quad \begin{aligned} V[w_{j+1}(t) - w_{j+1}(t-h), t] &= V\left(\int_{t-h}^t [f(x_{j+1}(s), s) - f(x_j(s), s)] ds, t\right) \\ &\leq \int_{t-h}^t V_t[x_{j+1}(s) - x_j(s), s] ds + \int_{t-h}^t V_x \cdot [f(x_{j+1}(s), s) - f(x_j(s), s)] ds, \end{aligned}$$

by (15), and by (4), this is no greater than  $\int_{t-h}^t \omega[V(w_j(s), s)] ds$ . Given any  $\delta > 0$ , there exists an integer  $N(\delta)$ , independent of  $s$  and  $j$ , such that

$$(18) \quad V[w_j(s), s] < m(s) + \delta, \quad j > N(\delta).$$

This follows from the fact that  $m$  is uniformly continuous and  $w_j$  is equicontinuous. Since  $\omega(r, t)$  is assumed non-decreasing in  $r$ , it follows from (18) that

$$\int_{t-h}^t \omega[V(w_j(s), s)] ds \leq \int_{t-h}^t \omega[m(s) + \delta, s] ds, \quad j > N(\delta).$$

Using (17), we obtain

$$(19) \quad V[w_{j+1}(t) - w_{j+1}(t-h), t] \leq \int_{t-h}^t \omega[m(s) + \delta, s] ds, \quad j > N(\delta).$$

From the definition of  $m(t)$  and (19), it is easy to see that

$$(20) \quad m(t) - m(t-h) \leq \int_{t-h}^t \omega[m(s) + \delta, s] ds.$$

Since  $\omega(r, t)$  is continuous in  $r$ ,

$$\omega[m(s) + \delta, s] \rightarrow \omega[m(s), s], \quad \delta \rightarrow 0,$$

and this together with (20) yields

$$(21) \quad m(t) - m(t-h) \leq \int_{t-h}^t \omega[m(s), s] ds.$$

This implies that  $m^*(t) = \limsup_{h \rightarrow 0} [m(t) - m(t-h)]/h \leq \omega[m(t), t]$ . The same argument used in the proof of Theorem 1 from (6) on shows that  $m(t)$  vanishes identically, which proves the theorem.

The condition that  $\omega(r, t)$  be monotone non-decreasing in  $r$  can be removed by means of the following lemma.

LEMMA. Let  $\omega_1(r, t)$  and  $\omega(r, t)$  be two continuous non-negative functions on  $0 < t < a$ ,  $r \geq 0$ , such that  $\omega_1(r, t) \leq \omega(r, t)$  on this domain. Suppose that the only solution of (2), (3) on  $0 \leq t \leq \alpha$ ,  $0 < \alpha < a$ , is the identically zero solution. Then the only solution of

$$(22) \quad r'(t) = \omega_1(r, t), \quad r(0) = r'(0) = 0,$$

on  $0 \leq t \leq \alpha$ ,  $0 < \alpha < a$ , is the identically zero solution.

Proof. Suppose  $m_1(t)$  is a solution of (22) which does not vanish identically, so that there exists  $\sigma$ ,  $0 < \sigma < a$ , such that  $m_1(\sigma) > 0$ . Through the point  $(\sigma, m_1(\sigma))$  there is a solution  $m(t)$  of (2) existing on some interval to the left of  $\sigma$ . Since  $\omega_1(r, t) \leq \omega(r, t)$ , the proof used to show (7) shows that as far to the left of  $\sigma$  as  $m(t)$  exists, it satisfies

$$(23) \quad 0 < m(t) \leq m_1(t).$$

Thus  $m(t)$  can be continued to the whole interval  $0 < t \leq \sigma$ . It is easy to see that  $\lim_{t \rightarrow 0} m(t) = 0$ , so that we may define  $m(0) = 0$ . It is also easy to show that  $m'(0) = 0$ . Thus  $m(t)$  is a solution of (2) which satisfies (3). By hypothesis,  $m(t)$  must vanish identically on  $0 \leq t \leq \sigma$ , contradicting the choice of  $m_1(\sigma)$ , and this proves the lemma.

Now, to remove the monotone condition in Theorem 4, we use

$$(24) \quad \omega_1(r, t) = \sup_{V(x-y, t) \leq r} (V_t(x-y, t) + V_x[f(x, t) - f(y, t)])$$

in place of  $\omega(r, t)$  if  $\omega(r, t)$  is not monotone. It is clear that (24) is monotone in  $r$  and satisfies the hypotheses of the lemma. The lemma shows that the hypotheses of Theorem 4 are satisfied with  $\omega_1$  in place of  $\omega$ , and the conclusion follows.

The use of  $V(x, t) = \sum_{i=1}^n |x_i|$  in Theorem 4 yields the result of Coddington and Levinson [1]:

*Let the hypotheses of Kamke's general uniqueness theorem be satisfied. Then the successive approximations (13) converge uniformly on some interval to the solution of (1).*

The monotonicity condition imposed in [1] is unnecessary, as we have remarked. The additional condition (15) takes the form

$$(25) \quad \sum_{i=1}^n \left| \int_{t-h}^t [f_i(x(s), s) - f_i(y(s), s)] ds \right| \\ \leq \int_{t-h}^t \sum_{i=1}^n |f_i(x(s), s) - f_i(y(s), s)| ds$$

in this case, and this condition is obviously satisfied.

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## RECIPROCITY AND CORRESPONDENCES.\*

By SERGE LANG.

*To Artin on his 60th birthday*

The reciprocity theorem  $f((g)) = g((f))$  on a curve arose first in Weil's investigations of Artin's reciprocity law on curves [4]. It has recently been used to deal with various questions of duality, for instance by Igusa [1]. Furthermore, Tate has just given a pairing of certain Galois cohomology groups of abelian varieties [3]. To prove the pairing well defined and bilinear on Jacobians, he used the above theorem.

In this manner, we are therefore led in a natural way to formulate it on abelian varieties, and by pull back on varieties. Aside from its application to the Tate pairing, we get an obvious complement to Kummer theory arising from the theory of divisorial correspondences.

**1. Notation.** Let  $U, V$  be two complete varieties, non-singular in codimension 1. This insures that a divisor which is linearly equivalent to 0 has this property over a given field of rationality and, also, that such a divisor determines uniquely the function of which it is the divisor, up to a multiplicative constant.

A cycle of codimension 1 on a variety will always be called a divisor. A cycle of dimension 0 will be called simply a cycle. Let  $\alpha = \sum n_i(P_i)$  be a cycle on  $U$ . Let  $\phi$  be a function on  $U$  such that  $\phi$  is defined at every point of  $\alpha$  and does not take on the value 0. Then we define

$$\phi(\alpha) = \prod \phi(P_i)^{n_i}.$$

If  $\alpha$  is of degree 0, and  $\psi$  is a function such that  $\psi = c\phi$  where  $c$  is constant, then  $\psi(\alpha) = \phi(\alpha)$ .

Let  $D$  be a divisor on the product  $U \times V$ . Let  $\alpha, \beta$  be two cycles of degree 0 on  $U$  and  $V$  respectively. Assume that  $D(\alpha)$  is defined and that it is the divisor of a function  $f$  on  $V$ . If in addition  $f(\beta)$  is defined, then we put  $D(\alpha, \beta) = f(\beta)$  and say that  $D(\alpha, \beta)$  is defined.

Suppose that, for every point  $P$  in  $\alpha$  and every point  $Q$  in  $\beta$ , the point

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$(P, Q)$  does not lie in the support of  $D$  (or as we shall say more briefly, in  $D$ ). Then first,  $D(a)$  is defined, and  $Q$  does not lie in the divisor of the function  $f$ . Hence  $f(Q)$  is defined and not equal to 0, and consequently,  $D(a, b)$  is defined. This sufficient condition will be used constantly in the sequel.

**2. The theorem of the square.** The proof of our main reciprocity theorem (Theorem 4) is based on the theorem of the square, which runs as follows.

*Let  $U, V, W$  be three varieties and assume that  $W$  is complete and non-singular in codimension 1. Let  $D$  be a divisor on  $U \times V \times W$ , and let  $u_i, v_j$  ( $i, j = 0, 1$ ) be simple points of  $U, V$  respectively, such that  $D(u_i, v_j)$  is defined. Then  $\sum_{i,j} (-1)^{i+j} D(u_i, v_j)$  is linearly equivalent to 0 on  $W$ .*

This is an immediate consequence of the existence of the Picard variety for  $W$ . Indeed, if  $a, b$  are two simple points of  $U, V$  respectively, such that  $D(a, b)$  is defined, then for  $u, v$  generic we have a rational map

$$(u, v) \rightarrow \text{Cl}[D(u, v) - D(a, b)]$$

of  $U \times V$  into the Picard variety of  $W$ . (As usual, Cl denotes the point associated with the linear equivalent class on the Picard variety.) A rational map of a product into an abelian variety splits into a sum of mappings of the factors. It is then clear that in our alternating sum, the constants will cancel, and the alternating sum will map on 0 in the Picard variety. It is therefore linearly equivalent to 0.

Conversely, as Weil has shown, one can give a direct proof for the theorem of the square, and one can base on it the construction of the Picard variety. For this as well as all further information concerning abelian varieties, the reader is referred to [2].

**3. The theorem of the hypercube.** In order to avoid too many indices, we restrict ourselves to four-fold products, which will be all that is needed for the applications we have in mind to abelian varieties. The notation we now describe will remain fixed throughout this section.

Let  $U, V, W, T$  be four varieties, defined over a field  $k$ . We assume them complete and non-singular in codimension 1.

Let  $D$  be a divisor on the product  $U \times V \times W \times T$ , rational over  $k$ . Let  $i, j, k, l$  range over 0 and 1, and take two copies of each one of our four varieties,  $U_i, V_j, W_k, T_l$ . On the double product

$$U_0 \times U_1 \times V_0 \times V_1 \times W_0 \times W_1 \times T_0 \times T_1 = U^{(2)} \times V^{(2)} \times W^{(2)} \times T^{(2)},$$

consider the divisor  $D_{ijkl}$  consisting of  $D$  on the partial product

$$U_i \times V_j \times W_k \times T_l$$

taken with the full varieties on the others. (In other words, it is the inverse image of  $D$  under the projection of the eight-fold product to the four-fold product.) We can take a coboundary of  $D$ , by setting

$$E = \sum (-1)^{i+j+k+l} D_{ijkl},$$

and observe that this formula could have been generalized to arbitrary products.

Let

$$P = (u_0, u_1, v_0, v_1) \quad \text{and} \quad Q = (w_0, w_1, t_0, t_1)$$

be two independent generic points of  $U^{(2)} \times V^{(2)}$  and  $W^{(2)} \times T^{(2)}$  over  $k$ . By the theorem of the square, there exists a function  $f_P$  on  $W \times T$ , defined over  $k(P)$ , and a function  $g_Q$  on  $U \times V$ , defined over  $k(Q)$  such that

$$(1) \quad (f_P) = \sum (-1)^{i+j} D(u_i, v_j) \quad \text{and} \quad (g_Q) = \sum (-1)^{k+l} D(w_k, t_l).$$

Furthermore, it is obvious from the definitions that we have

$$(2) \quad E(P) = \sum (-1)^{k+l} (f_P)_{kl} = \sum_{i,j,k,l} (-1)^{i+j+k+l} D(u_i, v_j)_{kl}$$

$$(3) \quad {}^t E(Q) = \sum (-1)^{i+j} (g_Q)_{ij} = \sum_{i,j,k,l} (-1)^{i+j+k+l} {}^t D(w_k, t_l)_{ij}.$$

All further reciprocity theorems derived from the correspondence  $D$  between  $U \times V$  and  $W \times T$  arise from the symmetry of our divisor  $E$ .

**THEOREM 1.** *The divisor  $E$  above is linearly equivalent to 0.*

*Proof.* There exists a function  $f^*$  on  $U^{(2)} \times V^{(2)} \times W \times T$  defined over  $k$  such that

$$f^*(P, w, t) = f_P(w, t),$$

and we have

$$P \times (f_P) = (f^*) \cdot (P \times W \times T).$$

There exists a function  $F$  on the eight-fold product such that

$$(4) \quad F(u_0, u_1, v_0, v_1, w_0, w_1, t_0, t_1) = \prod f^*(P, w_k, t_l)^{(-1)^{k+l}},$$

and from the definitions, we see that  $(F)(P) = E(P)$ . Hence  $(F)$  and  $E$  differ by a divisor which is degenerate on  $U^{(2)} \times V^{(2)}$ . Inducing  $(F)$  and  $E$  on the variety  $U^{(2)} \times V^{(2)} \times W^{(2)} \times \Delta_T$ , we obtain 0. Hence the degenerate component is equal to 0, and we have  $(F) = E$ , as desired.

Note that the function  $F$  such that  $E = (F)$  is defined at the point obtained by setting  $t_0 = t_1$ , and takes the value 1. This gives us a way of normalizing it, since two functions representing  $E$  differ by a multiplicative constant.

On the other hand, from the symmetry of  $F$ , we see that if we go from right to left in the correspondence given by  $D$ , then we obtain the following reciprocity formula, from (4).

**THEOREM 2.** *Let  $F$  be a function on the eight-fold product, defined over  $k$ , such that  $E = (F)$ , and normalized as above. Then*

$$F(P, Q) = \prod f_P(w_k, t_l)^{(-1)^{k+l}} = \prod g_Q(u_i, v_j)^{(-1)^{i+j}}.$$

As usual, such a formula at generic points has its counterpart for special points.

**THEOREM 3.** *Let  $(u'_0, u'_1, v'_0, v'_1)$  and  $(w'_0, w'_1, t'_0, t'_1)$  be two simple points of  $U^{(2)} \times V^{(2)}$  and  $W^{(2)} \times T^{(2)}$  respectively. Put*

$$\alpha = \sum (-1)^{i+j} (u'_i, v'_j) \quad \mathfrak{b} = \sum (-1)^{k+l} (w'_k, t'_l).$$

*Assume that none of the points  $(u'_i, v'_j, w'_k, t'_l)$  lies in the support of  $D$ . Then  $D(\alpha, \mathfrak{b}) = {}^tD(\mathfrak{b}, \alpha)$ .*

*Proof.* It is immediate from our hypothesis that  $(P', Q')$  does not lie in  $(F) = E$ , and hence  $F(P', Q')$  is defined. Our equality is now obvious.

**4. Application to abelian varieties.** Let  $A, B$  be two abelian varieties defined over  $k$ , and let  $D$  be a divisor on  $A \times B$ , rational over  $k$ . Let  $\alpha$  be a cycle on  $A$  of degree 0, such that  $D(\alpha)$  is defined. If  $\alpha$  is in the kernel of Albanese, i.e., if  $S(\alpha) = 0$ , then  $D(\alpha)$  is linearly equivalent to 0 on  $B$ , say  $D(\alpha) = (f)$ . If now  $\mathfrak{b}$  is a cycle of degree 0 on  $B$ , also in the kernel of Albanese on  $B$ , such that no point of  $\alpha \times \mathfrak{b}$  is contained in  $D$ , then we may form symmetrically either  $D(\alpha, \mathfrak{b})$  or  ${}^tD(\mathfrak{b}, \alpha)$ .

**THEOREM 4.** *Let  $A, B$  be two abelian varieties. Let  $D$  be a divisor on  $A \times B$ . Let  $\alpha, \mathfrak{b}$  be cycles on  $A, B$  respectively, of degree 0, such that  $S(\alpha) = 0$  and  $S(\mathfrak{b}) = 0$ . Assume no point of  $\alpha \times \mathfrak{b}$  is contained in  $D$ . Then  $D(\alpha, \mathfrak{b})$  and  ${}^tD(\mathfrak{b}, \alpha)$  are defined, and they are equal.*

*Proof.* Suppose first that  $D$  is the divisor of a function  $\phi$  on  $A \times B$ . Our hypotheses imply that  $\phi$  is defined at every point of  $\alpha \times \mathfrak{b}$ , and our theorem simply asserts  $\phi(\alpha, \mathfrak{b}) = \phi(\mathfrak{b}, \alpha)$ . Taking into account an obvious linearity, this remark allows us to change  $D$  by suitable linear equivalence.

We shall reduce our theorem to the case where  $\alpha$  and  $\beta$  consist of four points each. This will then allow us to use Theorem 3.

Suppose that  $\alpha$  is written

$$\alpha = (a_1) - (a_1') + (a_2) - (a_2') + \cdots + (a_n) - (a_n').$$

Let  $x$  be any point of  $A$ . We can write

$$\begin{aligned} \alpha &= (a_1) - (a_1') + (x) - (x + a_1 - a_1') \\ &+ (a_2) - (a_2') + (x + a_1 - a_1') - (x + a_2 - a_2' + a_1 - a_1') + \cdots, \end{aligned}$$

so that we correct each successive step to cancel the preceding term. We must come back at the end with  $-(x)$  using the fact that  $S(\alpha) = 0$ . Thus our cycle can be written as a sum of cycles of degree 0, in the kernel of Albanese, and consisting of four points.

From the theory of linear equivalence, one knows that it is possible to find a function  $\phi$  on  $A \times B$  such that none of the points of a given finite set of points on  $A \times B$  lies in the support of  $D + (\phi)$ . (We reproduce a proof of this fact in an appendix.) We apply this to the product of the set of points entering in the expression of  $\alpha$  obtained above, with a similar set for  $\beta$ . In view of our previous remark, and of an obvious linearity when all necessary expressions are defined, we see that it indeed suffices to prove our theorem when  $\alpha$  and  $\beta$  consist of four points.

Let us therefore write

$$\alpha = (a_0) - (a_1) + (a_2) - (a_3),$$

$$\beta = (b_0) - (b_1) + (b_2) - (b_3).$$

Let  $\lambda_A: A \times A \rightarrow A$  be the modified law of composition, such that  $\lambda_A(u, v) = u - v$ , and let  $\lambda = (\lambda_A, \lambda_B)$ . Let  $D^* = \lambda^{-1}(D)$ . It is a divisor on the product  $A \times A \times B \times B$ . Let  $u$  be a generic point of  $A$ , and let

$$u_0' = u + a_0, \quad u_1' = u + a_3, \quad v_0' = u, \quad v_1' = u + a_0 - a_1.$$

Do a similar construction for the points of  $\beta$ , to obtain points  $w_k'$  and  $t_l'$ . By hypothesis, no point of  $\alpha \times \beta$  lies in  $D$ , and hence none of the points  $(u_i', v_j', w_k', t_l')$  lies in  $D^*$ . We have

$$\sum (-1)^{i+j} D^*(u_i', v_j') = \lambda_B^{-1} [\sum (-1)^p D(a_p)].$$

If we put  $\alpha^* = \sum (-1)^{i+j} (u_i', v_j')$ , we can also write

$$D^*(\alpha^*) = \lambda_B^{-1} D(\alpha).$$

This comes from commutativity in the diagram

$$\begin{array}{ccc}
 (x, y) \times B \times B & \xrightarrow{\lambda} & (x - y) \times B \\
 \text{inj} \downarrow & & \downarrow \text{inj} \\
 A \times A \times B \times B & \xrightarrow{\lambda} & A \times B
 \end{array}$$

where  $\text{inj}$  is the injection, together with the formalism of the inverse mappings of cycles applied to the divisor  $D$  (cf. the appendix of [2]).

Similarly, we have a cycle  $b^*$  such that

$${}^tD^*(b^*) = \lambda_A^{-1} {}^tD(b).$$

If  $(f) = D(a)$ , we have a function  $f^* = f\lambda_B$ , and if  $(g) = {}^tD(b)$  we have a function  $g^* = g\lambda_A$ . Then

$$(f^*) = \lambda_B^{-1}(f) = \lambda_B^{-1}D(a) \quad \text{and} \quad (g^*) = \lambda_A^{-1}(g) = \lambda_A^{-1}{}^tD(b).$$

By Theorem 3,  $f^*(b^*) = g^*(a^*)$  and hence clearly,  $f(b) = g(a)$ . This proves our theorem.

By pull back, we now obtain the following result for varieties.

**THEOREM 5.** *Let  $V, W$  be two complete varieties, non-singular in co-dimension 1, and such that any finite set of points on them can be represented on an affine open subset. Let  $\phi: V \rightarrow A$  and  $\psi: W \rightarrow B$  be two canonical maps into their Albanese varieties. Let  $a, b$  be two cycles on  $V, W$  respectively, of degree 0, and in the kernel of Albanese. Let  $D$  be a divisor on  $V \times W$  such that no point of  $a \times b$  lies in  $D$ . Then  $D(a, b)$  and  ${}^tD(b, a)$  are defined and they are equal.*

*Proof.* If the theorem is true for one divisor  $D$  in a correspondence class on  $V \times W$ , then it is true for every divisor in that class not containing any point of  $a \times b$ . This is first checked for linear equivalence, and then for degenerate divisors. A representative divisor in a correspondence class can always be obtained as an inverse image of a divisor on  $A \times B$ , and our result is now an immediate consequence of the formalism of inverse mappings of cycles, applicable in the present case to the commutative diagram

$$\begin{array}{ccc}
 a \times W & \xrightarrow{(\phi, \psi)} & \phi(a) \times B \\
 \text{inj} \downarrow & & \downarrow \text{inj} \\
 V \times W & \xrightarrow{(\phi, \psi)} & A \times B.
 \end{array}$$

**COROLLARY.** *Let  $C$  be a complete non-singular curve, and let  $f, g$  be*

two functions on  $C$  whose divisors have no point in common. Then  $f((g)) = g((f))$ .

*Proof.* Take the diagonal on  $C \times C$ . Our hypothesis means that no point of  $(f) \times (g)$  lies in it, and we can apply the theorem.

**5. The Tate pairing.** Let  $A, B$  be two abelian varieties, defined over the field  $k$ . Let  $D$  be a divisor on  $A \times B$ , rational over  $k$ , and let  $K$  be a finite Galois extension of  $k$ , with group  $G$ . If  $M$  is a  $G$ -module, we have cohomology groups  $H^r(G, M)$ . Since  $G$  will be fixed throughout, we shall also write  $H^r(M)$ .

The groups  $A_K$  and  $B_K$  of rational points of  $A$  and  $B$  respectively, in  $K$  are such  $G$ -modules. We wish to define a pairing of  $H^1(A_K)$  and  $B_K$  into  $H^2(K^*)$ , where  $K^*$  is the multiplicative group of  $K$ . This is done as follows.

Let  $(a_\sigma)$  be a 1-cocycle in  $A_K$  and let  $b \in B_K$ . Denote by  $Z_0(V, K)$  the group of zero cycles on a variety  $V$ , of degree 0, rational over  $K$ . Let  $\alpha_\sigma$  be in  $Z_0(A, K)$ , and such that  $S(\alpha_\sigma) = a_\sigma$ . We can form the coboundary

$$\alpha = (\delta a)_{\sigma, \tau} = \sigma \alpha_\tau - \alpha_{\sigma\tau} + \alpha_\sigma,$$

and we have  $S(\alpha) = 0$ . Select  $b$  in  $Z_0(B, k)$  such that  $S(b) = b$ . We put

$$\alpha_{\sigma, \tau} = D(\alpha, b)$$

after changing  $D$  by the divisor of a function over  $k$  if necessary, to make this expression defined.

It is clear that  $(\alpha_{\sigma, \tau})$  is a 2-cocycle of  $G$  in  $K^*$ . We contend that its cohomology class depends only on the linear equivalence class of  $D$ , and, in fact, on its correspondence class, and that the pairing

$$[(a_\sigma), b] \rightarrow (\alpha_{\sigma, \tau})$$

induces a well defined bilinear map of  $H^1(A_K)$  and  $B_K$  into  $H^2(K^*)$ . In addition, if an element of  $B_K$  is a trace of an element of  $B_k$ , it is orthogonal to  $H^1(A_K)$  so that we may replace  $B_K$  by  $H^0(B_K)$ . This will be proved in the following sequence of assertions.

a.) If  $D$  and  $D'$  differ by a trivial divisor, of type  $X \times B + A \times Y + (\phi)$ , and if  $\alpha_{\sigma, \tau}$  and  $\alpha'_{\sigma, \tau}$  are obtained, respectively, from  $D$  and  $D'$  as above, then they are cohomologous.

*Proof.* To begin with, we must insure that if  $D$  and  $D'$  differ by the divisor of a function  $(\phi)$ ,  $\phi \in k(A \times B)$ , then our cocycles are cohomologous.

Indeed,  $\alpha_{\sigma,\tau}$  and  $\alpha_{\sigma,\tau'}$  differ by the coboundary of  $\phi(\alpha_\sigma, b)$ . The rest is now trivially verified.

b.) If  $\alpha_\sigma$  is changed by a cycle of type  $\alpha_\sigma' + (\sigma - 1)c$  with  $\alpha_\sigma' \in Z_0(A, K)$  and  $S(\alpha_\sigma') = 0$ , then  $\alpha_{\sigma,\tau}$  changes by a coboundary.

*Proof.* In fact, that coboundary is that of  $D(\alpha_\sigma', b)$ .

c.) If we change  $b$  by a cycle  $c \in Z_0(B, k)$  such that  $S(c) = 0$ , then  $\alpha_{\sigma,\tau}$  changes by a coboundary.

*Proof.* Using the reciprocity theorem, one sees that the coboundary is that of  ${}^tD(c, \alpha_\sigma)$ .

The above statements show that our pairing is well defined. That it is bilinear now follows from the bilinearity of  $D(a, b)$ .

d.) If  $b$  is a trace,  $b = \sum \rho b'$ , then  $\alpha_{\sigma,\tau}$  is a coboundary.

*Proof.* We can choose  $b = \sum (\rho' b') - n(0)$ . Then the coboundary is that of

$$c_\sigma = \prod_{\rho} f_{\sigma,\rho}(\sigma \rho b') / f_{\sigma,\rho}(0).$$

**6. Kummer theory.** Let  $A, B$  be two abelian varieties and  $D$  a divisor on the product. Let  $a, b$  be two cycles on  $A$  and  $B$  respectively, of degree 0. We write  $a \sim 0$  if  $S(a) = 0$ . Assume that there is an integer  $n$  such that  $na \sim 0$  and  $nb \sim 0$ . Let  $D_1$  be linearly equivalent to  $D$ , and such that no point of  $a \times b$  lies in  $D_1$ . Then one verifies immediately that  ${}^tD_1(nb, a) / D_1(na, b)$  is an  $n$ -th root of unity which is independent of the auxiliary divisor  $D_1$  selected in the linear equivalence class of  $D$ . We shall denote it by  $\epsilon_{n,D}(a, b)$ . Furthermore, if  $a_1 \sim a$  and  $b_1 \sim b$ , then  $\epsilon_{n,D}(a, b) = \epsilon_{n,D}(a_1, b_1)$ . Thus our root of unity defines a bilinear pairing of the points of order  $n$  on  $A$  and  $B$  respectively. If  $a = S(a)$  and  $b = S(b)$ , so that  $na = 0$  and  $nb = 0$ , we can also write  $\epsilon_{n,D}(a, b)$  instead of  $\epsilon_{n,D}(a, b)$ . Note finally that it depends only on the correspondence class of  $D$ .

On the other hand, we can obviously generalize the root of unity defined by Weil which is usually denoted by  $e_n(a, b)$  in the special case of divisors on an abelian variety, i.e., when  $B = \hat{A}$  and  $D$  is a Poincaré divisor. Namely, it is clear that  $(n\delta)^{-1} {}^tD(b)$  is linearly equivalent to 0 on  $A$  if  $nb \sim 0$ . Say  $(\omega) = (n\delta)^{-1} {}^tD(b)$ . Let  $u$  be a generic point of  $A$ . Then  $\omega(u + a) / \omega(u)$  is by definition, the  $n$ -th root of unity  $e_{n,D}(a, b)$ . (See [2], Ch. 7.)

**THEOREM 6.** Let  $D$  be a divisor on a product of abelian varieties  $A \times B$ .

Let  $\alpha, \beta$  be two cycles on  $A$  and  $B$  respectively, of degree 0, such that  $n\alpha \sim 0$  and  $n\beta \sim 0$ . Let  $a = S(\alpha)$  and  $b = S(\beta)$ . Then  $e_{n,D}(\alpha, \beta) = e_{n,D}(a, b)$ .

*Proof.* Consider the rational map  $n\delta: A \rightarrow A$  followed by the divisorial correspondence  $D$  on  $A \times B$ . Just as in the appendix of [2], we form the composed divisor on  $A \times B$ , which we denote by  $E = D \circ (n\delta)$ . Then it is obvious that  $E$  and  $nD$  are in the same correspondence class on  $A \times B$ , and thus that

$$E = nD + (F) + X \times B + A \times Y$$

for some function  $F$  on  $A \times B$ , and divisors  $X$  on  $A$  and  $Y$  on  $B$ . Let  $k$  be a field of rationality for  $D, E, F, X$  and  $Y$ , and let  $u, w$  be independent generic points of  $A, B$  over  $k$ . We put

$$\alpha = (u + a) - (u) \quad \text{and} \quad \beta = (w + b) - (w).$$

Then

$$e_{n,D}(\alpha, \beta) = {}^tE(\beta, \alpha) = {}^tD(n\beta, \alpha)F(\alpha, \beta).$$

On the other hand,  $E(\alpha) = D(nu + na) - D(nu) = 0$ , and hence  $E(\alpha, \beta) = 1$ . Thus  $D(n\alpha, \beta)^{-1} = F(\alpha, \beta)$ . This proves our theorem.

One should observe that, just as we proved Theorem 5 for arbitrary varieties, we can generalize our root of unity to arbitrary varieties by pull back. The details are obvious and are left to the reader.

### Appendix.

From the theory of linear equivalence, one knows that, given two cycles on a projective non-singular variety, it is possible to move one of them by linear equivalence so that the intersection is defined. Samuel has pointed out to me that this can be done rationally over a given field of rationality for all objects involved (at least over an infinite field). We need the result here only for divisors, and I shall reproduce below a proof due to Chevalley. We first deal with the local problem.

**THEOREM.** Let  $V$  be an affine variety defined over a field  $k$ . Let  $D$  be a divisor on  $V$ , rational over  $k$ . Let  $S$  be a finite set of simple points of  $V$ . Then there exists a function  $\phi$  on  $V$ , defined over  $k$ , such that no point of  $S$  lies in the support of  $D + (\phi)$ .

*Proof.* We may obviously assume that  $D$  is positive and, in fact, a prime rational cycle. Furthermore, we may assume that the points of  $S$  are algebraic over  $k$ , because  $D$  contains the specializations over  $k$  of all of its points.

Let  $R = k[v]$  be a coordinate ring for  $V$  over  $k$  and let  $I$  be the ideal of  $R$  consisting of those functions  $f \in R$  such that  $(f) \geq D$ . This ideal has a finite basis,  $I = (f_1, \dots, f_n)$ . Let  $\mathfrak{p}$  be a maximal ideal of  $R$ , and  $\mathfrak{o} = R_{\mathfrak{p}}$  its local ring with maximal ideal  $\mathfrak{m}$ . We assume that  $\mathfrak{p}$  belongs to a simple point, so that  $\mathfrak{o}$  is a unique factorization domain. Then there is a function  $t$  in  $\mathfrak{o}$  which represents  $D$  locally at  $\mathfrak{p}$ . We can write  $t = \sum a_i f_i$  with  $a_i \in \mathfrak{o}$ . I contend that if  $b_i \in \mathfrak{o}$  is such that  $b_i \equiv a_i \pmod{\mathfrak{m}}$  then  $\sum b_i f_i$  differs from  $t$  by a unit in  $\mathfrak{o}$ . Indeed,

$$\sum b_i f_i = \sum (b_i - a_i) f_i + t.$$

Since  $(f_i) \geq D$ , we can write  $f_i = t g_i$  with  $g_i \in \mathfrak{o}$ . Hence  $\sum b_i f_i \in t[1 + \mathfrak{m}]$ , thereby proving our contention.

Now let  $\mathfrak{p}_j$  be a finite number of maximal ideals of  $R$ , belonging to simple points. Let  $\mathfrak{o}_j = R_{\mathfrak{p}_j}$  be their local rings with maximal ideals  $\mathfrak{m}_j$ . Given elements  $x_j$  in  $\mathfrak{o}_j$ , it is possible to find  $x \in R$  such that  $x \equiv x_j \pmod{\mathfrak{m}_j}$ , by the well known Chinese remainder theorem, applicable since  $\mathfrak{p}_j + \mathfrak{p}_{j'} = R$  for  $j \neq j'$ . We now see that it suffices to approximate at each  $\mathfrak{o}_j$  by an element of  $R$ , the coefficients of an element  $t$ , representing  $D$  at  $\mathfrak{o}_j$  in terms of the  $f_i$ . This proves our theorem.

If we wish to apply the local result to an abstract variety  $V$ , then we must assume that the given finite set of points (algebraic over  $k$ ) can be represented on an affine  $k$ -open subset of  $V$ . This is the case on a projective variety, although, if the ground field is finite, we may have to change the projective embedding (for instance, by finding first a hypersurface section of  $V$  which does not pass through any of the given points, and then dehomogenizing  $V$  at this hypersurface, in the projective embedding in which this hypersurface becomes a hyperplane).

PARIS.

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# SPHERICAL FUNCTIONS OVER $\mathfrak{P}$ -ADIC FIELDS, I.\*

By F. I. MAUTNER.

**1. Introduction and preliminaries.** Let  $\Omega$  be a field which is complete under a given discrete real valued valuation. For every element  $a$  of  $\Omega$ , the order of  $a$  is defined and denoted by  $\text{ord}(a)$ . If  $a \neq 0$ , then  $\text{ord}(a)$  is a rational integer and  $a \rightarrow \text{ord}(a)$  is a homomorphism of the multiplicative group  $\Omega^\times$  of  $\Omega$  onto the additive group of rational integers; if  $a = 0$ , we put  $\text{ord}(0) = +\infty$ . Let  $\mathfrak{O}$  be the valuation ring of  $\Omega$ , i. e., the set of elements  $a$  of  $\Omega$  for which  $\text{ord}(a) \geq 0$ , and  $\mathfrak{P}$  the unique maximal ideal of  $\mathfrak{O}$  defined by  $\text{ord}(a) > 0$ . The ideal  $\mathfrak{P}$  is principal. Let  $\tau$  be a generator of  $\mathfrak{P}$ :

$$(1.1) \quad \mathfrak{P} = (\tau) = \tau\mathfrak{O}, \text{ord}(\tau) = 1.$$

Let  $\mathfrak{U}$  be the group of units of the ring  $\mathfrak{O}$ ; this is the set of elements of order zero. Every element  $a \neq 0$  of  $\Omega$  can be written uniquely in the form

$$(1.2) \quad a = \tau^n u, \text{ where } n = \text{ord}(a) \text{ and } u \in \mathfrak{U},$$

so that  $\Omega^\times$  is the direct product of  $\{\tau^n\}$  and  $\mathfrak{U}$ . We shall assume throughout that the residue class field  $\mathfrak{O}/\mathfrak{P}$  is finite:

$$(1.3) \quad q = \mathfrak{N}\mathfrak{P} = \text{number of elements of } \mathfrak{O}/\mathfrak{P} < \infty.$$

We normalize the valuation  $|\cdot|$  on  $\Omega$  by putting

$$(1.4) \quad |a| = (\mathfrak{N}\mathfrak{P})^{-\text{ord}(a)} = q^{-\text{ord}(a)}.$$

With  $|a - b|$  as distance,  $\Omega$  is a metric space; it is well known and readily verified that with the topology defined by this metric,  $\Omega$  is a locally compact topological field,  $\mathfrak{O}$  a compact open subring of  $\Omega$  and each ideal  $\mathfrak{P}^n$  compact and open. The ideals  $\mathfrak{P}^n$  ( $n = 1, 2, 3, \dots$ ) form a base for the neighborhoods of 0 in  $\Omega$ , so that  $\Omega$  is totally disconnected.

The additive group  $\Omega^+$  of  $\Omega$  being a locally compact topological group, there exists a Haar-measure on it, which we denote by  $\mu$ . One has

$$(1.5) \quad d\mu(x + a) = d\mu(x), \quad d\mu(ax) = |a| d\mu(x).$$

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The first of these two equations expresses merely the invariance of the Haar-measure  $\mu$  on  $\Omega^*$ , while the second equation in (1.5) is easily verified using the uniqueness of Haar-measure and the normalization (1.4) of the valuation.

We shall study, in this paper, certain function spaces on the group  $G = PG(2, \Omega)$  of fractional linear transformations over  $\Omega$ . Let  $\mathfrak{G}$  be the group  $GL(2, \Omega)$  of all non-singular  $2 \times 2$  matrices with coefficients in  $\Omega$  and  $\mathfrak{J}$  the center of  $\mathfrak{G}$ , i. e., the set of scalar matrices  $\neq 0$ . Both  $\mathfrak{G}$  and  $G$  are locally compact topological groups, and  $G$  is topologically isomorphic to  $\mathfrak{G}/\mathfrak{J}$ . We denote the natural homomorphism of  $\mathfrak{G}$  onto  $G$  by  $\mathfrak{G} \rightarrow G$ .

$$(1.6) \quad G \cong \mathfrak{G}/\mathfrak{J}; \quad \mathfrak{G} \rightarrow G.$$

Let  $\mathfrak{K}$  be the group  $GL(2, \mathfrak{D})$  of all  $2 \times 2$  matrices with coefficients in  $\mathfrak{D}$  and determinant in  $\mathfrak{U}$ . Clearly  $\mathfrak{K}$  is a compact open subgroup of  $\mathfrak{G}$ . Let  $K$  be the image of  $\mathfrak{K}$  under the natural homomorphism of  $\mathfrak{G}$  onto  $G$ , so that  $K$  is an open and compact subgroup of  $G$ .

*Definition 1.1.* Let  $S$  be the class of all complex-valued functions  $f(g)$  on  $G$  which satisfy

$$(1.7) \quad f(kgk') = f(g) \text{ for all } g \in G, k \text{ and } k' \in K.$$

We shall call  $S$  the space of spherical functions.

For any positive real number  $\rho$ , let  $\mathfrak{L}^\rho = \mathfrak{L}^\rho(G)$  be the space of equivalence classes of complex valued Haar-measurable functions  $f(g)$  on  $G$  whose  $\rho$ -th power is Lebesgue-integrable with respect to Haar-measure on  $G$ . We put  $S^\rho = S \cap \mathfrak{L}^\rho$ ; in particular,  $S^1$  is a closed subalgebra of  $\mathfrak{L}^1$ , which will be seen to be commutative. We shall study the spaces  $S^\rho$  for  $\rho = 1$  or  $2$  and also the algebra  $S^0$  of all spherical functions  $f(g)$  which are of compact support on  $G$ .

We obtain an explicit characterization of  $S^0$  by showing that, under the Fourier-transform on  $G$ , the algebra  $S^0$  is mapped isomorphically onto the algebra of all even Fourier polynomials and the isomorphism is given explicitly. As far as  $S^1$  is concerned, the situation is analogous to known properties of spherical functions on non-compact semi-simple Lie groups. In particular, it is again true that the Fourier transform of any function in  $S^1$  is a function  $F(s)$ , regular analytic in the strip  $0 < \Re s < 1$  and satisfying the functional equation  $F(s) = F(1-s)$ . We determine all maximal ideals of  $S^1$ , showing that they correspond to points in the strip  $0 \leq \Re s \leq 1$ . For this purpose, it is useful to compute the elementary spherical functions explicitly. This is done in § 7; it seems remarkable that they are elementary

functions (in fact rational functions of  $q^s$ ). Thus the situation is in this respect more analogous to that of complex semi-simple Lie groups than to the real case.

The results of this paper will be applied elsewhere to the analytic theory of quadratic forms, where it will be shown how they can be used to establish identities between certain of C. L. Siegel's representation functions of quadratic forms and to obtain functions of  $q^s$  associated with quadratic forms whose zeros are all on the line  $Rs = \frac{1}{2}$  or on the intervals  $s + 2\pi i n / \log q$  real,  $0 < Rs < 1$ , also, functions of  $s$  whose zeros lie on the line  $Rs = \frac{1}{2}$  or on the real interval  $0 < s < 1$ . These functions can be developed into Dirichlet series absolutely convergent for  $Rs > 1$ , have a functional equation ( $s \rightarrow 1 - s$ ) and an analogue of the Euler product; they are different from A. Selberg's generating functions for the order of magnitude of the norms of the primitive elements in discontinuous groups of motion of the hyperbolic plane. Indeed, the coefficients in the Dirichlet series are given in terms of Siegel's representation functions  $A(\mathfrak{S}, \mathfrak{T})$  and  $\alpha_p(\mathfrak{S}, \mathfrak{T})$  of quadratic forms.

**2. The groups  $GL(2, \Omega)$  and  $PGL(2, \Omega)$ .** We put on the group  $\mathfrak{G} = GL(2, \Omega)$  the topology which it inherits as a subset of the four-dimensional locally compact topological vector space of all  $2 \times 2$  matrices over  $\Omega$  (the latter, of course, with the Cartesian product topology of  $\Omega \times \cdots \times \Omega$ ). Clearly  $\mathfrak{G}$  is a locally compact topological group and  $\mathfrak{K} = GL(2, \mathfrak{D})$ , a compact and open subgroup of  $\mathfrak{G}$ . Hence the factor space  $\mathfrak{G}/\mathfrak{K}$  is discrete in its natural topology.

Using elementary divisors, one sees that every element  $g$  of  $\mathfrak{G}$  can be written in the form

$$(2.1) \quad g = k_1 \begin{pmatrix} \tau^m & 0 \\ 0 & \tau^n \end{pmatrix} k_2, \quad k_j \in \mathfrak{K}.$$

To this, we apply the natural homomorphism  $\mathfrak{G} \rightarrow G$  of  $\mathfrak{G}$  onto  $G$ , whose kernel is  $\mathfrak{J}$  (see (1.6) above) and note that there exists an element  $c$  of  $\mathfrak{J}$  such that  $c \begin{pmatrix} \tau^m & 0 \\ 0 & \tau^n \end{pmatrix} = \begin{pmatrix} \tau^{m-n} & 0 \\ 0 & 1 \end{pmatrix}$ . Let

$$(2.2) \quad y = \text{image in } G \text{ of } \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix} \text{ under } \mathfrak{G} \rightarrow G.$$

We conclude that every element  $g$  of  $G$  can be written in the form

$$(2.3) \quad g = k_1 y^n k_2, \quad k_j \in K,$$

since  $K$  was defined to be the image of  $\mathfrak{K}$  under  $\mathfrak{G} \rightarrow G$ .

Now  $Ky^mK = Ky^nK$  is equivalent to  $\mathcal{K} \begin{pmatrix} \tau^m & 0 \\ 0 & 1 \end{pmatrix} \mathcal{K} = \mathcal{J}\mathcal{K} \begin{pmatrix} \tau^n & 0 \\ 0 & 1 \end{pmatrix} \mathcal{K}$ .  
Hence

$$\begin{pmatrix} 0 & u^r \\ 0 & 1 \end{pmatrix} = k_1 \begin{pmatrix} c\tau^n & 0 \\ 0 & c \end{pmatrix} k_2, \quad c \in \Omega^\times, k_j \in \mathcal{K}.$$

By the uniqueness of the elementary divisors up to permutations and multiplications by units, this implies  $\tau^m = u\tau^n c$  and  $1 = vc$  with  $u, v \in \mathcal{U}$  or else  $\tau^m = uc$  and  $1 = v^n c$ . In the first case, we have  $\text{ord}(c) = 0$ , hence  $m = n$ ; in the second case,  $\text{ord}(c) = -n$ , hence  $m = -n$ . This proves

$$(2.4) \quad Ky^mK = Ky^nK \text{ if and only if } m = \pm n.$$

Thus  $G$  is the disjoint union

$$(2.5) \quad G = \bigcup_{n=0}^{\infty} Ky^nK.$$

Let  $\mathcal{A}$  be the subgroup of non-singular diagonal matrices of  $\mathcal{G}$  and  $\mathcal{B}$  the subgroup of all matrices  $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ , so that  $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$  is the subgroup of non-singular triangular matrices. Let  $A$  be the image of  $\mathcal{A}$ ,  $B$  the image of  $\mathcal{B}$  under the natural homomorphism (1.6) of  $\mathcal{G}$  onto  $G$ . Then  $AB = BA$  is a closed subgroup of  $G$ ,  $B$  a closed normal subgroup of  $AB$ ,  $A \cap B = (1)$  and  $AB/B \cong A \cong \Omega^\times$ . The restriction of the homomorphism (1.6) to  $\mathcal{B}$  is clearly a topological isomorphism of  $\mathcal{B}$  onto  $B$  and  $\mathcal{B}$  is obviously topologically isomorphic to the additive group  $\Omega^+$  of the field  $\Omega$ ; hence  $B \cong \Omega^+$ . Using elementary divisors, one sees that

$$(2.6) \quad G = BAK.$$

If  $g \in \mathcal{G}$  has matrix-coefficients  $x_{ij}$ , then

$$|\det x_{ij}|^{-2} d\mu(x_{11}) d\mu(x_{12}) d\mu(x_{21}) d\mu(x_{22})$$

is easily seen to be a Haar-measure on  $\mathcal{G}$  which is both left- and right-invariant. Hence the Haar-measure on  $G$  is both left- and right-invariant, as follows at once from elementary properties of measures on homogeneous spaces. We denote the Haar-measure on  $G$  by  $\mathfrak{M}$  and write often  $dg$  instead of  $d\mathfrak{M}g$ .

We wish to compute the Haar-measure  $\mathfrak{M}(Ky^nK)$  of the compact open subset  $Ky^nK$  of  $G$ . Clearly,  $\bigcup_{k \in K} kgK$  is a compact subset of the coset space  $G/K$ ; but  $G/K$  is discrete, so this is a finite subset of  $G/K$ . Hence there exist a finite number  $r$  of elements  $k_j$  of  $K$  such that  $KgK = \bigcup_{j=1}^r k_j gK$ .

Choosing the  $k_i$  such that  $k_i g K \neq k_j g K$  for  $i \neq j$ , we see that  $\mathfrak{M}(KgK) = r\mathfrak{M}(K)$ , since  $\mathfrak{M}(K)$  is left-invariant. Let  $K^g$  be the subgroup of those elements  $k$  of  $K$  which satisfy  $kgK = gK$  for this element  $g$ . Then  $r = \text{index of } K^g \text{ in } K$ . Now let  $g = y^n$ ; then this index is easily computed to be  $q^{n-1}(q+1)$ . Thus

$$\mathfrak{M}(Ky^nK) = q^{n-1}(q+1)\mathfrak{M}(K) \text{ for } n > 0.$$

Hence, for  $f(g) \in S^1$ , we obtain

$$\int_G f(g) dg = \mathfrak{M}(K) \{f(1) + (q+1) \sum_{n=1}^{\infty} f(y^n) q^{n-1}\}.$$

Since  $K$  is a compact subgroup of  $G$ , there exists on the factor space  $G/K$  an invariant measure  $d\dot{g}$  such that

$$(2.7) \quad \int_G f(g) dg = \int_{G/K} d\dot{g} \int_K f(gk) dk,$$

where  $dk$  refers to Haar measure on  $K$ . Now by (2.6),  $G/K$  is homeomorphic to  $AB/K \cap AB$ ; under this homeomorphism there must correspond to the invariant measure  $d\dot{g}$  on  $G/K$  the invariant measure on  $AB/K \cap AB$  to which we refer by  $d(\dot{a}\dot{b})$ . Hence

$$(2.8) \quad \int_G f(g) dg = \int_{AB/K \cap AB} d(\dot{a}\dot{b}) \int_K f(abk) dk,$$

where  $g = abk$ . It is convenient to normalize the measure  $d(\dot{a}\dot{b})$  by putting

$$(2.9) \quad \int_{AB/K \cap AB} h(\dot{a}\dot{b}) d(\dot{a}\dot{b}) = \int_{AB} h(ab) d_l(ab),$$

where  $d_l(ab)$  refers to left-invariant measure on  $AB$ . With this convention, we have, formally,

$$(2.10) \quad dg = d_l(ab) dk, \text{ where } g = abk.$$

Now one sees easily that  $d_l(ab) = da db$ , where  $da$  refers to Haar-measure on  $A$ ,  $db$  to Haar-measure on  $B$ . It remains to find the relation between the Haar-measure on  $A$ ,  $B$ ,  $K$  and  $G$ . We normalize the measure on  $K$  by letting the measure on  $K$  be the restriction of the measure on  $G$  (noting that  $K$  is open in  $G$ ).

Now let  $f_0(g)$  be the characteristic function of  $K$ , so that  $f_0(g)$  is a continuous function on  $G$  of compact support. Using (2.10), we have

$$\int_A f_0(g) dg = \int_{AB} \int_K f_0(abk) dk d_l(ab),$$

thus

$$\mathfrak{M}(K) = \int_{K \cap AB} \int_K dk d_l(ab).$$

Now we use  $B \cong \Omega^+$  and  $A \cong \Omega^\times$  and see that

$$\int_{K \cap AB} d_l(ab) = \int_{\mathfrak{U}} d\mu^\times(u) \int_{\mathfrak{D}} d\mu(\mathfrak{D}),$$

where  $\mu$  is Haar-measure on  $\Omega^+$ ,  $\mu^\times$  Haar-measure on  $\Omega^\times$ . Thus we obtain  $\mathfrak{M}(K) = \mu^\times(\mathfrak{U})\mu(\mathfrak{D})\mathfrak{M}(K)$ , i. e.,

$$(2.11) \quad \mu(\mathfrak{D})\mu^\times(\mathfrak{U}) = 1.$$

**3. The principal series.** We shall now introduce the principal series of representations of  $G$  as induced representations following the by now classical procedure of Frobenius, Schur, Bargmann, Gelfand, Naimark, Mackey and others. As above, let  $\mathcal{A}$  be the subgroup of  $\mathcal{G}$  of non-singular diagonal matrices and  $\mathcal{B}$  the subgroup of all matrices  $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ , so that  $\mathcal{A}\mathcal{B}$  is the subgroup of triangular matrices. Let  $A$  be the image of  $\mathcal{A}$  in  $G$  under the natural homomorphism  $\mathcal{G} \rightarrow G$ , similarly  $B$  the image of  $\mathcal{B}$ . Then  $AB$  is a closed subgroup of  $G$ ,  $B$  a normal closed subgroup of  $AB$ ,  $A \cap B = (1)$  and  $A \cong \Omega^\times$ . Now let  $\alpha$  be a continuous homomorphism of  $AB$  to the multiplicative group of complex numbers  $\neq 0$  satisfying

$$(3.1) \quad \alpha(ab) = \alpha(a), \text{ i. e., } \alpha(b) = 1 \text{ for all } b \in B.$$

Since  $AB/B \cong A$ , the set of all  $\alpha$  satisfying (3.1) is in natural one-one correspondence with the group of all (not necessarily unitary) characters of the multiplicative group  $\Omega^\times$ .

Let us consider now the space of all complex-valued (measurable or continuous) functions  $\psi(g)$  defined on  $G$  which satisfy

$$(3.2) \quad \psi(abg) = \alpha(a)\psi(g) \text{ for all } g \in G, a \in A, b \in B.$$

This is clearly a linear space, say  $\mathfrak{V}_\alpha$ . For any  $\gamma \in G$ , we define a linear transformation  $M(\gamma, \alpha)$  in  $\mathfrak{V}_\alpha$  by putting

$$(3.3) \quad [M(\gamma, \alpha)\psi](g) = \psi(g\gamma).$$

Consider now the linear transformation  $\mathfrak{F}(\alpha)$  of  $\mathfrak{V}_\alpha$  defined by

$$(3.4) \quad [\mathfrak{F}(\alpha)\psi](\bar{g}) = \int_G f(\gamma)\psi(g\gamma)d\gamma,$$

or, formally,

$$(3.5) \quad \mathcal{F}(\alpha) = \int_G f(\gamma) M(\gamma, \alpha) d\gamma.$$

Of course, (3.4) will exist only for suitably restricted functions  $f$  and  $\psi$ . Let us use the fact (elementary divisors) that every element  $g \in G$  can be written in the form  $g = ba\kappa$  with  $\kappa \in K$ , and we obtain

$$\begin{aligned} [\mathcal{F}(\alpha)\psi](k) &= \int f(\gamma)\psi(k\gamma) d\gamma = \int f(k^{-1}g)\psi(g) dg \\ &= \int f(k^{-1}ba\kappa)\psi(ba\kappa) d\kappa = \int \int f(k^{-1}ba\kappa)\alpha(a)\psi(\kappa) d_r(ba) d\kappa, \end{aligned}$$

where  $d\kappa$  refers to the Haar-measure on  $K$ ,  $d_r(ba)$  to the right-invariant Haar-measure on  $BA = AB$ , normalized so that

$$(3.6) \quad dg = d_r(ba) d\kappa \text{ when } g = ba\kappa.$$

Hence, if we put

$$(3.7) \quad F_\alpha(k, \kappa) = \int_{AB} f(k^{-1}ba\kappa)\alpha(a) d_r(ba)$$

and note that any function  $\psi(g)$  satisfying (3.2) is determined by its restriction to  $K$ , we see that the linear transformation  $\mathcal{F}(\alpha)$  (assuming it exists in a suitable sense) is an integral operator with kernel  $F_\alpha(k, \kappa)$  given by (3.7). Thus

$$(3.8) \quad [\mathcal{F}(\alpha)\psi](k) = \int_K F_\alpha(k, \kappa)\psi(\kappa) d\kappa.$$

If, for example,  $\psi$  is continuous and  $f$  continuous of compact support, the above integrals certainly exist. And our integral operator  $\mathcal{F}(\alpha)$  has a trace given by

$$(3.9) \quad \text{trace } \mathcal{F}(\alpha) = \int_K F_\alpha(k, k) dk = \int_K \int_{AB} f(k^{-1}ba\kappa)\alpha(a) d_r(ba) dk.$$

In particular, if  $f(k^{-1}gk) = f(g)$  for any  $g \in G$ ,  $k \in K$ , we obtain

$$(3.10) \quad \text{trace } \mathcal{F}(\alpha) = \int_K dk \int_{AB} f(ba)\alpha(a) d_r(ba) = \mathfrak{M}(K) \int_{AB} f(ba)\alpha(a) d_r(ba).$$

This well known formula goes, of course, back to Gelfand and Naimark who obtained it first for some of the classical groups over the field of complex numbers.

Using (3.2) and  $g = bak$ , we see that every  $\psi(g) \in \mathfrak{V}_\alpha$  is uniquely determined by its restriction to  $K$ . Let us now consider only those functions

$\psi \in \mathcal{V}_\alpha$  for which  $\psi(k) \in L^2(K)$ . They form a Hilbert space which we denote by  $\mathcal{H}_\alpha$ . One sees easily that  $M(g, \alpha)$  defines a unitary operator of  $\mathcal{H}_\alpha$  if and only if

$$(3.11) \quad \alpha(y^n) = q^{n(t-1)} \text{ with } t \text{ real.}$$

Let us decompose  $\mathcal{H}_\alpha$  with respect to the operators  $M(k, \alpha)$  with  $k$  varying only over  $K$ . Since  $K$  is compact, we obtain a decomposition of  $\mathcal{H}_\alpha$  into a direct sum of finite dimensional pairwise orthogonal  $K$ -invariant  $K$ -irreducible subspaces. Let us introduce in each of them an orthonormal basis; their union is a complete orthonormal set in  $\mathcal{H}_\alpha$ . With respect to this basis of  $\mathcal{H}_\alpha$ , we form the matrix coefficients of the operator  $\mathcal{F}(\alpha)$ . Using the Schur orthogonality relations, one sees easily that if  $f(g) \in S^0$ , then all matrix-coefficients of  $\mathcal{F}(\alpha)$  are zero except  $\langle \mathcal{F}(\alpha)\psi_\alpha, \psi_\alpha \rangle$ , where  $\psi_\alpha$  is an element of  $\mathcal{H}_\alpha$  satisfying  $M(k, \alpha)\psi_\alpha = \psi_\alpha$  for all  $k \in K$  and  $\langle \psi_\alpha, \psi_\alpha \rangle = 1$ . It is known and easily verified that the subspace of  $M(k, \alpha)$ -invariant elements of  $\mathcal{H}_\alpha$  is either 1- or 0-dimensional according as  $\alpha(k) = 1$  for all  $k \in A \cap K$  or not. Hence, if  $f(g) \in S^0$ , we have  $\text{trace } \mathcal{F}(\alpha) = \langle \mathcal{F}(\alpha)\psi_\alpha, \psi_\alpha \rangle$ , hence

$$(3.12) \quad \langle \mathcal{F}(\alpha)\psi_\alpha, \psi_\alpha \rangle = \mathfrak{M}(K) \int_{AB} f(ba) \alpha(a) d_r(ba).$$

Now let us write

$$(3.13) \quad \phi(g, \alpha) = \langle M(g, \alpha)\psi_\alpha, \psi_\alpha \rangle,$$

and we obtain

$$(3.14) \quad \text{trace } \mathcal{F}(\alpha) = \langle \mathcal{F}(\alpha)\psi_\alpha, \psi_\alpha \rangle = \int_G f(g) \phi(g, \alpha) dg.$$

**4. The mapping  $f \rightarrow \tilde{f}$ .** We wish to compute the integral  $\tilde{f}(a) = \int_B f(ba) db$  for integrable spherical functions  $f$ , where  $db$  refers to the Haar-measure on the group  $B$ . The restriction of the homomorphism (1.6)  $\mathcal{G} \rightarrow G$  to the subgroup  $\mathcal{B}$  of  $\mathcal{G}$  is a topological isomorphism onto  $B$ , so that  $B$  is isomorphic to the additive group  $\Omega^*$  of  $\Omega$ ,  $b \leftrightarrow x$ ; hence  $db = d\mu(x)$ , where  $\mu$  is Haar-measure on  $\Omega^*$ .

Consider now the matrix  $C$  defined by

$$C = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} \tau^n & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tau^n & 0 \\ x\tau^n & 1 \end{pmatrix}.$$

Let  $n \geq 0$ ; using elementary divisors, one sees that  $C \in \mathcal{K} \begin{pmatrix} \tau^m & 0 \\ 0 & 1 \end{pmatrix} \mathcal{K}$ , where

$$(4.1) \quad m = \begin{cases} n & \text{if } \text{ord}(x) \geq -n \\ n + 2 \text{ord}(x) & \text{if } \text{ord}(x) < -n. \end{cases}$$

To this we apply the homomorphism (1.6) of  $\mathcal{G}$  onto  $G$  and see that

$$(4.2) \quad by^n \in Ky^mK, \text{ where } b \leftrightarrow x$$

and  $m$  is given by (4.1). Let  $r$  be any non-negative integer and  $f_r(g)$  the characteristic function of the double coset  $Ky^rK$ , so that  $f_r(g)$  is a continuous function on  $G$  of compact support. We have

$$(4.3) \quad f_r(by^n) = \delta_{r, |m|} = \begin{cases} 1 & \text{if } r = |m| \\ 0 & \text{if } r \neq |m|. \end{cases}$$

Hence

$$\tilde{f}_r(y^n) = \int_B f_r(by^n) db = \int_{\text{ord}(x) \geq -n} \delta_{r,n} d\mu(x) + \int_{\text{ord}(x) < -n} \delta_{-r, n+2\text{ord}(x)} d\mu(x).$$

Now

$$\int_{\text{ord}(x) \geq -n} d\mu(x) = q^n \mu(\mathfrak{D})$$

and

$$\int_{\text{ord}(x) < -n} \delta_{-r, n+2\text{ord}(x)} d\mu(x) = \begin{cases} 0 & \text{if } r \leq n \\ \int_{\text{ord}(x) = -(r+n)/2} d\mu(x) & \text{if } r > n \end{cases}$$

since  $\text{ord}(x) = -\frac{1}{2}(r+n) < -n$  implies  $r > n$ . In particular, the last integral is zero if  $r+n$  is odd since  $\text{ord}(x)$  takes on only rational integers as values. Therefore

$$\int_{\text{ord}(x) < -n} \delta_{-r, n+2\text{ord}(x)} d\mu(x) = \begin{cases} 0 & \text{if } r \leq n \text{ or } r+n \text{ odd} \\ (1-1/q)q^{(r+n)/2} \mu(\mathfrak{D}) & \text{if } r > n \text{ and } r+n \text{ even.} \end{cases}$$

Hence

$$(4.4) \quad \tilde{f}_r(y^n) = \begin{cases} q^n \mu(\mathfrak{D}) & \text{if } r = n \geq 0 \\ 0 & \text{if } 0 \leq r < n \text{ or if } r+n \text{ is odd} \\ (1-1/q)q^{(r+n)/2} \mu(\mathfrak{D}) & \text{if } r > n \geq 0 \text{ and } r+n \text{ is even.} \end{cases}$$

Now let  $f(g)$  be any integrable spherical function, then  $f(g) = \sum_{r=0}^{\infty} f(y^r) f_r(g)$ , and we obtain for  $n \geq 0$  and any  $f \in S^1$

$$\begin{aligned} \tilde{f}(y^n) &= \sum_{r=0}^{\infty} f(y^r) \tilde{f}_r(y^n) \\ &= \mu(\mathfrak{D}) \{ q^n f(y^n) + (1-1/q) \sum_{\substack{r > n \\ r+n \text{ even}}} q^{(r+n)/2} f(y^r) \} \\ &= \mu(\mathfrak{D}) \{ q^n f(y^n) + (1-1/q) \sum_{j=1}^{\infty} q^{n+j} f(y^{n+2j}) \}. \end{aligned}$$

Thus

$$(4.5) \quad \tilde{f}(y^n) = \mu(\mathfrak{D}) q^n \{ f(y^n) + (1-1/q) \sum_{j=1}^{\infty} f(y^{n+2j}) q^j \}.$$

Now consider  $by^{-n}$  for  $n \geq 0$ . We have, again by elementary divisors,  $by^{-n} \in Ky^mK$  where

$$m = \min \{-n, -n + 2 \operatorname{ord}(x)\} = \begin{cases} -n & \text{if } \operatorname{ord}(x) \geq 0 \\ -n + 2 \operatorname{ord}(x) & \text{if } \operatorname{ord}(x) < 0. \end{cases}$$

Hence, if  $f_r(g)$  is again the characteristic function of the double coset  $Ky^rK$  with  $r \geq 0$ , we obtain for  $n \geq 0$

$$\int_B f_r(by^{-n}) db = \int_{\operatorname{ord}(x) \geq 0} \delta_{r,n} d\mu(x) + \int_{\operatorname{ord}(x) < 0} \delta_{r, n-2\operatorname{ord}(x)} d\mu(x).$$

Now

$$\int_{\operatorname{ord}(x) < 0} \delta_{r, n-2\operatorname{ord}(x)} d\mu(x) = \begin{cases} \int_{\operatorname{ord}(x) = (n-r)/2} d\mu(x) & \text{if } n-r \text{ is even and } < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus for  $r \geq 0$  and  $n \geq 0$

$$\tilde{f}_r(y^{-n}) = \begin{cases} \mu(\mathfrak{D}) & \text{if } r = n \\ (1 - 1/q) q^{(r-n)/2} \mu(\mathfrak{D}) & \text{if } n-r \text{ is even and } < 0 \\ 0 & \text{otherwise.} \end{cases}$$

We compare this with the above expression (4.4) for  $\tilde{f}_r(y^n)$  and see that

$$(4.6) \quad \tilde{f}_r(y^{-n}) = q^{-n} \tilde{f}_r(y^n).$$

We wish to invert the mapping  $f \rightarrow \tilde{f}$ . For this purpose, we note that (4.5) implies

$$\tilde{f}(y^{n+2}) - q\tilde{f}(y^n) = \mu(\mathfrak{D}) q^{n+1} \{f(y^{n+2}) - f(y^n)\},$$

hence

$$\begin{aligned} f(y^n) &= [f(y^n) - f(y^{n+2})] + [f(y^{n+2}) - f(y^{n+4})] \\ &\quad + [(y^{n+4}) - f(y^{n+6})] + \dots \\ &= \mu(\mathfrak{D})^{-1} q^{-(n+1)} [q\tilde{f}(y^n) - \tilde{f}(y^{n+2})] + q^{-(n+3)} [q\tilde{f}(y^{n+2}) \\ &\quad - \tilde{f}(y^{n+4})] + \dots \\ &= \mu(\mathfrak{D})^{-1} \{\tilde{f}(y^n) + (1/q - 1)[\tilde{f}(y^{n+2})q^{-1} + \tilde{f}(y^{n+4})q^{-3} \\ &\quad + \tilde{f}(y^{n+6})q^{-5} + \dots]\}. \end{aligned}$$

Thus

$$(4.7) \quad f(y^n) = \mu(\mathfrak{D})^{-1} q^{-n} \{\tilde{f}(y^n) + (1 - q) \sum_{j=1}^{\infty} \tilde{f}(y^{n+2j}) q^{-2j}\}.$$

Let us note that  $f(g) \in \mathcal{S}^0$  if and only if  $f(y^n)$  is zero except for a finite number of  $n$  and that (4.5), (4.6) and (4.7) show that this is equivalent

to  $\bar{f}(y^n)$  vanishing for a finite number of  $n$ . Hence, we obtain the following result.

PROPOSITION 4.1. *The mapping  $f \rightarrow \bar{f}$  maps the space  $S^0$  of all spherical functions of compact support one-one linearly onto the space of all finite sequences  $\bar{f}(y^n)$  which satisfy  $\bar{f}(y^n) = q^{-n}\bar{f}(y^n)$ . The explicit form of the mapping  $f \rightarrow \bar{f}$  is given by (4.5), it inverse by (4.7).*

5. The mapping  $f \rightarrow F(s)$ . Now we consider

$$F(\alpha) = \mathfrak{M}(K) \int_A \bar{f}(a) \alpha(a) d\mu^\times(a),$$

where  $\mu^\times$  is the Haar-measure on  $\Omega^\times \cong A$  and  $\alpha(a)$  a (not necessarily unitary) character of  $A$  such that

$$(5.1) \quad \alpha(ky^n) = \alpha(y^n) \text{ for } k \in A \cap K.$$

We write

$$(5.2) \quad \alpha(y^n) = \alpha_s(y^n) = q^{n(s-1)}$$

and  $F(s)$  instead of  $F(\alpha)$ :

$$(5.3) \quad F(s) = \mathfrak{M}(K) \int_A \bar{f}(a) \alpha_s(a) d\mu^\times(a) = \mathfrak{M}(K) \mu^\times(\mathfrak{U}) \sum_{n=-\infty}^{\infty} \bar{f}(y^n) q^{n(s-1)}$$

since  $f(ky^n) = f(y^n)$  for  $k \in A \cap K$ . Now we use (4.6) and obtain

$$(5.4) \quad F(s) = \mathfrak{M}(K) \mu^\times(\mathfrak{U}) \{ \bar{f}(1) + \sum_{n=1}^{\infty} \bar{f}(y^n) [q^{n(s-1)} + q^{-n}q^{-n(s-1)}] \} \\ = \mathfrak{M}(K) \mu^\times(\mathfrak{U}) \{ \bar{f}(1) + \sum_{n=1}^{\infty} \bar{f}(y^n) [q^{n(s-1)} + q^{-ns}] \}.$$

This proves

LEMMA 5.1. *If  $f \in S^1$ , then  $F(s)$  defined by (5.3) satisfies*

$$(5.5) \quad F(s) = F(1-s)$$

for any complex number  $s$  for which (5.3) exists.

Now let  $f$  vary over  $S^0$ ; then we know by Proposition 4.1 that  $\bar{f}(y^n)$  varies over all finite sequences satisfying  $\bar{f}(y^n) = q^{-n}\bar{f}(y^n)$  for  $n \geq 0$ . Hence (5.4) shows that  $F(s)$  varies over all Fourier polynomials which satisfy  $F(s) = F(1-s)$ . It is clear that the mapping  $f \rightarrow F(s)$  is linear. Since  $g \rightarrow M(g, \alpha)$  is a representation of  $G$ , it follows that to the convolution of any two elements of  $S^0$  corresponds the ordinary product of the corresponding functions  $F(s)$ . This proves

THEOREM 5.1. *The mapping  $f \rightarrow F(s)$  is an isomorphism of the algebra  $S^0$  onto the algebra of all polynomials  $F(s) = F(1-s)$  of the form  $\sum c_n(q^{n(s-\frac{1}{2})} + q^{-n(s-\frac{1}{2})})$ . Hence every maximal ideal of  $S^0$  is given by a complex number  $c$  and is the set  $J_c$  of all those  $f \in S^0$  for which  $F(s)$  vanishes for this  $c$ . Two maximal ideals  $J_c, J_{c'}$  are equal if and only if  $c' = c + 2\pi i n / \log q$  or  $c' = 1 - c + 2\pi i n / \log q$ .*

In (5.4), let  $s = \frac{1}{2} + it$ , where  $t$  is real. We obtain

$$\begin{aligned} F(\tfrac{1}{2} + it) &= \mathfrak{M}(K) \mu^\times(\mathbb{U}) \{f(1) + \sum_{n=1}^{\infty} \bar{f}(y^n) q^{-n/2} [q^{nit} + q^{-nit}]\} \\ &= \mathfrak{M}(K) \mu^\times(\mathbb{U}) \{f(1) + 2 \sum_{n=1}^{\infty} \bar{f}(y^n) q^{-n/2} \cdot \cos(nt \log q)\}. \end{aligned}$$

Hence, by Fourier series,

$$\begin{aligned} (5.6) \quad \bar{f}(y^n) &= \frac{q^{n/2} \log q}{2\pi \mu^\times(\mathbb{U}) \mathfrak{M}(K)} \int_0^{2\pi/\log q} F(\tfrac{1}{2} + it) \cos(nt \log q) dt \\ &= \frac{q^{n/2} \log q}{\pi \mu^\times(\mathbb{U}) \mathfrak{M}(K)} \int_0^{\pi/\log q} F(\tfrac{1}{2} + it) \cos(nt \log q) dt \text{ for } n \geq 0. \end{aligned}$$

In (4.5), let us replace  $f(g)$  by  $f^*(g) = \bar{f}(g^{-1})$  and \* note that  $f(g) = f(g^{-1})$  for  $f(g) \in S$ . We obtain

$$\bar{f}^*(y^n) = \mu(\mathfrak{D}) q^n \{\bar{f}(y^n) + (1 - 1/q) \sum_{j=1}^{\infty} \bar{f}(y^{n+2j}) q^j\}.$$

Hence, if we write  $h(y^n) = \bar{f}(y^n)$ , we have

$$\bar{f}^*(y^n) = \bar{h}(y^n).$$

Now we put that in (5.4) and write  $F^*(s) = \mathfrak{M}(K) \int_A \bar{f}^*(a) \alpha_s(a) d\mu^\times(a)$ . We obtain

$$F^*(s) = \mathfrak{M}(K) \mu^\times(\mathbb{U}) \{\bar{h}(1) + \sum_{n=1}^{\infty} \bar{h}(y^n) [q^{n(s-1)} + q^{-ns}]\}.$$

Therefore

$$\bar{F}^*(\bar{s}) = \mathfrak{M}(K) \mu^\times(\mathbb{U}) \{\bar{f}(1) + \sum_{n=1}^{\infty} \bar{f}(y^n) [q^{n(s-1)} + q^{-ns}]\}.$$

Hence

$$(5.7) \quad F^*(s) = \bar{F}(\bar{s});$$

in particular,

$$F^*(\tfrac{1}{2} + it) = \bar{F}(\tfrac{1}{2} - it) = \bar{F}(\tfrac{1}{2} + it).$$

\*  $\bar{f}$  denotes the complex conjugate of  $f$ , whereas  $\bar{f}$  is defined at the beginning of § 4.

If we combine this with Proposition 4.1, we see that  $F^*(s) = \bar{F}(s)$  for all  $f \in S^0$  if and only if  $Rs = \frac{1}{2}$  or  $s + 2\pi i n / \log q$  is real.

**6. The dual measure.** We shall now compute the value of  $f \in S^0$  at the unit element 1 of  $G$  in terms of  $F(s)$ . By (4.7), we have

$$f(1) = \mu(\mathfrak{D})^{-1} \{ \bar{f}(1) + (1-q) \sum_{j=1}^{\infty} \bar{f}(y^{2j}) q^{-2j} \}.$$

Hence, using (5.6), we get

$$f(1) = \frac{\log q}{2\pi\mu(\mathfrak{D})\mu^\times(\mathfrak{U})\mathfrak{M}(K)} \int_0^{2\pi/\log q} F(\tfrac{1}{2} + it) [1 + (1-q) \sum_{n=1}^{\infty} q^{-n} \cos(2nt \log q)] dt.$$

Now we use (2.11), i.e.,  $\mu(\mathfrak{D})\mu^\times(\mathfrak{U}) = 1$ , and get

$$f(1) = \frac{\log q}{\pi\mathfrak{M}(K)} \int_0^{\pi/\log q} F(\tfrac{1}{2} + it) [1 + (1-q) \sum_{n=1}^{\infty} q^{-n} \cos(2nt \log q)] dt.$$

Noting that

$$\sum_{n=1}^{\infty} q^{-n} \cos(2nt \log q) = \tfrac{1}{2} \left\{ \frac{q^{2i}}{q - q^{2it}} + \frac{q^{-2it}}{q - q^{-2it}} \right\},$$

we obtain

$$(6.1) \quad f(1) = \int_0^{\pi/\log q} F(\tfrac{1}{2} + it) \mathfrak{M}^*(\tfrac{1}{2} + it) dt,$$

where

$$\begin{aligned} \mathfrak{M}^*(s) &= \frac{\log q}{\pi\mathfrak{M}(K)} \left[ 1 + \frac{1-q}{2} \left( \frac{q^{2s-1}}{q - q^{2s-1}} + \frac{q^{1-2s}}{q - q^{1-2s}} \right) \right] \\ &= -\frac{\log q}{\pi\mathfrak{M}(K)} \cdot \frac{q(q+1)}{2} \cdot \frac{(q^{s-\frac{1}{2}} - q^{\frac{1}{2}-s})^2}{(q^{2s-1} - q)(q^{1-2s} - q)}. \end{aligned}$$

In particular, for  $s = \frac{1}{2} + it$ ,

$$\mathfrak{M}^*(\tfrac{1}{2} + it) = -\frac{\log q}{\pi\mathfrak{M}(K)} \cdot \frac{q(q+1)}{2} \cdot \frac{(q^i - q^{-it})^2}{(q^{2it} - q)(q^{-2it} - q)},$$

thus

$$(6.3) \quad \mathfrak{M}^*(\tfrac{1}{2} + it) = \frac{2q(q+1)\log q}{\pi\mathfrak{M}(K)} \cdot \frac{\sin^2(t \log q)}{1 - 2q \cos(2t \log q) + q^2}.$$

Hence

$$(6.4) \quad \mathfrak{M}^*(\tfrac{1}{2} + it) \geq 0 \text{ for real } t.$$

We note that

$$(6.5) \quad \mathfrak{M}^*(\tfrac{1}{2} + \frac{i\pi n}{\log q}) = 0 \quad \text{for } n = 0, \pm 1, \pm 2, \dots$$

LEMMA 6.1. The function  $\mathfrak{M}^*(s)$  is a meromorphic function of  $s$  (in fact, a rational function of  $q^s$ ), regular in  $s$  except when  $q^s = \pm q$  or  $q^s = \pm 1$  (i. e.,  $s = \pi i n / \log q$  or  $s = 1 + \pi i n / \log q$ ) where it has a pole of order 1 with residue

$$(6.6) \quad \frac{1-q}{4\mathfrak{M}(K)\pi}.$$

It satisfies the functional equation  $\mathfrak{M}^*(1-s) = \mathfrak{M}^*(s)$ . Moreover,  $\mathfrak{M}^*(s) \geq 0$  for  $\text{Re } s = \frac{1}{2}$ .

7. **The Plancherel formula for  $S^2$ .** Let  $f(g) \in S^0$ , and put, as usual  $f^*(g) = \bar{f}(g^{-1})$ ; then  $(f * f^*)(g) \in S^0$ , where  $*$  denotes convolution on  $G$ . If to  $f$  corresponds  $F(\frac{1}{2} + it)$  in the sense of (5.3), then to  $f^*$  corresponds  $\bar{F}(\frac{1}{2} + it)$ , because  $g \rightarrow M(g, \frac{1}{2} + it)$  is a unitary representation of  $G$  if  $t$  is real (see § 3 or equation (5.7) above). Hence to  $f * f^*$  corresponds the function  $* |F(\frac{1}{2} + it)|_\infty^2$ . To this we apply (6.1) and obtain

$$(f * f^*)(1) = \int_0^{\pi/\log q} |F(\frac{1}{2} + it)|_\infty^2 \mathfrak{M}^*(\frac{1}{2} + it) dt.$$

Next we observe that  $(f * f^*)(1) = \int_G |f(g)|_\infty^2 dg$  and that  $F(\frac{1}{2} + it)$  is even in  $t$  and periodic with period  $2\pi/\log q$ . Hence

$$(7.1) \quad \int_G |f(g)|_\infty^2 dg = \int_0^{\pi/\log q} |F(\frac{1}{2} + it)|_\infty^2 \mathfrak{M}^*(\frac{1}{2} + it) dt.$$

Clearly  $S^0$  is dense in  $S^2$ . On the other hand, by Theorem 5.1,  $F(\frac{1}{2} + it)$  varies over all even Fourier polynomials in  $t$  with period  $2\pi/\log q$  as  $f$  varies over  $S^0$ . Hence (7.1) shows that the mapping  $f(g) \rightarrow F(\frac{1}{2} + it)$  can be extended uniquely to a unitary mapping of  $S^2$  onto the Hilbert space of all equivalence classes of complex-valued functions which are square integrable over the interval  $0 \leq t \leq \pi/\log q$  relative to the measure  $\mathfrak{M}^*(\frac{1}{2} + it) dt$ .

8. **The elementary spherical function  $\varphi(g, s)$ .** Combining (4.7) with (5.6), we obtain

$$\begin{aligned} f(y^n) &= \frac{\log q}{\pi q^n \mathfrak{M}(K) \mu(\mathfrak{D}) \mu^\times(\mathfrak{U})} \int_0^{\pi/\log q} F(\frac{1}{2} + it) \{q^{n/2} \cos(nt \log q) \\ &\quad + (1-q) \sum_{j=1}^{\infty} q^{n/2-j} \cos[(n+2j)t \log q]\} dt \\ &= \frac{q^{-n/2} \log q}{\pi \mathfrak{M}(K)} \int_0^{\pi/\log q} F(\frac{1}{2} + it) \{q^{n/2} \cos(nt \log q) \\ &\quad + (1-q) \sum_{j=1}^{\infty} q^{-j} [(n+2j)t \log q]\} dt, \end{aligned}$$

\*  $|\cdot|_\infty$  denotes the ordinary absolute value of a complex number.

using (2.11), i. e.,  $\mu(\mathfrak{D})\mu^\times(1) = 1$ . Now

$$\begin{aligned} \cos(nt \log q) + (1-q) \sum_{j=1}^{\infty} q^{-j} \cos[(n+2j)t \log q] \\ = \frac{1}{2} \left\{ q^{int} + q^{-int} + (1-q) \left[ q^{int} \cdot \frac{q^{2it}}{q - q^{2it}} + q^{-int} \cdot \frac{q^{-2it}}{q - q^{-2it}} \right] \right\} \\ = \frac{1}{2} \left\{ q^{int} \cdot \frac{q - q^{2it+1}}{q - q^{2it}} + q^{-int} \cdot \frac{q - q^{1-2it}}{q - q^{-2it}} \right\}. \end{aligned}$$

We divide this expression by  $\mathfrak{M}^*(\frac{1}{2} + it)$  and obtain

$$-\frac{\pi \mathfrak{M}(K)}{q(q+1) \log q} \left\{ q^{int} \cdot \frac{q^{1-it} - q^{2+it}}{q^i - q^{-it}} + q^{-int} \cdot \frac{q^{1+it} - q^{2-it}}{q^{-it} - q^{it}} \right\}.$$

Hence

$$(8.1) \quad f(y^n) = \int_0^{\pi/\log q} F(\tfrac{1}{2} + it) \phi(y^n, \tfrac{1}{2} + it) \mathfrak{M}^*(\tfrac{1}{2} + it) dt,$$

where, for  $n \geq 0$ ,

$$(8.2) \quad \phi(y^n, s) = \frac{q^{n(s-\frac{1}{2})} [q^{\frac{3}{2}+s} - q^{\frac{3}{2}-s}] - q^{-n(s-\frac{1}{2})} [q^{\frac{5}{2}-s} - q^{\frac{3}{2}+s}]}{(q+1)q^{n/2+1}(q^{s-\frac{1}{2}} - q^{\frac{1}{2}-s})}.$$

One sees at once that  $\phi(y^n, s)$  is, for fixed  $n$ , an entire function of  $s$  (including at  $s = \frac{1}{2} + \pi i n / \log q$ ). Hence, we obtain, for each fixed complex number  $s$ , a spherical function  $\phi(g, s)$  on  $G$  by putting  $\phi(k_1 y^n k_2, s) = \phi(y^n, s)$  for any  $k_j \in K$  and using (2.3), (2.4) and (2.5). By putting  $n=0$  in (8.2), one sees that

$$(8.3) \quad \phi(1, s) = 1 \text{ for all } s.$$

By putting  $s=0$  or  $1$  in (8.2), one sees that

$$(8.4) \quad \phi(g, 0) = \phi(g, 1) = 1 \text{ for all } g \in G.$$

Moreover, (8.2) implies immediately the functional equation

$$(8.5) \quad \phi(g, s) = \phi(g, 1-s) \text{ for all } g \in G.$$

Also,

$$(8.6) \quad \bar{\phi}(g, \tfrac{1}{2} + it) = \phi(g, \tfrac{1}{2} + it) \text{ for real } t.$$

Hence, if we combine (7.1) and (8.1), we may conclude that

$$(8.7) \quad F(\tfrac{1}{2} + it) = \int_G f(g) \phi(g, \tfrac{1}{2} + it) dg$$

for all  $f(g) \in S^0$ , or for all  $f(g) \in S^2$  if one introduces the usual l.i.m.  $\int$ .

So we see that if  $f(g) \in S^0$ , then

$$(8.8) \quad F(s) = \int_G f(g) \phi(g, s) dg.$$

If we combine this with (3.14), we see that  $\phi(g, s) = \phi(g, \alpha)$  provided  $\alpha$  satisfies (5.1) and (5.2). Now from (8.2), we obtain at once that

$$(8.9) \quad |\phi(g, s)| \leq \phi(g, \sigma), \text{ where } \sigma = Rs.$$

Hence, combining (8.4) with (8.9), we obtain

$$(8.10) \quad |\phi(g, it)| \leq 1 \quad \text{and} \quad |\phi(g, 1 + it)| \leq 1.$$

It follows from this that, for all  $g \in G$ ,

$$(8.11) \quad |\phi(g, s)| \leq 1 \quad \text{for } 0 \leq Rs \leq 1.$$

We now prove that  $\phi(g, s)$  is an unbounded function of  $g$  for any fixed  $s$  for which  $Rs > 1$  or  $Rs < 0$ . Indeed,

$$\frac{q^{n(s-\frac{1}{2})}[q^{\frac{3}{2}+s} - q^{\frac{3}{2}-s}]}{(q+1)q^{n/2+1}(q^{s-\frac{1}{2}} - q^{\frac{1}{2}-s})} + \frac{q^{n(s-1)}[q^s - q^{-s}]}{(q+1)(q^{s-\frac{1}{2}} - q^{\frac{1}{2}-s})} \rightarrow +\infty$$

as  $n \rightarrow +\infty$  if  $Rs = \sigma > 1$ , whereas

$$\frac{q^{-n(s-\frac{1}{2})}[q^{\frac{3}{2}-s} - q^{\frac{3}{2}+s}]}{(q+1)q^{n/2+1}(q^{s-\frac{1}{2}} - q^{\frac{1}{2}-s})} + \frac{q^{-ns}[q^{\frac{5}{2}-s} - q^{\frac{3}{2}+s}]}{q(q+1)(q^{s-\frac{1}{2}} - q^{\frac{1}{2}-s})} \rightarrow 0$$

as  $n \rightarrow +\infty$  if  $Rs = \sigma > 0$ . Therefore  $\phi(y^n, s)$  is unbounded as  $n \rightarrow +\infty$  for any fixed  $s$  with  $Rs > 1$ , hence, because of (8.5), also for  $Rs < 0$ .

On the other hand, if  $0 < Rs < 1$ , then  $q^{zn(\sigma-\frac{1}{2})-in\tau} q^{-n/2} \rightarrow 0$  as  $n \rightarrow +\infty$ . Therefore

$$(8.12) \quad \lim_{n \rightarrow +\infty} \phi(y^n, s) = 0 \quad \text{if } 0 < Rs < 1$$

uniformly in  $s$  for  $\epsilon \leq Rs \leq 1 - \epsilon$ ,  $\epsilon > 0$ . This proves

LEMMA 8.1.  $\phi(g, s)$  is an unbounded function of  $g$  for any fixed complex number  $s$  for which  $Rs > 1$  or  $Rs < 0$ .

On the other hand,  $|\phi(g, s)| \leq 1$  for all  $g \in G$  if  $0 \leq Rs \leq 1$ , and (8.12) holds.

From this we deduce

THEOREM 8.1. Let  $f(g) \in S^1$ , then  $F(s) = \int f(g) \phi(g, s) dg$  is a regular analytic function in the strip  $0 < Rs < 1$ , continuous in  $0 \leq Rs \leq 1$ , satisfying

$$(8.13) \quad F(s) = F(1-s) = F(s + \frac{2\pi in}{\log q}).$$

Any homomorphism of the commutative normed algebra  $S^1$  to the complex numbers is of the form  $f(g) \rightarrow \int f(g)\phi(g, c)dg = F(c)$ , where  $c$  is a complex number in the strip  $0 \leq Rc \leq 1$ . This homomorphism maps  $f^*(g) = \bar{f}(g^{-1})$  to  $\bar{F}(c)$  if and only if  $Rc = \frac{1}{2}$  or  $c + 2\pi i n / \log q$  is real. Two homomorphisms  $f \rightarrow F(c)$  and  $f \rightarrow F(c')$  are identical if and only if  $c' = c + 2\pi i n / \log q$  or  $c' = 1 - c + 2\pi i n / \log q$ .

*Proof.* The regularity and continuity of  $F(s)$  follow immediately from the above properties of  $\phi(g, s)$ , and so does (8.13). Now given any homomorphism of  $S^1$  to the complex numbers, its restriction to  $S^0$  is, of course, a homomorphism of  $S^0$  to the complex numbers, hence, by Theorem 5.1 above, is of the form  $f \rightarrow F(c)$  for some complex number  $c$ . In order that this homomorphism can be extended to a homomorphism of  $S^1$ , it is necessary and sufficient that  $\phi(g, c)$  be a bounded function of  $g$  for this number  $c$ . Hence, by Lemma 8.1, we must have  $0 \leq Rc \leq 1$ . The assertion about the identity of two such homomorphisms follows from the corresponding assertion about  $S^0$  (see again Theorem 5.1) above). The assertion about  $f^*(g)$  follows immediately from the last assertion in § 5 above (see equation (5.7) above) and Theorem 5.1.

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## CHARACTERISTIC CLASSES AND HOMOGENEOUS SPACES, I.\*

By A. BOREL and F. HIRZEBRUCH.

**Introduction.** It is known that the characteristic classes of a real or complex vector bundle may be interpreted as elementary symmetric functions in certain variables, which are 1, 2 or 4 dimensional cohomology classes. If we consider the tangent bundle of the coset space  $G/U$  of a compact connected Lie group modulo a closed subgroup, it turns out that these variables may be identified with certain roots of  $G$  (or their squares). Our first purpose is to establish this connection between roots and characteristic classes, which is the basis of this paper, and to compute the characteristic classes of certain well-known homogeneous spaces. These results are then applied in particular to  $G/T$  ( $T$  maximal torus of  $G$ ), and to other algebraic homogeneous spaces, where they lead to relations between characteristic classes, Betti numbers, the Riemann-Roch theorem and representation theory; they are also used to discuss multiplicative properties of the Todd genus and other genera in fibre bundles with  $G/U$  as fibre. As an application, we get a divisibility property of the Chern class of a complex vector bundle over an even dimensional sphere which yields some information about certain homotopy groups of Lie groups.

We now give a summary of the different chapters. For the notions and notations used without further comments, the reader is referred to [2, 19].

Chapter I. The first three Sections give a survey of standard properties of roots and linear representations; § 4 gives two characterizations of systems of positive roots which will be used in Chapter IV. In § 5 we introduce the roots of a Lie group with respect to a commutative subgroup of type  $(2, 2, \dots, 2)$ , which will occur in the description of Stiefel-Whitney classes.

Chapter II recalls those concepts of fibre bundle theory which are most often used in this paper, such as restriction and extension of the structural group (with respect to homomorphisms), integration over the fibre, to be denoted by  $\int$ , and the bundle of vectors tangent to the fibres of a bundle whose typical fibre admits a differentiable structure invariant under the structural group, to be called hereafter the "bundle along the fibres".

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Chapter III starts with a review of the definition by means of symmetric functions of the characteristic classes of a bundle having a classical group as structural group (§ 9). In § 10, we consider the  $\lambda$ -extension  $\eta^1$  of a principal  $G$ -bundle  $\eta$  by means of a unitary representation  $\lambda: G \rightarrow U(n)$ ; the bundle  $\eta^1$  is a principal  $U(n)$ -bundle, whose Chern classes are shown to be the elementary symmetric functions in the weights of  $\lambda$ , suitably interpreted as 2-dimensional classes; analogous statements are proved for the real orthogonal representations and Pontrjagin classes (10.3). Now, the real tangent bundle to  $G/U$  is the  $\iota$ -extension of the principal  $U$ -bundle  $(G, G/U, U)$ , with respect to the linear isotropy representation  $\iota$  of  $U$  in the tangent space at a point of  $G/U$  fixed under  $U$  (Proposition 7.5). Applying 10.3 to this situation yields the relation between roots and characteristic classes mentioned at the beginning of this introduction, which, in fact, holds more generally for the bundle along the fibres of  $(E/U, B, G/U)$ , where  $(E, B, G)$  is a principal  $G$ -bundle.

Chapter IV. The homogeneous space  $G/U$  admits an invariant almost complex structure  $J$  if and only if the isotropy representation  $\iota$  can be factorized through the standard inclusion of  $U(n)$  in  $SO(2n)$ , ( $2n = \dim G/U$ ); we obtain in this case a unitary representation  $\iota_c: U \rightarrow U(n)$ , whose weights are certain roots of  $G$ , to be called the roots of  $J$ ; they allow us to compute the Chern classes of  $J$  using 10.3 and to discuss the integrability of  $J$  using the results of § 4; among the applications in § 13 we give new proofs of some results of H. C. Wang.

The invariant complex structures of  $G/U$  (where  $U$  is the centralizer of a torus in  $G$ ), can be obtained directly by using the complexification of  $G$ ; the space  $G/U$  is also homogeneous kählerian [5] and even rational algebraic, and there is a close connection between its projective embeddings and the linear representations of  $G$ . For later use, we include in § 14 a short discussion of some of these results; moreover, we prove (14.10) that the real cohomology classes of these algebraic homogeneous spaces are all of type  $(p, p)$  and for  $p = 1$  describe those which are positive in the sense of Kodaira.

Chapter V is devoted to some special cases; in particular, to projective spaces.

Chapter VI.† In § 20 a formula for the homomorphism  $\mathfrak{h}$  in the bundle  $\xi = (B_T, B_G, G/T, \rho(T, G))$  is established (20.3); it shows, in particular (22.2), that  $\mathfrak{h}$ , applied to the total Todd class of the bundle along the fibres of  $\xi$  endowed with the complex vector bundle structure defined by means of an

† §§ 20 to 30 will be published in a later issue of this Journal.

invariant almost complex structure  $J$  on  $G/T$ , gives a zero-dimensional term, which is 1 or 0, according to whether  $J$  is integrable or not; it follows that the Todd genus  $T(G/T)^1$  of  $G/T$  with respect to  $J$  is 1 or zero respectively and that (22.5) in certain bundles  $(E, B, G/T)$ , with almost complex  $E$  and  $B$  the Todd genus, "behaves multiplicatively," i.e., that we have  $T(E) = T(B) \cdot T(F)$ . These results are generalized to the  $T_y$ -genus and to homogeneous almost complex spaces  $G/U$  ( $\text{rank } U = \text{rank } G$ ).

In § 23 we consider the  $A$ -genus of homogeneous spaces  $G/U$  with  $\text{rank } U = \text{rank } G$  and, in particular, prove it to be equal to 0 when the second Betti number of  $G/U$  vanishes; moreover, in a differentiable bundle  $(E/U, B, G/U)$  with  $A(G/U) = 0$ , we also have  $A(E) = 0$ .

In § 24,  $G/U$  ( $\text{rank } U = \text{rank } G$ ) is assumed to be algebraic. The value of the  $T_y$ -genus found in § 22 and (14.10) yield a formula for the Betti numbers of  $G/U$  in terms of the action of the Weyl groups of  $G$  and  $U$  on the roots of  $G$ . The dimension of the vector space of holomorphic cross-sections of a complex line bundle  $F$  on  $G/U$  is computed by means of the Riemann-Roch theorem and is shown to be either zero or equal to the degree of the irreducible representation of  $G$  having the first Chern class of  $F$  as highest weight (24.7); (this fact has led to the results of [7a] and has been further generalized by R. Bott [7b]). The degree (in the sense of algebraic geometry) of the projective embedding of  $G/U$  given by this linear representation, or equivalently, by the complete linear system of divisors belonging to  $F$ , is also explicitly calculated.

Chapter VII. In § 25, the  $A$ -genus is proved to be an integer. More generally, we introduce, for a complex vector bundle  $\eta$  over a differentiable manifold  $X$  and for an arbitrary element  $d \in H^2(X, \mathbf{Z})$ , a rational number  $\hat{A}(X, d, \eta)$ , in analogy with the Riemann-Roch formula, and prove that it is an integer after multiplication by a suitable power of 2. It follows that the  $q$ -th Chern class of a complex vector bundle over the  $2q$ -dimensional sphere  $S_{2q}$  is divisible by the greatest odd factor of  $(q-1)!$ ; applications of this last fact to the homotopy of Lie groups are given in § 26.

As is well known, the index of a differentiable manifold  $X$  equals the  $L$ -genus of  $X$ , which is a linear combination of Pontrjagin numbers [19]. It was recently proved [12] that the index "behaves multiplicatively" in differentiable bundles. This fact has certain consequences for the Pontrjagin classes of the bundle along the fibres of a differentiable bundle, from which

<sup>1</sup> We allow ourselves to denote by  $T$  the Todd genus as well as a maximal torus, since this is unlikely to bring any confusion.

we conclude that the sequence of  $L$ -polynomials is essentially uniquely characterized by the property of giving rise to a genus which behaves multiplicatively in differentiable fibre bundles.

In Appendix I, we compare the different definitions of Chern classes known to us, with particular emphasis on the signs.

In Appendix II, it is first proved that the torsion coefficients of  $H^*(B_{O(n)}, \mathbf{Z})$  and  $H^*(B_{so(n)}, \mathbf{Z})$  are all of order 2. This allows us to characterize the universal integral Pontrjagin class  $p_i$  by its canonical images in  $H^{4i}(B_{O(n)}, \mathbf{R})$  and  $H^{4i}(B_{O(n)}, \mathbf{Z}_2)$ , the latter being equal to the square of the universal  $2i$ -th Stiefel-Whitney class. It is also shown that, up to 2-torsion, the integral  $i$ -th Pontrjagin class of a principal  $O(n)$ -bundle  $\xi$  can be defined by means of the transgression in a certain bundle  $\eta_i$  associated with  $\xi$ ; in particular,  $\eta_i$  has the typical fibre  $O(n)/O(2i-1)$  when  $n$  is odd.

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## Chapter I. Compact Lie Groups.

## 1. Generalities.

1.1. *Coset spaces.* Let  $G$  be a Lie group,  $U$  a closed subgroup of  $G$ ,  $G/U$  the space of left cosets of  $G \bmod U$ ,  $\pi$  the natural projection of  $G$  onto  $G/U$ , and  $\mathfrak{g}, \mathfrak{u}$  the Lie algebras of  $G, U$  identified as usual with the tangent spaces at the neutral element. Left translation by  $g \in G$  induces a homeomorphism of  $G/U$  which will be denoted by the same letter; if  $u \in U$ , it leaves  $o = \pi(e)$  invariant and induces an automorphism  $\tilde{u}$  of the tangent space  $(G/U)_o$  of  $G/U$  at  $o$ . The homomorphism  $\iota_u: u \rightarrow \tilde{u}$  is called the *isotropy representation* and its image the *linear isotropy group*  $\tilde{U}$ . For

connected  $G$ , its kernel is the subgroup of those elements of  $G$  which act trivially on  $G/U$  or, also, the largest subgroup of  $U$  invariant in  $G$ .

$\text{Ad } g$  or  $\text{Ad}_{\mathfrak{g}} g$  will denote the automorphism of  $\mathfrak{g}$  induced by the inner automorphism  $x \rightarrow gxg^{-1}$  of  $G$ . If  $\pi_e$  is the differential of  $\pi$  at  $e$ , we have clearly

$$\pi_e \circ \text{Ad}_{\mathfrak{g}} u = \bar{u} \circ \pi_e \quad (u \in U);$$

in particular, since the kernel of  $\pi_e$  is  $\mathfrak{u}$ ,  $\pi_e$  allows us to identify  $(G/U)_e$  with any subspace  $\mathfrak{m}$  of  $\mathfrak{g}$  supplementary to  $\mathfrak{u}$ , invariant under  $\text{Ad}_{\mathfrak{g}} u$ , in such a way that  $\bar{u}$  is carried over to the restriction of  $\text{Ad}_{\mathfrak{g}} u$  to  $\mathfrak{m}$ . If  $\sigma$  is an automorphism of  $G$  and  $\sigma_e$  its differential at  $e$ , then

$$(1) \quad \text{Ad } \sigma(g) = \sigma_e \circ \text{Ad } g \circ \sigma_e^{-1}.$$

1.2. From now on,  $G$  is a compact Lie group. We recall that its maximal toral subgroups are conjugate to each other by inner automorphisms and are maximal abelian subgroups of  $G$  if  $G$  is connected; their common dimension is the *rank* of  $G$ , to be denoted here by  $l$  or  $l(G)$ . The letter  $T$  will be reserved for a maximal torus of  $G$ , and  $S$  for an arbitrary toral subgroup;  $V_S$  will be the universal covering of  $S$  and  $\Gamma_S$  the "unit lattice", i.e., the inverse image of the identity element of  $S$ . The latter is a free commutative group of rank  $k$  ( $k = \dim S$ ), which spans  $V_S$ . A real valued linear form on  $V_S$  is said to be *integral* if it takes integral values on  $\Gamma_S$ .

1.3. *Roots, diagram.* The representation  $s \rightarrow \text{Ad}_{\mathfrak{g}} s$  of  $S$  in  $\mathfrak{g}$  is fully reducible and there is a direct sum decomposition

$$(2) \quad \mathfrak{g} = \mathfrak{a}_1 + \cdots + \mathfrak{a}_k + \mathfrak{b} + \mathfrak{s} \quad (\dim \mathfrak{a}_i = 2)$$

of  $\mathfrak{g}$  into subspaces invariant under  $\text{Ad}_{\mathfrak{g}} S$ , where  $\mathfrak{b} + \mathfrak{s}$  is the largest subspace on which  $S$  operates trivially. We may then write, for  $s \in S$ ,

$$(3) \quad \text{Ad}_{\mathfrak{g}} s|_{\mathfrak{a}_i} = \begin{pmatrix} \cos 2\pi a_i(s) & -\sin 2\pi a_i(s) \\ \sin 2\pi a_i(s) & \cos 2\pi a_i(s) \end{pmatrix}$$

where  $a_i(p(x))$ ,  $p$  the projection of  $V_S$  onto  $S$ , is a non-zero integral linear form.<sup>2</sup> The linear forms  $\pm a_i$  are the roots of  $G$  with respect to  $S$ . We shall be concerned mainly with the case where  $S = T$  is a maximal torus. Then  $\mathfrak{b} = 0$ , and the  $2m$  linear forms  $\pm a_i$ , ( $i = 1, \dots, m$ ;  $\dim G = l + 2m$ ), are simply the roots of  $G$ .<sup>3</sup> Clearly, if  $S \subset T$ , the roots relative to  $S$  are the restriction to  $V_S$  of the roots of  $G$  which do not vanish identically on  $S$ .

<sup>2</sup> In the sequel, there will be no notational distinction between  $a(p(x))$  and  $a(t)$ .

<sup>3</sup> We call them roots in spite of the facts that the roots in the sense of the

An element  $t \in T$  is *singular* if its centralizer has dimension  $> l(G)$ , *regular* otherwise; in  $V_T$  the singular elements are represented by the points of the hyperplanes  $a_i \equiv 0 \pmod{1}$ ,  $(1 \leq i \leq m)$ , which form the *diagram* of  $G$ .

In case  $S$  is a toral subgroup of  $U$ , the decomposition (2) may be chosen in such a way that  $u$  is spanned by a subspace  $\mathfrak{h}_1$  of  $\mathfrak{h}$ ,  $\mathfrak{s}$  and some of the  $\mathfrak{a}_i$ , say  $\mathfrak{a}_1, \dots, \mathfrak{a}_q$ ; the  $\pm a_i$  ( $q < i \leq m$ ) will be called *the roots of  $G$  relative to  $S$  complementary to those of  $U$* , or simply the *complementary roots* if there is no danger of confusion.

**1.4. The Weyl Group.** We choose once and for all a positive definite metric on  $\mathfrak{g}$  invariant under  $\text{Ad } G$ , and consider on  $V_T$  the metric which is induced by it in the obvious way; it allows us to define in the standard way a canonical isomorphism between  $V_T$  and its dual space  $V_T^*$  and a metric on  $V_T^*$ ; the scalar product on  $V_T$  or  $V_T^*$  will be denoted  $(\ , \ )$ . An element  $a \in V_T^*$  is called *singular* if there exists a root  $a_i$  such that  $(a, a_i) = 0$ ; its image in  $V_T$  under the canonical isomorphism is then singular in the above sense. Finally, we remark that the symmetry  $S_a$  of  $V_T$  with respect to the hyperplane  $a = 0$  induces a symmetry of  $V_T^*$ , to be denoted also by  $S_a$ , defined by

$$S_a(b) = b - 2(a, b)(a, a)^{-1} \cdot a.$$

The Weyl group  $W(G)$  of  $G$  is the group of automorphisms of  $T$  induced by inner automorphisms of  $G$  leaving  $T$  invariant; it is a finite group and a quotient of  $N_T/T$ , where  $N_T$  is the normalizer of  $T$  in  $G$ ; it may also be viewed as a group of isometries of  $V_T$  leaving  $\Gamma_T$  and the diagram invariant. For connected  $G$ , it is isomorphic to  $N_T/T$  and is generated by the symmetries to the hyperplanes  $a_i = 0$  ( $i = 1, \dots, m$ ).

**2. Standard properties of roots.**  $G$  is a compact *connected* Lie group of dimension  $n$  and rank  $l$ ,  $T$  a maximal torus,  $V$  its universal covering and  $\pm a_i$  ( $1 \leq i \leq m, n = l + 2m$ ) are the roots of  $G$ . The proofs of the statements in §§ 2, 3 may be found e.g. in [8, 13, 23, 27, 28].

**2.1.** An element  $v \in V$  is in the inverse image of the center of  $G$  if and only if  $a_i(v) \equiv 0 \pmod{1}$  ( $i = 1, \dots, m$ ); in particular,  $G$  is semi-simple if and only if it has  $l$  linearly independent roots.

**2.2.**  $a_i$  and  $a_j$  are linearly independent if  $i \neq j$ .

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infinitesimal theory (i.e., the roots of the Killing equation), are the forms  $\pm 2\pi i a_j$  and the zero form; the forms  $a_j$  were called "paramètres angulaires" by E. Cartan [8]. Also, unless otherwise specified, the zero form will not be considered as a root.

2.3. The number  $2(a, b) \cdot (a, a)^{-1}$  is an integer for any two roots  $a, b$  and the linear forms  $b - k \cdot a$  are also roots for  $k$  integral varying between 0 and  $2(a, b)(a, a)^{-1}$ .

2.4. *Simple roots.* Let  $(x^i)$ ,  $(1 \leq i \leq l)$ , be a base of  $V^*$ . We define a total ordering  $\mathcal{S}$  on  $V^*$  by saying that  $a = a_1x^1 + \cdots + a_lx^l$  is  $> 0$  if its first non vanishing coefficient is  $> 0$  and that  $a > b$  if  $a - b > 0$ . A root is *simple*, relative to  $\mathcal{S}$ , if it is positive and cannot be represented as the sum of two positive non-zero roots. The simple roots are linearly independent, the scalar product of any two of them is  $\leq 0$ , and every root is a linear combination, with integral coefficients of the same sign, of simple roots. It follows, in particular, that, when  $G$  is semi-simple, there are  $l$  simple roots; also, a simple root is not a linear combination with positive coefficients of other positive roots.

2.5. A set of elements of  $V$  or  $V^*$  is said to be decomposable if it is the union of two non-empty mutually orthogonal subsets. A semi-simple group  $G$  is simple if and only if its root system or the system of its simple roots is indecomposable. Assume  $G$  to be simple, let  $a_i$  ( $1 \leq i \leq l$ ) be the simple roots and let  $b = b_1a_1 + \cdots + b_la_l$  be the highest root with respect to an ordering  $\mathcal{S}$ . Then we have

$$(b, a_i) \geq 0, \quad b_i > 0, \quad (i = 1, \cdots, l),$$

and  $b_i$  majorizes the coefficient of  $a_i$  for all roots.

2.6. *The sign of an element of  $W(G)$ .* Let  $a_i$  ( $1 \leq i \leq m$ ), be the positive roots with respect to  $\mathcal{S}$ . Since  $W(G)$  leaves the diagram invariant, any  $w \in W(G)$  induces a permutation of the system  $(\pm a_i)$  ( $1 \leq i \leq m$ ), and transforms  $(a_i)$  into a system of roots  $(\epsilon_i a_i)$  with  $\epsilon_i = \pm 1$ . We shall denote by  $s(w)$  the number of  $\epsilon_i$ 's equal to  $-1$  and by  $\text{sgn } w$  the product of the  $\epsilon_i$ 's. We contend that  $\text{sgn } w$  is equal to the determinant of  $w$  viewed as a linear transformation of  $V$ , and, in particular, does not depend on  $\mathcal{S}$ . In fact, let  $e_i, e_{-i}$  be the orthonormal base of  $\alpha_i$  with respect to which we have (3) of § 1, and let  $g \in N_T$  belong to the coset of  $w$ . It follows from 2.2 that  $w$  permutes the  $\alpha_i$ , and then from (1) § 1 that if  $w(a_i) = \epsilon_i a_j$ , then  $\text{Ad } g(e_i \wedge e_{-i}) = \epsilon_i e_j \wedge e_{-j}$ ; moreover, the natural isomorphism of  $\mathfrak{t}$  on  $V$  carries the restriction of  $\text{Ad } g$  to  $\mathfrak{t}$  over to  $w$ ; our contention follows readily from this and from the fact that,  $G$  being connected, we have  $\det \text{Ad } g = 1$ .

2.7. *The Weyl chambers.* Let  $a_1, \cdots, a_r$  be the simple roots belonging to the order  $\mathcal{S}$ . A Weyl chamber in  $V$  or  $V^*$  is a connected component of

the set of regular elements. In particular, the set of points  $v \in V$  (resp.  $x \in V^*$ ), such that  $a_i(v) > 0$ , (resp.  $(x, a_i) > 0$ ) ( $1 \leq i \leq r$ ), is one and will be called the positive Weyl chamber with respect to  $\mathcal{J}$ . The Weyl group acts simply transitively on the Weyl chambers, and, in particular, the integer  $s(w)$  defined in 2.6 is zero if and only if  $w$  is the identity; it is generated by the symmetries to the hyperplanes  $a_i = 0$  ( $1 \leq i \leq r$ ).

2.8. *Remark on orderings.* Let  $a \in V^*$  be such that  $(a, a_i) \neq 0$  for all  $i$  ( $1 \leq i \leq m$ ), and let us say that a root is positive if  $(a, a_i) > 0$ . Then the order relation thus obtained between roots is induced by an ordering on  $V^*$  as considered in (2.4) and has, therefore, the properties (2.4), (2.5). To define  $\mathcal{J}$ , one takes a base  $(x^i)$  of  $V^*$  dual to a base  $(e_i)$  ( $1 \leq i \leq l$ ), of  $V$  where  $e_i$  ( $i \geq 2$ ) is contained in the hyperplane  $a = 0$ .

2.9. *Classification.* We recall that a compact connected Lie group  $G$  has a finite covering  $\tilde{G}$  which is the direct product of a torus  $S$  by a semi-simple and simply connected group  $\tilde{G}'$ ; the group  $\tilde{G}$  is uniquely determined, up to an isomorphism, if we require, moreover, that the kernel of the projection of  $\tilde{G}$  onto  $G$  intersects  $S$  only at the identity. The image of  $\tilde{G}'$  in  $G$  is the derived group  $G'$  of  $G$  and is also its largest semi-simple subgroup; the image of  $S$  is, of course, the connected identity component of the center of  $G$ .

The semi-simple groups are locally isomorphic to products of simple non-commutative groups. For the classification of the Lie algebras of compact simple Lie groups, we refer to [13, 23]. For a list of their roots, see for instance [25a]. For the simple Lie groups and the classical linear groups we follow here the standard notations.

3. **Linear representations.**  $a_i$  ( $1 \leq i \leq m$ ) are the positive roots of the compact connected Lie group  $G$  of rank  $l$ , with respect to an ordering  $\mathcal{J}$ .  $a$  is the sum of the  $a_i$ 's and  $a_1, \dots, a_r$  are the simple roots,  $r$  being the rank of the semi-simple part  $G'$  of  $G$ .

3.1. **LEMMA.** *We have  $(a, a_j) = (a_j, a_j)$ , ( $1 \leq j \leq r$ ).*

Let  $S_j$  be the symmetry with respect to  $a_j = 0$ . We have

$$S_j a_i = a_i - 2(a_i, a_j) \cdot (a_j, a_j)^{-1} \cdot a_j.$$

Hence  $S_j a_i$  and  $a_i$  ( $j \leq r$ ), when expressed as linear combinations of simple roots, differ at most by the coefficient of  $a_j$ ; thus, if  $a_i \neq a_j$ ,  $S_j a_i$  has at least

one positive coefficient, and is a positive root by (2.4); this means that  $S_j$  permutes among themselves the positive roots different from  $a_j$ . But we have

$$(a_j, S_j a_i + a_i) = 0,$$

whence the lemma.

3.2. To abbreviate, we write  $E(b)$ , ( $b \in V^*$  or  $b \in V^* \otimes \mathbf{C}$ ), for

$$\sum_{w \in W(G)} (\text{sgn } w) \exp(2\pi\sqrt{-1} \cdot w(b))$$

and  $E(b, x)$  for the value of this function on  $x \in V$ . If  $b^*$  and  $x_*$  are the images of  $b$  and  $x$  under the canonical isomorphism between  $V$  and  $V^*$  defined by a metric invariant under  $W(G)$ , we have clearly  $E(x_*, b^*) = E(b, x)$ .

By a standard result of representation theory, we have

$$(1) \quad E(a/2) = \prod_{i=1}^{i=m} 2\sqrt{-1} \sin \pi a_i.$$

The computation of the  $m$ -th orders terms on each side yields the equality

$$(2) \quad m! \prod a_i = \sum_{w \in W(G)} (\text{sgn } w) \cdot w(a/2)^m.$$

Let us denote by  $E_x^{(m)}(b, 0)$  the value at 0 of the  $m$ -th derivative of  $E(b, y)$  in the direction  $x$ ; then we have

$$(3) \quad E_x^{(m)}(a/2, 0) = m! (2\pi\sqrt{-1})^m \prod_1^m \langle a_i, x \rangle.$$

It follows directly from the definition that  $E(b)$  vanishes identically when  $b$  is singular. Conversely, the equality  $E(x_*, b^*) = E(b, x)$ , recalled above, and (3) imply

$$(4) \quad E_{a^*/2}^{(m)}(b, 0) = m! (2\pi\sqrt{-1})^m \prod_1^m \langle a_i, b \rangle = m! (2\pi\sqrt{-1})^m \prod_1^m (a_i, b);$$

hence, if  $E(b) = 0$ , the element  $b$  must be singular.

3.3. *The weights.* An element  $b \in V^*$  is called a weight of  $G$  if it is integral on the unit lattice of the connected identity component of the center of  $G$  and is such that  $2(b, a_j) \cdot (a_j, a_j)^{-1}$  is an integer ( $j \leq m$ ). The weights form a free commutative group of rank  $l$ . The previous condition may also be expressed by saying that a weight is a linear form which is integral on the unit lattice  $\Gamma_0$  corresponding to the covering  $\bar{G}$  of  $G$  which has the form  $S \times \bar{G}'$  where  $S$  is a torus,  $\bar{G}'$  is semi-simple and simply connected, and such that the kernel of the projection  $\bar{G} \rightarrow G$  intersects  $S$  only at the identity (2.9).

For  $b, c \in V^*$ , let us put  $q(b, c) = 2 \cdot (b, c) (c, c)^{-1}$ . Let  $S_d$  be the sym-

metry to the hyperplane  $d=0$ . We have  $(b, S_d c) = (S_d b, c)$ ,  $(S_d c, S_d c) = (c, c)$  and  $S_d b = b - q(b, d)d$ , and therefore

$$(5) \quad q(b, S_d c) = q(b, c) - q(b, d)q(d, c).$$

PROPOSITION. *Let  $b$  be an element of  $V^*$ . Then  $q(b, a_j)$  is an integer for  $1 \leq j \leq m$  if and only if it is so for  $1 \leq j \leq r$ . In particular,  $a/2$  is a weight.*

The Weyl group  $W(G)$  is generated by the symmetries  $S_j$  to the hyperplanes  $a_j=0$  ( $1 \leq j \leq r$ ), and every root is the image of a simple root under some transformation of  $W(G)$  (this follows from 2.7). Since  $q(d, c)$  is an integer when  $c$  and  $d$  are roots (2.3), the equality (5) shows that  $q(b, S_j a_k)$  is integral if  $q(b, a_k)$  and  $q(b, a_j)$  are, and our first assertion follows by an obvious induction. The second one is then a consequence of (3.1) and of the fact that  $a$  is zero on the identity component of the center of  $G$ .

3.4. *Characters.* It follows from the results of H. Weyl and from standard facts about direct products, that the characters of the irreducible representations of the group  $\bar{G}$  introduced in 3.3, restricted to the maximal torus  $\bar{T}$ , are the functions

$$(6) \quad \chi(t) = E(b) \cdot E(a/2)^{-1},$$

where  $b$  runs through the weights contained in the positive Weyl chamber defined by  $\mathcal{O}$ . In other words, the  $b$ 's are the weights which verify

$$2(a_j, b) = k_j(a_j, a_j), \quad (k_j > 0, \text{ integral}, j=1, \dots, r).$$

By dividing out in (6), one may write  $\chi(t)$  as a finite sum of exponentials  $\exp 2\pi\sqrt{-1}c_s$ , where the  $c_s$ 's are weights. The highest one is  $b - (a/2)$ ; it has multiplicity one and characterizes the linear representations up to an equivalence. In view of the foregoing, the highest weights are those which satisfy

$$(7) \quad 2(a_j, c) = k_j(a_j, a_j), \quad (k_j \geq 0, \text{ integral}, j=1, \dots, r).$$

Assume now  $G$  to be semi-simple, hence  $r=l=\text{rank } G$ . Let  $\varpi_i$  be the linear form defined by  $q(\varpi_i, a_j) = \delta_{ij}$  ( $i=1, \dots, l$ ). By (3.3), the  $\varpi_i$ 's are weights, to be called the fundamental weights, and form a basis of the group of weights. By (7), the highest weights are the linear combinations of the  $\varpi_i$ 's with integral non negative coefficients.

Let again  $\bar{G}$  be compact, not necessarily semi-simple. The degree  $d$  of

the representation with highest weight  $b - (a/2)$  is  $\chi(0)$ . In the right hand side of (6), this appears in the form  $0/0$ , but by taking suitable  $m$ -derivatives at the origin, and using (2), (3), (4), one arrives easily at the formulas:

$$(8) \quad d \cdot (m!) \cdot \prod_1^m a_i = \sum_{w \in W(G)} (\text{sgn } w) w(b)^m$$

$$d = \prod_1^m (a_i, b) \cdot (a_i, a/2)^{-1}.$$

Finally, we remark that the representation of  $\bar{G}$  with highest weight  $b - (a/2)$  is single-valued on  $G$  if and only if  $b - (a/2)$  is integral on the unit lattice corresponding to  $G$ .

**4. Two characterizations of systems of positive roots.** We assume here  $G$  to be semi-simple, and denote its rank by  $l$ , but otherwise follow the notations of § 3. We discuss here two conditions under which given roots are positive relative to a suitable ordering; the first one is the object of (4.3), which will be preceded by two lemmas.

**4.1. LEMMA.** *Let  $u_j$  be the integers such that the sum  $a$  of the positive roots is equal to  $u_1 a_1 + \dots + u_l a_l$ . Then the only solution of the system of inequalities*

$$(1) \quad (a_j, x) \geq (a_j, a_j); \quad 0 \leq x_j \leq u_j, \quad (x = x_1 a_1 + \dots + x_l a_l; j = 1, \dots, l),$$

*is  $a$  itself.*

That  $a$  is a solution follows from (3.1).

Let  $y = y_1 a_1 + \dots + y_l a_l$  be a solution of (1) for which the sum of the  $y_i$  is minimum; such a  $y$  clearly exists. The  $y_i$ 's are  $> 0$ , because if, e.g.,  $y_k = 0$ , then we would have  $(a_k, y) \leq 0$  by (2.4).

Put  $r_j = (a_j, y)$ . Since the  $a_j$ 's ( $1 \leq j \leq l$ ) form a base, it is enough to show that  $r_j = (a_j, a_j)$ . Suppose otherwise; then there exists a  $k$  such that  $r_k > (a_k, a_k)$ . Consider  $z = y - c \cdot a_k$ , where  $c$  is a small positive constant. Then

$$z = \sum z_j a_j, \quad (z_j = y_j, (j \neq k); z_k = y_k - c)$$

$$(a_j, z) = (a_j, y) - c(a_j, a_k) = r_j - (a_j, a_k)c.$$

Since  $(a_j, a_k) \leq 0$  for  $j \neq k$  by (2.4),  $z$  is, for suitably small  $c > 0$ , a solution of (1) for which the sum of the coefficients is strictly smaller than for  $y$ , contradicting the latter's definition.

**4.2. LEMMA.** *Let  $(\epsilon_i)$ , ( $i = 1, \dots, m$ ), be a sequence of integers of*

absolute value 1, and let  $a^* = \sum_i \epsilon_i a_i$ . If  $(a^*, a_j) > 0$  for  $j = 1, \dots, l$ , then  $a^* = a$  and  $\epsilon_i = 1$  ( $1 \leq i \leq m$ ).

Let  $c = c_1 a_1 + \dots + c_l a_l$  (resp.  $d = d_1 a_1 + \dots + d_l a_l$ ), be the sum of the  $a_i$  for which  $\epsilon_i = 1$  (resp.  $\epsilon_i = -1$ ). We have

$$(2) \quad a^* = c - d; \quad a = c + d; \quad c_j + d_j = u_j; \quad c_j, d_j \geq 0, \quad (j = 1, \dots, l).$$

By assumption,

$$(a_j, c) - (a_j, d) = (a_j, a^*) > 0, \quad (j = 1, \dots, l),$$

and by (3.1),

$$(a_j, c) + (a_j, d) = (a_j, a) = (a_j, a_j), \quad (j = 1, \dots, l),$$

whence

$$2(a_j, c) > (a_j, a_j), \quad (j = 1, \dots, l).$$

But it follows from (2.3) that  $2(a_j, c) \cdot (a_j, a_j)^{-1}$  is an integer; therefore the preceding inequality implies that

$$(a_j, c) \geq (a_j, a_j), \quad (j = 1, \dots, l),$$

which, together with (2), shows that  $c$  is a solution of (1). By (4.1), this gives  $c = a$ ,  $d = 0$ ,  $a = a^*$  and  $d = 0$  implies (2.4) that no  $\epsilon_i$  equals  $-1$ .

**4.3. THEOREM.** Let  $(\epsilon_i)$ , ( $i = 1, \dots, m$ ), be a sequence of integers of absolute value 1. Then the set  $(\epsilon_i a_i)$  is the system of positive roots with respect to some ordering  $\mathcal{S}'$  if and only if  $a^* = \sum_i \epsilon_i a_i$  is a regular element.

*Necessity:* Suppose that  $(\epsilon_i a_i)$  are the positive roots with respect to some ordering; by (2.7), there exists  $w \in W(G)$  which sends the  $\epsilon_i a_i$  onto the  $a_i$  and, therefore,  $a^*$  onto  $a$ . Since the Weyl group leaves the set of regular elements invariant, it suffices to show that  $a$  is regular; but this follows from (3.1).

*Sufficiency:* Suppose  $a^*$  to be regular, and let  $\mu_i = \pm 1$  be such that  $(a^*, \mu_i a_i) > 0$ . Then, as remarked in (2.8),  $(\mu_i a_i)$  is the system of positive roots with respect to some ordering  $\mathcal{S}_1$ , and it will therefore be enough to show that  $\mu_i = \epsilon_i$ , ( $i = 1, \dots, m$ ).

By (2.7), we may find  $w \in W(G)$  carrying  $(\mu_i a_i)$  onto  $(a_i)$  and, therefore,  $a^*$  onto  $a^{**} = \sum_i \epsilon_i \mu_i a_{\sigma(i)}$ , where  $\sigma$  is a permutation of  $(1, 2, \dots, m)$  since  $(a^*, \mu_i a_i) > 0$  for all  $i$  and since  $w$  preserves the scalar product, we have

$(a^{**}, a_j) > 0$ , ( $j = 1, \dots, l$ ), and (4.2) gives then  $a^{**} = a$  and  $\mu_i \epsilon_i = 1$ ,  
 $(i = 1, \dots, m)$ .

4.4. COROLLARY. Let  $J$  be a non-empty set of integers belonging to the interval  $[1, m]$ . Suppose we have signs  $\epsilon_j$ , ( $j \in J$ ), such that  $\sum_{j \in J} \epsilon_j a_j = 0$ . Then for any choice of the remaining signs  $\epsilon_i$ , the form  $\sum_1^m \epsilon_i a_i$  is a singular element.

Suppose otherwise; then by (4.3), the  $(\epsilon_i a_i)$  are the positive roots in some suitable ordering, but, obviously, a sum of positive roots cannot vanish.

4.5. DEFINITION. A set  $B$  of roots of  $G$  is said to be closed if it contains the sum of any two of its elements whenever this sum is a root of  $G$ .

Our main purpose will be to show that a closed system containing one root from each pair  $\pm a_i$  is positive for some ordering.

4.6. LEMMA. Let  $B$  be a closed system which for no  $i$  ( $1 \leq i \leq m$ ), contains both  $a_i$  and  $-a_i$ . Then if a linear combination  $b$  of elements of  $B$  with positive integral coefficients is a root, it belongs to  $B$ .

Proof by induction on the sum  $k$  of the coefficients of  $b$ . For  $k = 2$ , it is an assumption; assume the lemma to be true for  $k - 1$ , and let

$$b = c_1 b_1 + \dots + c_q b_q, \quad (c_i > 0, c_i \text{ integer}, \sum c_i = k, b_i \in B).$$

We distinguish two cases. (a) For some  $j \leq q$ ,  $b_1 + b_j$  is a root; it is then in  $B$  by definition, and we have

$$b = (b_1 + b_j) + (c_1 - 1)b_1 + c_2 b_2 + \dots + (c_j - 1)b_j + c_{j+1} b_{j+1} + \dots + c_q b_q.$$

Therefore,  $b$  may also be written as

$$b = c'_1 b'_1 + \dots + c'_r b'_r, \quad (c'_i > 0, \text{ integral}, \sum c'_i = k - 1, b'_i \in B),$$

and is in  $B$  by the induction assumption. (b) No element  $b_1 + b_j$  is a root; then  $(b_1, b_j) \geq 0$  for  $j = 2, \dots, q$ , (see 2.3), whence  $(b_1, b) > 0$ , and  $b - b_1$  is a root. By induction, it is in  $B$ , and  $b$  then belongs to  $B$  by definition.

4.7. LEMMA. We keep the same assumption on  $B$ . Then any sum  $\sum c_i b_i$  ( $b_i \in B, c_i > 0$ , integral) is  $\neq 0$ .

Otherwise, we would have

$$-b_1 = (c_1 - 1)b_1 + c_2b_2 + \cdots + c_qb_q,$$

contradicting (4.5) and (4.6).

4.8. A sequence  $(b_1, \dots, b_k)$  of elements of  $B$  such that  $b_i - b_{i+1} \in B$  for  $i = 1, \dots, k-1$  will be said to be decreasing of length  $k$ . The height  $h(b)$  of  $b \in B$  will be the maximal length of decreasing sequences starting with  $b$ . By Lemma 4.7, any two elements in a decreasing sequence are different, and hence  $h(b)$  is always finite. Let  $k = h(b)$  and  $b, b_2, \dots, b_k$  a decreasing sequence. If we add to  $b$  a decreasing sequence for  $b_2$  or for  $b - b_2$ , we clearly get a decreasing sequence starting with  $b$ . Therefore  $h(b) = k$  implies that  $h(b_2), h(b - b_2)$  are  $\leq k-1$ ; thus an element of height  $k \geq 2$  is sum of two elements of heights  $\leq k-1$ .

4.9. THEOREM. Let  $B$  be a closed system of roots which, for each  $i$ ,  $(1 \leq i \leq m)$ , contains exactly one of the two roots  $\pm a_i$ . Then  $B$  is the set of positive roots for a suitable ordering.

Let  $b_1, \dots, b_s$  be the elements of height 1 in  $B$ . By induction on the height, it follows from the last assertion in (4.8) that every element of  $B$  is a linear combination of the  $b_i$ 's ( $i = 1, \dots, s$ ) with positive integral coefficients. Therefore, it suffices to show that the  $b_i$ 's are linearly independent. Since they span all roots, their rank is  $l$ ; assume that  $b_1, \dots, b_l$  are independent and that, contrary to our contention,  $s \neq l$ . We have then a relation

$$(3) \quad b_{l+1} = c_1b_1 + \cdots + c_lb_l, \quad (c_i \text{ real, not all zero}).$$

The  $c_i$ 's are also a solution of the linear system

$$(4) \quad (x_1(b_1, b_j) + \cdots + x_l(b_l, b_j))(b_j, b_j)^{-1} = (b_{l+1}, b_j)(b_j, b_j)^{-1}$$

( $j = 1, \dots, l$ ), whose determinant is, up to a positive factor, the determinant of the products  $(b_i, b_j)$  and is therefore  $\neq 0$ . Thus the  $c_i$ 's are the unique solution of (4) and are rational numbers since the coefficients are rational by (2.3). The given relation is then equivalent to a relation

$$d_1b_1 + \cdots + d_lb_l + d_{l+1}b_{l+1} = 0 \quad (d_i \text{ integers, } d_{l+1} \neq 0).$$

By (4.7) the coefficients do not all have the same sign, and, after a change in numeration of the  $b_i$  ( $i \leq l$ ), we arrive at an equality

$$(5) \quad e_1 b_1 + \cdots + e_p b_p = e_{p+1} b_{p+1} + \cdots + e_{l+1} b_{l+1} = b$$

( $e_i \geq 0$ , integral,  $(e_1, \dots, e_p) \neq (0, \dots, 0)$ ). On the other hand, we have  $(b_i, b_j) \leq 0$  for  $(1 \leq i < j \leq s)$ ; because otherwise, by (2.3), the elements  $\pm(b_i - b_j)$  would be roots, one of them would belong to  $B$  and either  $b_i$  or  $b_j$  would have a height  $\geq 2$ . Therefore we have

$$(b, b) = (e_1 b_1 + \cdots + e_p b_p, e_{p+1} b_{p+1} + \cdots + e_{l+1} b_{l+1}) \leq 0$$

and  $b = 0$ , in contradiction to (4.7), which proves that  $l = s$ .

4.10. COROLLARY. Let  $B$  be a closed system of roots which for each  $i$ ,  $(1 \leq i \leq m)$ , contains at least one of the roots  $\pm a_i$ . Then  $B$  contains the set of all positive roots relative to a suitable ordering.<sup>4</sup>

Let us number the roots in such a way that  $B$  consists of  $\pm a_1, \dots, \pm a_q, \epsilon_{q+1} a_{q+1}, \dots, \epsilon_m a_m$  ( $\epsilon_i = \pm 1$ ). Then the system  $B'$  consisting of the  $a_i$  ( $1 \leq i \leq q$ ) and the  $\epsilon_j a_j$  ( $q < j \leq m$ ) is closed; in fact, if  $a_s + a_t$  ( $s, t \leq q$ ) is a root, it is positive and in  $B$ , hence in  $B'$ . If  $a_s + \epsilon_t a_t = -a_p$  ( $s, p \leq q < t$ ), then  $a_s + a_p = -\epsilon_t a_t$  and  $B$  would not be closed; if  $\epsilon_s a_s + \epsilon_t a_t = -a_p$ , ( $p \leq q < s, t$ ), then  $a_p + \epsilon_t a_t = -\epsilon_s a_s$  and  $B$  is again not closed. Therefore, by the theorem,  $B'$  is the set of positive roots for some ordering.

4.11. Remark. Using complex semi-simple Lie algebras, one can also prove more generally than 4.9 that a closed system of roots  $B$  which for each  $i$  contains at most one of the roots  $\pm a_i$  is positive for some ordering. In fact, in the notations of (12.2), it follows readily from (4.7) that the subspace of  $\mathfrak{g}^c$  spanned by  $\mathfrak{t}^c$  and the  $v_b$ , ( $b \in B$ ), is a solvable subalgebra. It is then conjugate by an inner automorphism  $\alpha$  to a subalgebra of the algebra spanned by  $\mathfrak{t}^c$  and the  $v_{a_i}$  by a result of Morosow (C. R. Acad. Sci. U. R. S. S. (N. S.), 36 (1942), pp. 83-86), also proved in A. Borel, *Annals of Math.*, 64 (1956), pp. 20-80, § 16. Moreover, by the conjugacy of Cartan subalgebras in solvable Lie algebras, we may assume that  $\alpha(\mathfrak{t}^c) = \mathfrak{t}^c$  which means that  $B$  is transformed onto a subset of the  $a_i$ 's by an element of the Weyl group.

5. The 2-roots of a compact Lie group. We define here certain linear forms with values in  $\mathbb{Z}_2$ , analogous to the roots, which are useful in the study of Stiefel-Whitney classes.

5.1. Let  $G$  be a Lie group. We denote by  $Q$  or  $Q_s$  a subgroup of  $G$

<sup>4</sup> Another, completely different, proof of this corollary has been given by Harish-Chandra, *American Journal of Mathematics*, vol. 77 (1955), pp. 743-777, § 2, Lemma 4.

isomorphic to the product of  $s$  copies of  $\mathbf{Z}_2$ . The real irreducible linear representations of  $Q$  are 1-dimensional, being defined by a character which may be viewed as an element of  $\text{Hom}(Q, \mathbf{Z}_2)$ . Let us now decompose the Lie algebra  $\mathfrak{g}$  of  $G$  into a direct sum

$$\mathfrak{g} = \mathfrak{b}_1 + \cdots + \mathfrak{b}_n, \quad (n = \dim \mathfrak{g})$$

of 1-dimensional subspaces invariant under  $\text{Ad}_g Q$ ; the characters  $b_1, \dots, b_n$  corresponding to these subspaces will be called the *2-roots of  $G$  with respect to  $Q$* . In case  $Q$  is contained in a maximal torus, these are just the restrictions to  $Q$  of the roots of  $G$  and the zero form with multiplicity  $l = \text{rank}(G)$ , but otherwise, they represent different elements of the group-theoretical structure of  $G$  and have a "global" character.

5.2. Let  $U$  be a closed subgroup of  $G$  containing  $Q$ . Then we can choose a decomposition into subspaces  $\mathfrak{b}_i$ , such that the  $n-k$  last ones generate the Lie algebra  $\mathfrak{u}$  of  $U$ . The 2-roots  $b_i$  ( $1 \leq i \leq k$ ) are then the *complementary 2-roots*; or, more explicitly, the 2-roots of  $G$  with respect to  $Q$  which are complementary to those of  $U$ .

#### Examples.

5.3.  $G = \mathbf{O}(n), \mathbf{SO}(n)$ . We consider in  $\mathbf{O}(n)$  the subgroup  $Q$  of diagonal matrices; it is a maximal commutative subgroup of type  $(2, 2, \dots, 2)$ , and any subgroup of this type is conjugate to a subgroup of  $Q$ . We take in  $\mathfrak{g}$  the usual basis consisting of the antisymmetric matrices having only two non-vanishing entries, equal to  $\pm 1$ . Let  $x_{ii}$  ( $1 \leq i \leq n$ ) be the diagonal matrix all of whose coefficients are equal to 1, except for the  $i$ -th one which is equal to  $-1$ , and let  $(y_i)$  be the dual basis of  $\text{Hom}(Q, \mathbf{Z}_2)$ . A straightforward computation shows that the basis of  $\mathfrak{g}$  mentioned above is invariant under  $\text{Ad}_g Q$  and that the 2-roots relative to  $Q$  are

$$y_i - y_j, \quad (1 \leq i < j \leq n).$$

In  $\mathbf{SO}(n)$ , the diagonal matrices also form a maximal commutative subgroup  $Q'$  of type  $(2, 2, \dots, 2)$ , isomorphic to  $(\mathbf{Z}_2)^{n-1}$ . It is convenient to consider it as a subgroup of  $Q$ , and, therefore,  $\text{Hom}(Q', \mathbf{Z}_2)$  as a quotient of  $\text{Hom}(Q, \mathbf{Z}_2)$ ; it is then generated by  $n$  elements  $y_i$  subject to the relation  $y_1 + \cdots + y_n = 0$ , and the 2-roots are again the differences  $y_i - y_j$  ( $1 \leq i < j \leq n$ ).

5.4.  $G = \mathbf{U}(n), \mathbf{SU}(n)$ . In  $\mathbf{U}(n)$ , all maximal commutative subgroups of type  $(2, 2, \dots, 2)$  are conjugate to the subgroup  $Q$  of diagonal matrices

with coefficients  $\pm 1$ , and are therefore contained in maximal tori. The 2-roots are then obtained from the usual roots and, with respect to the standard basis of skew-hermitian matrices, are the zero form with multiplicity  $n$  and the differences  $y_i - y_j$  ( $1 \leq i < j \leq n$ ), each with multiplicity two.

The 2-roots of  $SU(n)$  with respect to the subgroup of the elements of  $Q$  having determinant 1 will be the same, the  $y_i$ 's being subject to the relation  $y_1 + \cdots + y_n = 0$ , the zero form having multiplicity  $n - 1$ .

5.5.  $G = Sp(n)$ . Here again, all maximal commutative subgroups of type  $(2, 2, \dots, 2)$  are conjugate to the subgroup  $Q$  of diagonal matrices with coefficients  $\pm 1$  ( $G$  being considered as the group of unitary matrices with quaternionic coefficients) and are isomorphic to  $(\mathbf{Z}_2)^n$ . Since the usual roots are  $\pm y_i \pm y_j$  ( $1 \leq i < j \leq n$ ) and  $\pm 2y_i$  ( $1 \leq i \leq n$ ), we get as 2-roots: the root zero, with multiplicity  $3n$  and  $y_i - y_j$  ( $1 \leq i < j \leq n$ ), with multiplicity 4.

In §17, we shall also discuss the 2-roots of the exceptional group  $G_2$  with respect to a subgroup not contained in a maximal torus.

## Chapter II. Topological Preliminaries.

### 6. Fibre bundles.

6.1. *Notations.*  $p$  denotes a prime number or zero,  $K_p$  a field of characteristic  $p$ ,  $\mathbf{Z}_p$  ( $p \neq 0$ ),  $\mathbf{Z}_0$ ,  $\mathbf{R}$ ,  $\mathbf{C}$ , the fields of integers mod  $p$ , of rational, real, complex numbers respectively.

$H^i(X, A)$  (resp.  $H_i(X, A)$ ) is the  $i$ -th singular cohomology (resp. homology) group of the space  $X$  with coefficients in the commutative group  $A$ ,  $H^*(X, A)$  (resp.  $H_*(X, A)$ ) the direct sum of the cohomology (resp. homology) groups; for all spaces considered in this paper,  $H^i(X, \mathbf{Z})$  will be finitely generated and equal to the  $i$ -th Alexander-Spanier cohomology group. The map of cohomology (resp. homology) groups induced by a continuous map  $f$  is denoted by  $f^*$  (resp.  $f_*$ ).

When dealing with classifying spaces, it will sometimes be convenient to consider formal infinite sums of cohomology elements, and to this effect, we also introduce the *direct product*  $H^{**}(X, A)$  of the  $H^i(X, A)$ ; an element  $x \in H^{**}(X, A)$  may be identified with a sum  $x_0 + \cdots + x_i + \cdots$ , with  $x_i \in H^i(X, A)$  possibly  $\neq 0$  for infinitely many values of  $i$ . When  $A$  is a ring,  $H^{**}(X, A)$  also becomes an associative ring under the cup product. The homomorphism of  $H^{**}(Y, A)$  into  $H^{**}(X, A)$  induced by  $f: X \rightarrow Y$  will be denoted by  $f^{**}$ .

$A\{X_1, \dots, X_k\}$  will denote the ring of formal power series in the  $X_i$ 's, with coefficients in the commutative ring  $A$ .

Let  $U$  be a closed, connected subgroup of maximal rank of the compact, connected Lie group  $G$ , and let  $T$  be a maximal torus of  $U$ . We have then  $H^{**}(B_T, A) = A\{x_1, \dots, x_l\}$ , ( $x_i \in H^2(B_T, A)$ ,  $1 \leq i \leq l = \text{rank } G$ ). The results of [2, §§ 26, 27] imply that  $\rho^{**}(T, G)$  maps  $H^{**}(B_G, \mathbb{Z}_0)$  isomorphically onto the ring of invariants of the Weyl group, and that it is isomorphic to a ring of formal power series in  $l$  indeterminates; moreover,  $H^{**}(G/U, \mathbb{Z}_0)$  is the quotient of  $H^{**}(B_U, \mathbb{Z}_0)$ , regarded as a subring of  $H^{**}(B_T, \mathbb{Z}_0)$ , by the ideal  $(I^+_G)^*$  generated in  $H^{**}(B_U, \mathbb{Z}_0)$  by the (finite or infinite) sums of homogeneous invariants of  $W(G)$  with strictly positive degrees. Similar translations in cohomology over  $\mathbb{R}$ ,  $\mathbb{Z}_p$  or  $\mathbb{Z}$  of the results of [2, § 29] are left to the reader.

**6.2. Fibre bundles.** The fibre bundles occurring in this paper will be locally trivial; we follow the definitions of [19, 26]. We do not require the structural group to act effectively on the fibre [19, § 3.2c)]. A fibre bundle is denoted by  $(E, B, F, \pi)$  or  $(E, B, F)$ , where  $E$  is the total space,  $B$  the base space,  $F$  the typical fibre,  $\pi$  the projection, or just by one symbol, mostly  $\xi, \eta, \theta$ ; in the latter case, we often write  $E_\xi, B_\xi, F_\xi, \pi_\xi, G_\xi, \tau_\xi$  for  $E, B, F, \pi$ , the structural group and the transgression in  $\xi$  respectively. A bundle with structural group  $G$  will also be called a  $G$ -bundle.

Let  $\xi$  be a principal  $G$ -bundle and  $U$  a closed subgroup of  $G$ . The space of the cosets  $x \cdot U$  modulo  $U$  ( $x \in E_\xi$ ), is denoted by  $E_\xi/U$ ; it is the base space of the principal fibering  $(E_\xi, E_\xi/U, U)$  and the total space of the  $G$ -bundle  $(E_\xi/U, B_\xi, G/U)$ . Let  $F$  be a space operated upon by  $G$ . We denote by  $E_\xi \times_G F$  the quotient of  $E_\xi \times F$  by the equivalence relation  $(x, f) \approx (x \cdot g, g^{-1} \cdot f)$ . As is well known, it is the total space of a  $G$ -bundle  $(\xi, F)$  over  $B_\xi$ , with fibre  $F$ .

**6.3. Representations of fibre bundles.** Let  $\xi, \eta$  be two fibre bundles. A representation of  $\xi$  in  $\eta$  is a continuous map  $\phi: E_\xi \rightarrow E_\eta$  which sends fibres into fibres; it induces then a map  $\bar{\phi}: B_\xi \rightarrow B_\eta$  such that  $\bar{\phi} \circ \pi_\xi = \pi_\eta \circ \phi$ . We shall use without further comment the fact that  $\phi$  commutes with transgression and, more generally, induces a homomorphism of the spectral sequence of  $\eta$  into that of  $\xi$  (see e.g. [2], § 4).

**6.4. Homomorphisms of fibre bundles.** Let  $G, G'$  be topological groups,  $\lambda: G \rightarrow G'$  a homomorphism, and  $F$  (resp.  $F'$ ), a space on which  $G$ , (resp.  $G'$ ), operates. A  $\lambda$ -map of  $F$  into  $F'$  is a continuous map  $\psi$  such that

$\psi(g \cdot f) = \lambda(g) \cdot \psi(f)$ , (or  $\psi(f \cdot g) = \psi(f) \cdot \lambda(g)$  if  $G, G'$  operate on the right). Let  $\xi$  and  $\eta$  be principal  $G$ - and  $G'$ -bundles respectively. A homomorphism of  $\xi$  into  $\eta$  is a representation induced by a  $\lambda$ -map of  $E_\xi$  into  $E_\eta$ ; clearly every  $\lambda$ -map defines a homomorphism. A homomorphism of  $(\xi, F)$  into  $(\eta, F')$  is a representation defined by two  $\lambda$ -maps of  $E_\xi$  and  $F$  into  $E_\eta$  and  $F'$  respectively.

Let  $U, U'$  be closed subgroups of  $G$  and  $G'$  such that  $\lambda(U) \subset U'$ . Then we have a commutative diagram

$$\begin{array}{ccccc} E_\xi & \longrightarrow & E_\xi/U & \longrightarrow & B_\xi \\ \downarrow \phi & & \downarrow \phi_1 & & \downarrow \phi_2 \\ E_\eta & \longrightarrow & E_\eta/U' & \longrightarrow & B_\eta \end{array}$$

$\phi$  defines  $\lambda$ -homomorphisms  $(E_\xi, E_\xi/U, U) \rightarrow (E_\eta, E_\eta/U', U')$  and  $(E_\xi, B_\xi, G) \rightarrow (E_\eta, B_\eta, G')$ ; the map  $\phi_1$  is a  $\lambda$ -homomorphism of  $(E_\xi/U, B_\xi, G/U)$  into  $(E_\eta/U', B_\eta, G'/U')$ .

6.5. *Restriction and extension of the structural group.* Let  $\xi$  and  $\eta$  be two principal bundles over the same base space  $B$ , and let  $\lambda$  be a homomorphism of  $G_\xi$  into  $G_\eta$ . Assume that there exists a  $\lambda$ -homomorphism of  $\xi$  into  $\eta$  which induces the identity of  $B$ . Then we say that  $\eta$  is a  $\lambda$ -extension of  $\xi$  and that  $\xi$  is a  $\lambda$ -restriction of  $\eta$ . We recall that, given  $\xi$  and  $\lambda$ , there always exists a  $\lambda$ -extension which is unique up to equivalence, and which will be denoted by  $\lambda(\xi)$ ; whereas given  $\eta$  and  $\lambda$ , a  $\lambda$ -restriction does not always exist and, if it does, is not necessarily unique. The  $\lambda$ -extension  $\eta$  of  $\xi$  is defined as follows:  $E_\eta = E_\xi \times_G G'$ , where  $G$  operates on  $G'$  by  $g \cdot g' = \lambda(g) \cdot g'$ ; if  $\mu$  is the projection of  $E_\xi \times G'$  onto  $E_\xi$ , then  $\pi_\eta$  is induced by  $\mu(x, g') \rightarrow \pi_\xi(x)$ , and the  $\lambda$ -map  $\phi: E_\xi \rightarrow E_\eta$  is defined by  $\phi(x) = \mu(x, e)$ , where  $e$  is the neutral element in  $G'$ ; finally, the principal bundle operations on  $E_\eta$  are introduced by  $(x, y) \cdot g' = (x, y \cdot g')$ , ( $x \in E_\xi; y, g' \in G'$ ). These notions are defined in the same way for associated bundles; they generalize the standard concepts of extension and restriction of the structural group, to which they reduce when  $\lambda$  is the inclusion map of a closed subgroup. Clearly they can also be formulated for equivalence classes of bundles; if these are identified with the elements of the cohomology sets  $H^1(B, G_c)$  and  $H^1(B, G'_c)$ , in the notations of [19, § 3], then the  $\lambda$ -extension of  $\xi \in H^1(B, G_c)$  is its image under the natural coefficient map induced by  $\lambda$ .

6.6. *Characteristic map.*  $E_G$  (resp.  $B_G$ ) is a universal bundle (resp. classifying space) for the compact Lie group  $G$  ([26], § 19, [2], § 18; as in

[6] they will usually be taken as universal or classifying for all dimensions). Any principal  $G$ -bundle  $\xi$  over a base space  $B$  belonging to a suitable class of topological spaces is induced from the universal bundle by a map  $\sigma: B \rightarrow B_G$ , defined up to homotopy as the "characteristic map" for  $\xi$  [26, § 19].

To a homomorphism  $\lambda$  of  $G$  into a compact Lie group  $G'$  corresponds a map  $\rho(\lambda)$  of  $B_G$  into  $B_{G'}$ , defined up to homotopy, called the characteristic map for the  $\lambda$ -extension of  $(E_G, B_G, G)$  (see [6], § 1). It follows immediately from the definitions that a  $\lambda$ -restriction of a principal  $G'$ -bundle  $\eta$  exists if and only if the characteristic map  $\sigma$  of  $\eta$  can be written as  $\sigma = \rho(\lambda) \circ \sigma'$  where  $\sigma'$  is a map of  $B$  into  $B_G$ ; when  $\lambda$  is an inclusion,  $\rho(\lambda)$  reduces to the map  $\rho(G, G')$  introduced in [2].

6.7. *Fibre bundle over a fibre bundle.* We discuss here a generalization of the "bundle along the fibres" (see § 7), which allows us to put in its proper setting a useful fact about characteristic maps.

Let  $\xi, \eta$  be fibre bundles,  $\bar{\xi}, \bar{\eta}$  the corresponding principal bundles. We assume that  $F_\xi = B_\eta$  and that  $G_\xi$  is also a group of automorphisms of  $\bar{\eta}$ ; the latter condition means that there is a homomorphism  $g \rightarrow \bar{g}$  of  $G_\xi$  in the group of those homeomorphisms of  $E_{\bar{\eta}}$  which commute with the operations of  $G_\eta$ , and, of course, such that the induced homeomorphisms of  $B_\eta$  are those which define  $G_\xi$  as structural group for  $\xi$ . In particular, the homeomorphism  $\bar{g} \times \text{Id}$  of  $E_{\bar{\eta}} \times F_\eta$  is compatible with the equivalence relation which defines  $\eta$ , hence  $G_\xi$  is also a group of homeomorphisms of  $E_\eta$  commuting with  $\pi_\eta$ . By means of these operations, we define first a bundle  $\mu = (E_\mu, B_\xi, E_\eta)$  with structural group  $G_\xi$ , and typical fibre  $E_\eta$ , associated to  $\bar{\xi}$ ; its total space is then

$$E_\mu = E_{\bar{\xi}} \times_{G_\xi} E_\eta.$$

Since  $G_\xi$  commutes with  $\pi_\eta$ , this map induces a map  $\lambda$  of  $E_\mu$  onto  $E_{\bar{\xi}} \times_{G_\xi} B_\eta = E_\xi$ . Since  $G_\xi$ , operating on  $E_{\bar{\eta}}$ , and  $G_\eta$  commute, the space  $E_{\bar{\xi}} \times_{G_\xi} E_{\bar{\eta}}$  can be considered as a principal  $G_\eta$ -bundle over  $E_\xi$ , the operations of the group being defined by means of its action on the right factor, and moreover, we have the "associativity law"

$$(E_{\bar{\xi}} \times_{G_\xi} E_{\bar{\eta}}) \times_{G_\eta} F_\eta \approx E_{\bar{\xi}} \times_{G_\xi} (E_{\bar{\eta}} \times_{G_\eta} F_\eta).$$

From this, it follows immediately that  $\lambda$  is the projection in a fibre bundle  $(E_\mu, E_\xi, F_\eta) = \nu$ , in which  $G_\eta$  is the structural group, and whose corresponding principle bundle has total space  $E_{\bar{\xi}} \times_{G_\xi} E_{\bar{\eta}}$ . Therefore, we have obtained a bundle over  $E_\xi$  with fibre  $F_\eta$ . It is clear that the inclusion map of a fibre of  $\mu$  in  $E_\mu$  may be viewed as a homomorphism of  $\eta$  in  $\nu$ ; it induces a map  $i: B_\eta \rightarrow E_\xi$  of their base spaces which is the inclusion map of a fibre of  $\xi$ .

Therefore, if  $\sigma$  is the characteristic map for  $\nu$ , then  $\sigma \circ i$  is characteristic for  $\eta$ , and the characteristic ring of  $\eta$  is the image of  $i^* \circ \sigma^*$ . This proves in particular the following:

**6.8. PROPOSITION.** *Let  $\xi, \eta$  be two bundles with  $F_\xi = B_\eta$ , satisfying the conditions of 6.7., and let  $i$  be the injection of a fibre of  $\xi$ . Then the image of  $i^*: H^*(E_\xi, A) \rightarrow H^*(F_\xi, A)$  contains the characteristic ring of  $\eta$ .*

Let  $G$  be a topological group,  $U$  be a closed subgroup and  $\theta$  be a principal  $G$ -bundle. Then  $\xi = (E_\theta/U, B_\theta, G/U)$  and  $\eta = (G, G/U, U)$  satisfy the assumptions of 6.7,  $\eta$  being considered as a principal  $U$ -bundle, and  $G_\xi = G$  (resp.  $G_\eta = U$ ) acting by means of left (resp. right) translations on  $G$ ; in this case,  $\mu$  may be identified with  $\theta$  and  $\nu$  with  $(E_\theta, E_\theta/U, U)$ , and 6.8 reduces to the corollary to Prop. 18.3 of [2]. The application to differentiable bundles will be mentioned in § 7.

## 7. Vector bundles.

**7.1.** A real (resp. complex) vector bundle is a fibre bundle with an  $n$ -dimensional real (resp. complex) vector space as typical fibre, the structural group operating by means of linear transformations. Most often, we shall identify the typical fibre with  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , and the structural group with a subgroup of  $\mathbf{GL}(n, \mathbf{R})$  or  $\mathbf{LG}(n, \mathbf{C})$ . We refer to [19] for the notions of sub-bundle, quotient bundle of a vector bundle, of Whitney sum  $\xi \oplus \eta$  and tensor product  $\xi \otimes \eta$  of two vector bundles  $\xi, \eta$ . We recall that a principal bundle with group  $\mathbf{GL}(n, \mathbf{R})$  or  $\mathbf{GL}(n, \mathbf{R})^+$  or  $\mathbf{GL}(n, \mathbf{C})$  has a unique restriction (up to isomorphism) with group  $\mathbf{O}(n)$  or  $\mathbf{SO}(n)$  or  $\mathbf{U}(n)$  [26, § 12].

**7.2. Orientable real vector bundles.** A real vector bundle is orientable if its structural group can be reduced to  $\mathbf{GL}(n, \mathbf{R})^+$  or  $\mathbf{SO}(n)$ . If such a restriction has been made, we then endow each fibre with the orientation which is carried over from a fixed given orientation of the typical fibre  $V$  by the allowable homeomorphisms of the bundle structure; the bundle is then said to be oriented; if  $V$  has been identified with  $\mathbf{R}^n$ , we always take the natural orientation of  $\mathbf{R}^n$ .

**7.3. Almost complex structures.** A complex vector bundle  $(E, B, \mathbf{C}^q)$  defines in a natural fashion a real vector bundle  $(E', B, \mathbf{R}^{2q})$ , its  $\lambda$ -extension relative to the standard inclusion  $\lambda: \mathbf{GL}(q, \mathbf{C}) \rightarrow \mathbf{GL}(2q, \mathbf{R})$ ; it is oriented. Conversely, if a real vector bundle  $(E, B, \mathbf{R}^{2q})$  has a  $\lambda$ -restriction, we say that it admits a complex structure and that such a complex restriction is

a complex structure of the given bundle. A differentiable manifold admits an almost complex structure (resp. is almost complex) if its tangent bundle admits (resp. has been given) a complex structure.<sup>5</sup> To a complex structure of  $\xi = (E, B, \mathbf{R}^{2n})$  there is attached a section  $J$  in the real vector bundle  $\xi^* \otimes \xi = \text{Hom}(\xi, \xi)$ , where the value of  $J_b$  of  $J$  at  $b \in B$  is the linear map defined by multiplication by  $\sqrt{-1}$ ; conversely, given a section  $J$  of linear maps such that  $J_b^2 = -\text{Id}$  for all  $b \in B$ , we introduce on each fibre a complex structure by putting

$$(x + \sqrt{-1}y) \cdot v = x \cdot v + y \cdot J_b(v),$$

which gives a complex structure for the given real vector bundle.

7.4. *The bundle along the fibres.* Let  $\xi$  be a fibre bundle whose fibre is a differentiable manifold  $F$  of dimension  $n$ ,  $G_\xi$  being a group of differentiable homeomorphisms of  $F$ ; the group  $G_\xi$  is then also a group of automorphisms of the tangent bundle  $\eta = T(F_\xi)$  to  $F_\xi$  and of the bundle of frames  $\overline{\eta} = B(F_\xi)$  which have  $\mathbf{GL}(n, \mathbf{R})$  as structural group. We may apply 6.7, and the bundle corresponding to  $\nu$  of 6.7 will be called *the bundle along the fibres*. It is a real vector bundle over  $E_\xi$ , whose fibres are the tangent spaces to the fibres of  $\xi$ , and will be denoted by  $\hat{\xi}$ . If  $F$  has an almost complex structure which is invariant under  $G_\xi$ , in other words, if  $G_\xi$  is also a group of automorphisms for a complex structure  $\eta'$  of  $\eta$ , then the construction of 6.7 may also be applied to  $\xi$  and  $\eta'$  and yields a complex structure on  $\hat{\xi}$  which will then be called a *complex bundle along the fibres* of  $\xi$ . Also, if  $F_\xi$  carries an orientation invariant under  $G_\xi$ , then the structural group of  $\hat{\xi}$  may also be reduced to  $\mathbf{GL}(n, \mathbf{R})^+$ . Applying 6.8 to the basic elements of the characteristic ring of a  $\mathbf{O}(n)$ ,  $\mathbf{SO}(n)$  or  $\mathbf{U}(n)$ -bundle (see §9), we obtain the

**PROPOSITION.** *Let  $\xi$  be a fibre bundle whose typical fibre  $F_\xi$  has a differentiable structure invariant under  $G_\xi$ , and let  $i$  be the inclusion map of a fibre in  $E_\xi$ . Then the Pontrjagin and Stiefel-Whitney classes of  $F_\xi$ , its Euler-Poincaré class with respect to an orientation invariant under  $G_\xi$ , and its Chern classes with respect to a  $G_\xi$ -invariant almost complex structure are in the image of  $i^*$ .*

(A similar remark has already been made in A. Borel, Jour. math. pur. appl. (9) 35, 127-139 (1956), proof of 3.2.)

<sup>5</sup> In this terminology therefore, an-almost complex structure on a manifold corresponds to a complex structure of its tangent bundle.

$\xi$  is said to be differentiable if  $E_\xi$ ,  $B_\xi$ ,  $F_\xi$  are differentiable manifolds,  $\pi_\xi$  is a differentiable map, and the coordinate functions are differentiable; it follows then that  $G_\xi$  is a group of diffeomorphisms of  $F_\xi$ . In this case, the fibre of  $\hat{\xi}$  over  $x \in E_\xi$  may be identified with the subspace of the tangent space of  $E_\xi$  at  $x$  which is tangent to the fibre of  $\xi$  passing through  $x$ .

7.5. PROPOSITION. Let  $G$  be a Lie group,  $U$  a closed subgroup,  $\iota: U \rightarrow \mathbf{GL}(n, R)$  the isotropy representation (1.1), and let  $\xi$  be a principal  $G$ -bundle. Then the principal bundle  $\eta$  along the fibres of  $(E_\xi/U, B_\xi, G/U)$  is the  $\iota$ -extension of  $(E_\xi, E_\xi/U, U)$ .

We have to show the existence of a  $\iota$ -map:  $E_\xi \rightarrow E_\eta$  inducing the identity on  $E_\xi/U$ .

We recall first that  $(E_\xi/U, B_\xi, G/U)$  may be considered as the bundle with typical fibre  $G/U$  associated to  $\xi$ ; more precisely, there is a commutative diagram

$$\begin{array}{ccc} E_\xi & \xrightarrow{\alpha} & E_\xi \times G/U \\ \downarrow \gamma & & \downarrow \delta \\ E_\xi/U & \xrightarrow{\beta} & E_\xi \times_G G/U \end{array}$$

where  $\gamma$  and  $\delta$  are the natural projections,  $\alpha$  is defined by  $x \rightarrow (x, o)$ , the point  $o \in G/U$  being the image of  $U$  under the projection,  $\beta$  is determined by the other maps and is a homeomorphism. This also allows us to attach to each  $x \in E_\xi$  a homeomorphism  $\sigma_x$  of  $G/U$  onto the fibre  $\gamma(x \cdot G)$  of  $\gamma(x)$  in  $E_\xi/U$ , defined by

$$\sigma_x(y) = \gamma \cdot \alpha^{-1}(x \cdot g, o), \quad (y \in G/U, g \in G \text{ such that } g^{-1}(y) = o).$$

We have

$$\sigma_x(o) = \gamma(x); \quad \sigma_{x \cdot g} = \sigma_x \circ g.$$

All this is well known and easily checked. Let now  $R_0$  be a fixed base of the tangent space to  $G/U$  at  $o$ . Then  $\sigma_x(R_0)$  is a base of the tangent space to the fibre of  $(E_\xi/U, B_\xi, G/U)$  at  $\gamma(x)$ . We define  $\phi$  by  $\phi(x) = \sigma_x(R_0)$ ; from the relation  $\sigma_{x \cdot u} = \sigma_x \circ u$ , it follows readily that  $\phi(x \cdot u) = \phi(x) \cdot \iota(u)$ , in other words, that  $\phi$  is a  $\iota$ -map. Since by construction,  $\phi$  induces the identity on  $E_\xi/U$ , our contention is proved.

(7.5) shows in particular that the structural group of the tangent bundle to  $G/U$  may be  $\iota$ -restricted to  $U$ . Finally, we mention the following well known elementary fact:

7.6. PROPOSITION. *Let  $\xi$  be a differentiable bundle. Then the quotient of the tangent bundle to  $E_\xi$  by the bundle along the fibres  $\hat{\xi}$  is equivalent to the bundle induced by  $\pi_\xi$  from the tangent bundle to  $B_\xi$ .*

In fact,  $\pi_\xi$  induces a bundle map of this quotient onto the tangent bundle to  $B_\xi$ , and the proposition follows then from [26, § 10.3].

## 8. Integration over the fibre.

8.1. Let  $A$  be a commutative group,  $(E, B, F, \pi)$  a fibre bundle with connected fibres such that (i) there exists an integer  $q$  for which  $H^r(F, A) = 0$  for  $r > q$  and that (ii) the cohomology groups of the different fibres form a constant sheaf over  $B$ .

We want to define, in terms of the spectral sequence of the bundle, a homomorphism

$$\natural: H^k(E, A) \rightarrow H^{k-q}(B, H^q(F, A)) \quad (k = 0, 1, \dots),$$

the so-called "integration over the fibre". We put, of course,  $\natural = 0$  for  $k < q$  and assume from now on that  $k \geq q$ . By (i), no non-zero element of  $E_r^{k-q, q}$ , ( $r \geq 2$ ), is a coboundary, hence the subgroup of the elements in  $E_2^{k-q, q}$  which are cocycles for all differentials is canonically isomorphic to  $E_\infty^{k-q, q}$ , and we get a natural inclusion map

$$h_1: E_\infty^{k-q, q} \rightarrow E_2^{k-q, q} \cong H^{k-q}(B, H^q(F, A)).$$

Let now  $J^a$  ( $a = 0, 1, \dots$ ), be the decreasing sequence of submodules defining the filtration of  $H^*(E, A)$  attached to the fibration, and let us put, as usual,  $J^{a, b} = J^a \cap H^{a+b}(E, A)$ . Since  $E_\infty^{a, b} = 0$  for  $b > q$ , we have  $H^k(E, A) = J^{k-q, q}$ , whence

$$E_\infty^{k-q, q} = J^{k-q, q} / J^{k-q+1, q-1} = H^k(E, A) / J^{k-q+1, q-1}$$

and a natural projection

$$h_2: H^k(E, A) \rightarrow E_\infty^{k-q, q}.$$

$\natural$  is then defined by  $\natural = h_1 \circ h_2$ ; by linearity it extends to an additive homomorphism of  $H^*(E, A)$  into  $H^*(B, H^q(F, A))$  and of  $H^{**}(E, A)$  into  $H^{**}(B, H^q(F, A))$ . Whenever  $H^q(F, A)$  can be identified with  $A$ , for instance, when  $F$  is an oriented  $q$ -dimensional manifold, we consider it as a map from  $H^*(E, A)$  or  $H^{**}(E, A)$  to  $H^*(B, A)$  or  $H^{**}(B, A)$ , lowering by  $q$  the degree of homogeneous elements.

8.2. PROPOSITION. Let  $A$  be a commutative ring,  $\xi$  a bundle satisfying conditions (i), (ii) of (8.1),  $\natural$  the integration over the fibre. Then

$$(\pi_{\xi}^*(b) \cdot x)^{\natural} = b \cdot (x)^{\natural}, \quad (b \in H^*(B, A), x \in H^*(E, A)).$$

(Here  $b \cdot (x)^{\natural}$  means the product of  $b$  and  $(x)^{\natural}$  under the natural pairing of  $A$  and  $H^q(F_{\xi}, A)$  to  $H^q(F_{\xi}, A)$ .) For the proof, we may assume  $b$  and  $x$  to be homogeneous of degrees  $s, t$ . We identify  $b$  with its image in  $E_2^{s,0}$  under the canonical isomorphism with  $H^s(B_{\xi}, H^0(F_{\xi}, A)) = H^s(B_{\xi}, A)$ . Then we have

$$\begin{aligned} h_2(\pi_{\xi}^*(b) \cdot (x)) &= \kappa_{\infty}^2(b) \cdot h_2(x) \\ h_1(\kappa_{\infty}^2(b) \cdot h_2(x)) &= b \cdot (h_1 \circ h_2(x)) \end{aligned}$$

because  $E_{\infty}$  is the graded ring associated to  $H^*(E_{\xi}, A)$  filtered by the  $J^a$ , and  $b$  is a cocycle for all differentials.

8.3. PROPOSITION. Let  $\xi, \eta$  be two fibre bundles satisfying the conditions (i), (ii) of (8.1) and let  $\phi$  be a representation of  $\xi$  in  $\eta$ . Let  $\psi: H^*(B_{\eta}, H^q(F_{\eta}, A)) \rightarrow H^*(B_{\xi}, H^q(F_{\xi}, A))$  be the homomorphism which is induced by the map  $\bar{\phi}: B_{\xi} \rightarrow B_{\eta}$  defined by  $\phi$ , and by the map  $\nu: H^*(F_{\eta}, A) \rightarrow H^*(F_{\xi}, A)$  defined by the restriction of  $\phi$  to a fibre.<sup>9</sup> Then the following diagram is commutative

$$\begin{array}{ccc} H^k(E_{\eta}, A) & \xrightarrow{\phi^*} & H^k(E_{\xi}, A) \\ \downarrow \natural & & \downarrow \natural \\ H^{k-q}(B_{\eta}, H^q(F_{\eta}, A)) & \xrightarrow{\psi} & H^{k-q}(B_{\xi}, H^q(F_{\xi}, A)). \end{array}$$

This follows from the fact that  $\phi$  induces a homomorphism of the spectral sequence of  $\eta$  into that of  $\xi$ , reducing to  $\psi$  on the  $E_2$  terms [2, § 4].

*Remark.* For another discussion of the integration over the fibre, see [11]; it is also proved there, but we shall not need this fact, that in case  $E, B, F$  are oriented compact connected manifolds, then  $\natural$  is equivalent to the Gysin homomorphism defined by means of  $\pi$  [11, Theorem 3].

8.4. Let  $\xi$  be a fibre bundle satisfying (i), (ii) and: (iii)  $A$  is a principal ideal ring,  $H^*(F_{\xi}, A)$  is a free  $A$ -module of finite rank,  $H^q(F_{\xi}, A) \cong A$ , and  $F_{\xi}$  is totally non-homologous to zero in  $E_{\xi}$ .

As is well known, these conditions have the following consequences for the spectral sequence of  $\xi$ :

<sup>9</sup> Note that, by assumption (ii), the latter homomorphism has an invariant meaning, independent from the particular fibre to which we restrict  $\phi$ .

$$E_x = E_2 = H^*(B_\xi, A) \otimes H^*(F_\xi, A),$$

and  $\pi_\xi^*$  is injective; if  $H^*(E_\xi, A)$  is considered as a  $H^*(B_\xi, A)$ -module by means of the rule  $b \cdot x = \pi_\xi^*(b) \cup x$ , ( $b \in H^*(B_\xi, A)$ ,  $x \in H^*(E_\xi, A)$ ), and if  $h_i$  ( $1 \leq i \leq m = \text{rank } H^*(F_\xi, A)$ ) are homogeneous elements of  $H^*(E_\xi, A)$  inducing a module basis of  $H^*(F_\xi, A)$ , then  $H^*(E_\xi, A)$  is a free  $H^*(B_\xi, A)$ -module with base  $(h_i)$ .

Assume that  $h_1$  induces a generator  $\bar{h}_1$  of  $H^q(F_\xi, A)$ , and use  $\bar{h}_1$  to identify  $H^q(F_\xi, A)$  with  $A$ . Then we have clearly

$$(1) \quad x = \pi_\xi^*(x^\sharp) \cdot h_1 + \sum_2^m \pi_\xi^*(b_i) \cdot h_i,$$

and this characterizes  $x^\sharp$  completely.

Let  $\mu, \nu$  be two fibre bundles with the following properties:  $E_\mu = E_\xi$ ,  $B_\mu = E_\nu$ ,  $B_\nu = B_\xi$ ,  $\pi_\xi = \pi_\nu \circ \pi_\mu$ , and the restriction of  $\pi_\mu$  to a fibre of  $\xi$  is the projection map in a fibre bundle  $\theta = (F_\xi, F_\nu, F_\mu)$ . Assume that  $\xi, \mu, \nu, \theta$  satisfy (i), (ii), (iii), (with of course  $q$  depending on the fibre bundle); let  $h_\mu, h_\nu$  be homogeneous elements of  $H^*(E_\mu, A)$ ,  $H^*(E_\nu, A)$  whose restrictions to a fibre generate the highest non-vanishing cohomology groups. Then  $\pi_\mu^*(h_\nu) \cdot h_\mu = h_\xi$  has the same property in  $\xi$ . If these elements are used to identify the corresponding cohomology groups of the fibres with  $A$ , then it follows immediately from (1) that

$$(2) \quad \natural_\xi = \natural_\nu \circ \natural_\mu.$$

When  $\xi, \mu, \nu, \theta$  are fibre bundles satisfying (i), (ii) whose total spaces, fibres and base spaces are compact oriented manifolds, then (2) follows directly from the equivalence with the Gysin homomorphism mentioned in 8.3; it was shown to us to be true in general by Puppe, but since this is not needed here, we shall not reproduce the somewhat longer proof of this fact.

### Chapter III. Roots and Characteristic Classes.

**9. Characteristic classes.** We recall here the definitions of Chern, Stiefel-Whitney and Pontrjagin classes to be used in this paper, i.e., mainly the definitions which use universal bundles and flag manifolds.  $S(x_1, \dots, x_n)$  is the ring of symmetric polynomials in the  $x_i$ 's, with respect to a ring of coefficients which the context will make precise.  $S\{x_1, \dots, x_n\}$  is the corresponding ring of symmetric formal power series.

**9.1. Chern classes.** Let  $\xi$  be a principal  $U(n)$ -bundle. Its  $i$ -th Chern class is denoted by  $c_i$  or  $c_i(\xi)$ , ( $c_i \in H^{2i}(B_\xi, \mathbf{Z})$ ), and  $c$  or  $c(\xi)$  is the sum of

the  $c_i$ 's. It may be defined as follows: let  $d_j$  ( $1 \leq j \leq n$ ) be the complex lines spanned by the canonical basis vectors of  $\mathbf{C}^n$  and let  $\mathbf{T}$  be the group of diagonal matrices in  $\mathbf{U}(n)$ ; the group  $\mathbf{T}$  is a maximal torus of  $\mathbf{U}(n)$ , its largest subgroup leaving the  $d_j$ 's invariant. In the universal covering  $V$  of  $\mathbf{T}$ , we introduce coordinates  $x_j$  such that  $x = (x_1, \dots, x_n)$  operates on  $d_j$  by  $z \rightarrow z \cdot \exp(2\pi i x_j)$ ; in other words,  $x_j$  is such that, for small positive values of  $x_j$ , the product  $z \wedge x(z)$  defines the natural orientation of  $d_j$ . The restrictions of the  $x_j$  to the unit lattice define a basis of  $\text{Hom}(H_1(\mathbf{T}, \mathbf{Z}), \mathbf{Z})$  and thus a basis of  $H^1(\mathbf{T}, \mathbf{Z})$ , and they are, moreover, permuted by the Weyl group  $W(\mathbf{U}(n))$  of  $\mathbf{U}(n)$ . Let  $y_j = -\tau(x_j)$ , where  $\tau$  is the transgression in  $(E_\xi, E_\xi/\mathbf{T}, \mathbf{T})$ , and let  $\rho$  be the projection of  $E_\xi/\mathbf{T}$  onto  $B_\xi$ . Then  $y_i \in H^2(E_\xi/\mathbf{T}, \mathbf{Z})$  and  $c(\xi)$  is defined by

$$\rho^*(c(\xi)) = \prod_1^n (1 + y_i).$$

To legitimize this, we have to know that the right hand side is in the image of  $\rho^*$  and that  $\rho^*$  is injective. It suffices to prove the first point in the universal bundle, in view of the commutative diagram

$$\begin{array}{ccccc} E_\xi & \longrightarrow & E_\xi/\mathbf{T} & \xrightarrow{\rho} & B_\xi \\ \downarrow & & \downarrow & & \downarrow \sigma \\ E_{U(n)} & \longrightarrow & E_{U(n)}/\mathbf{T} & \xrightarrow{\rho'} & B_{U(n)} \end{array}$$

where  $\sigma$  is a characteristic map, but there it follows from [2, § 29] since  $\rho$  is by definition  $\rho^*(\mathbf{T}, \mathbf{U}(n))$ . As to the second point,  $H^*(\mathbf{U}(n)/\mathbf{T}, \mathbf{Z})$  is equal to its characteristic subalgebra [2, Prop. 29.2]; hence  $\mathbf{U}(n)/\mathbf{T}$  is totally non-homologous to zero in any fibre bundle of the type  $(E_\xi/\mathbf{T}, B_\xi, \mathbf{U}(n)/\mathbf{T})$ , where  $\xi$  is a principal  $\mathbf{U}(n)$ -bundle ([2], Cor. to Prop. 18.3), and this implies, in particular, that  $\rho^*$  is injective.

Let us call here a flag or, more precisely, a complex flag an ordered system of  $n$  mutually orthogonal 1-dimensional subspaces of  $\mathbf{C}^n$ . Then  $\mathbf{U}(n)/\mathbf{T}$  is the space of flags and  $E_\xi/\mathbf{T}$  is the total space of the bundle of flags in the complex vector bundle  $\xi_1$  associated to  $\xi$ ; the bundle  $\eta$  induced from  $\xi_1$  by  $\rho$ , with base space  $E_\xi/\mathbf{T}$ , decomposes into a Whitney sum of  $n$   $\mathbf{C}^1$ -vector-bundles with characteristic classes  $y_i$ . Thus the present definition of  $c(\xi)$  is quite analogous to that of [19, § 4] and, in fact, will be shown in Appendix I to be equivalent to it.

From the properties of  $\rho^*$  quoted above, it follows that  $\rho^{**}: H^{**}(B_\xi, \mathbf{R}) \rightarrow H^{**}(E_\xi/\mathbf{T}, \mathbf{R})$  is injective and has an image containing the formal power

series in the  $y_i$ 's which are symmetric. Thus we may introduce the *Chern character*  $ch(\xi)$  of  $\xi$  as an element of  $H^{**}(B_\xi, \mathbf{R})$  by

$$\rho^*(ch(\xi)) = \exp y_1 + \cdots + \exp y_n = \sum_{j \geq 0} (j!)^{-1} (y_1^j + \cdots + y_n^j).$$

Clearly,  $ch(\xi)$  and  $c(\xi)$ , both regarded as elements of  $H^{**}(B_\xi, \mathbf{R})$ , determine each other;  $ch(\xi)$  is denoted by  $t(\xi)$  in [19].

9.2. *The Stiefel-Whitney classes mod 2.* Let  $\xi$  be a principal  $\mathbf{O}(n)$ -bundle; its  $i$ -th Stiefel-Whitney class mod 2 is denoted by  $w_i$  or  $w_i(\xi)$ , ( $w_i \in H^i(B_\xi, \mathbf{Z}_2)$ ), and the sum of the  $w_i$  by  $w$  or  $w(\xi)$ . By naturality, it is enough to define it in the universal bundle. Let  $\mathbf{Q}$  be the subgroup of diagonal matrices in  $\mathbf{O}(n)$ ; we have  $H^*(B_{\mathbf{Q}(n)}, \mathbf{Z}_2) = \mathbf{Z}_2[u_1, \dots, u_n]$ , where the  $u_i$ 's are 1-dimensional classes, which may be assumed to be permuted among themselves by the normalizer of  $\mathbf{Q}$  in  $\mathbf{O}(n)$ , and  $\rho^*(\mathbf{Q}, \mathbf{O}(n))$  maps  $H^*(B_{\mathbf{O}(n)}, \mathbf{Z}_2)$  isomorphically onto  $S(u_1, \dots, u_n)$ . Then  $w$  is defined by

$$\rho^*(\mathbf{Q}(n), \mathbf{O}(n))(w) = \prod_1^n (1 + u_j)$$

(see [3]). This can also be expressed by means of flags. In fact,  $\mathbf{O}(n)/\mathbf{Q}$  is the space of flags (i.e., of ordered systems of  $n$  mutually orthogonal lines) in  $\mathbf{R}^n$  and  $(E_\xi/\mathbf{Q}, B_\xi, \mathbf{O}(n)/\mathbf{Q})$  is the bundle of flags in the vector bundle associated to  $\xi$ . Let  $u'_j$  be the image of  $u_j$  under the characteristic map of  $(E_\xi, E_\xi/\mathbf{Q}, \mathbf{Q})$ , and let  $\rho$  be the projection of  $E_\xi/\mathbf{Q}$  on  $B_\xi$ . Then

$$\rho^*(w(\xi)) = \prod_1^n (1 + u'_j),$$

and this characterizes  $w(\xi)$  since  $\rho^*$  is injective by [3], Remark on p. 177, and [2], Cor. to Prop. 18.3.

For an  $\mathbf{SO}(n)$  bundle, the Stiefel-Whitney classes mod 2 are defined as those of the extension to  $\mathbf{O}(n)$ .

9.3. *The Pontrjagin classes.* Let  $\xi$  be a principal  $\mathbf{O}(n)$ - or  $\mathbf{SO}(n)$ -bundle. Its  $i$ -th Pontrjagin class  $p_i$  or  $p_i(\xi)$  is the  $2i$ -th Chern class of the unitary extension of  $\xi$  multiplied by  $(-1)^i$ , and  $p$  or  $p(\xi)$  is the sum of the  $p_i$ 's. It may also be characterized in the following way: for  $n = 2m$ ,  $2m + 1$ , let  $d_j$  be the 2-dimensional subspaces of  $\mathbf{R}^n$  spanned by the  $(2j - 1)$ -th and  $2j$ -th canonical basis vectors, and let  $\mathbf{T}$  be the maximal subgroup of  $\mathbf{SO}(n)$  leaving the  $d_j$ 's invariant; it is a maximal torus. We choose coordinates  $x_j$  in its universal covering such that  $x = (x_1, \dots, x_m)$  operates on  $d_j$  by means of a rotation of angle  $2\pi x_j$ ; for  $n = 2m$ , we require that for small

positive values of the  $x_j$ 's, the exterior product  $v_j \wedge x(v_j)$  ( $v_j \in d_j$ ,  $v_j \neq 0$ ,  $j=1, \dots, m$ ) defines in  $d_j$  the same orientation as the  $(2j-1)$ -th and  $2j$ -th canonical basis vectors of  $\mathbf{R}^n$ . This determines the  $x_j$ 's completely. Let us consider the  $x_j$ 's as a basis of  $H^1(\mathbf{T}, \mathbf{Z})$  and put  $y_j = -\tau(x_j)$ , where  $\tau$  is the transgression in the universal bundle. It follows from the definition and from the computations made in [2], proof of Prop. 31.4 (see also 9.4), that

$$(1) \quad \rho_{\mathbf{Z}}^*(\mathbf{T}, G)(p) = \prod_1^m (1 + y_i^2).$$

In § 30, we shall see that

$$(2) \quad \rho_{\mathbf{Z}}^*(\mathbf{T}, G)(p_i) = w_{2i}^2, \quad (G = \mathbf{SO}(n), \mathbf{O}(n)),$$

and that  $p$  is completely characterized by (1) and (2). The Pontrjagin classes mod  $p$  ( $p \neq 2$ ), may also be defined by going over to a bundle of flags. In  $\mathbf{R}^n$  ( $n=2m, 2m+1$ ), we call a 2-flag an ordered system of  $m$  mutually orthogonal 2-dimensional oriented subspaces. Then the space of 2-flags in  $\mathbf{R}^n$  is  $\mathbf{O}(n)/\mathbf{T}$  for  $n$  even or  $\mathbf{O}(n)/\mathbf{T}'$  for  $n$  odd, where  $\mathbf{T}'$  is an extension of  $\mathbf{T}$  by  $\mathbf{Z}_2$ . Let  $\xi$  be a principal  $\mathbf{O}(n)$ -bundle and  $\rho$  be the projection of  $E_{\xi}/\mathbf{T}$  or  $E_{\xi}/\mathbf{T}'$  on  $B_{\xi}$ . Then we have

$$\rho^*(p(\xi)) = \prod (1 + \tau(x_i)^2),$$

where  $\tau$  is the transgression in the canonical principal  $\mathbf{T}$ -bundle over  $E_{\xi}/\mathbf{T}$  or  $E_{\xi}/\mathbf{T}'$  respectively. This is valid over the integers; however  $\rho^*$  is injective in general only for the cohomology mod  $p$  ( $p \neq 2$ ), (again by [2], § 29 and Cor. to Prop. 18.3, since  $\mathbf{SO}(n)$  and  $\mathbf{O}(n)$  have no  $p$ -torsion for  $p \neq 2$ ).

The  $(2i+1)$ -th Chern class of the complex extension will be denoted  $p_{i+1}$ ; it is an element of order 2, equal to the square of the integral  $(2i+1)$ -th Stiefel-Whitney class (see Appendix II);  $\bar{p}$  or  $\bar{p}(\xi)$  will be the sum of the  $p_i$  and  $p_{i+1}$ . We have  $\rho_{\mathbf{Z}}^*(\mathbf{T}, G)(p) = \rho_{\mathbf{Z}}^*(\mathbf{T}, G)(\bar{p})$ .

9.4. *Remark on the complex extension.* For computational convenience, we shall take as a complex extension of an  $\mathbf{O}(n)$ -bundle the  $\lambda$ -extension, where  $\lambda = \delta \circ \gamma$  is the product of the injection  $\gamma: \mathbf{O}(n) \rightarrow \mathbf{U}(n)$  and of the inner automorphism  $\delta: x \rightarrow gxg^{-1}$ , the element  $g$  being a direct sum of  $2 \times 2$  matrices

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

and of (1) for odd  $n$ ; since it is equivalent to the  $\gamma$ -extension, this does not alter the Chern classes. The maximal torus  $\mathbf{T}$  of  $\mathbf{O}(n)$ , previously described,

is then mapped onto the diagonal matrices with coefficients  $\exp(\pm 2\pi i x_j)$ , and 1 for odd  $n$ . We have

$$\begin{aligned}\lambda^*(x'_{2j+1}) &= -\lambda^*(x'_{2j}) = x_j, & (1 \leq j \leq [n/2]) \\ \lambda^*(x'_n) &= 0, & (n \text{ odd}),\end{aligned}$$

where  $(x_j)$ ,  $(x'_j)$  are the bases of the first integral cohomology groups of the standard maximal tori of  $O(n)$  and  $U(n)$  described before.

9.5. *The Euler-Poincaré class.* Let  $\xi$  be an oriented vector bundle with  $2m$ -dimensional fibre, structural group  $SO(2m)$ , and let  $\eta$  be the associated bundle of unit spheres. The Euler-Poincaré class  $W_{2m}(\xi)$  or  $W_{2m}$  of  $\xi$  is equal to  $-\tau(x)$ , where  $x$  is the generator of  $H^{2m-1}(S_{2m-1}, \mathbf{Z})$  defined by the positive orientation, and  $\tau$  is the transgression in  $\eta$ . In the universal case, it is also characterized by the two properties:

- (i)  $W_{2m}$ , reduced mod 2, is equal to  $w_{2m}$ ,
- (ii)  $\rho_{\mathbf{Z}}^*(T, SO(2m))(W_{2m}) = y_1 \cdots y_m$ .

For the tangent bundle to a differentiable, compact, connected, oriented manifold,  $W_{2n}$  is the fundamental class multiplied by the Euler-Poincaré characteristic.

9.6. *Symplectic Pontrjagin classes.* Let  $\xi$  be a  $Sp(n)$  bundle and  $\eta$  its extension under the standard inclusion of  $Sp(n)$  in  $U(2n)$ . Its  $i$ -th symplectic Pontrjagin class  $e_i(\xi)$  or  $e_i$  is by definition

$$e_i(\xi) = (-1)^i c_{2i}(\xi),$$

and its total symplectic Pontrjagin class  $e(\xi)$  or  $e$  is the sum of the  $e_i$ 's. The computations made in the proof of Prop. 31.3 in [2] show then that the universal symplectic Pontrjagin class satisfies

$$(3) \quad \rho_{\mathbf{Z}}^*(T, Sp(n))(e) = \prod_{i=1}^n (1 + y_i^2), \quad (T \text{ a maximal torus of } Sp(n)),$$

where the  $y_i$ 's form a base of  $H^2(B_T, \mathbf{Z})$  whose elements are permuted, up to sign, by  $W(Sp(n))$ . Moreover, it follows from ([2], § 9, 29) that  $H^*(B_{Sp(n)}, \mathbf{Z})$  is the ring of polynomials in the  $e_i$ 's and that  $\rho_{\mathbf{Z}}^*(T, Sp(n))$  is injective.

9.7. *The multiplication theorem.* Finally, we recall the Whitney multiplication theorem. Let

$$0 \rightarrow \xi' \rightarrow \xi \rightarrow \xi'' \rightarrow 0$$

be an exact sequence of real (resp. complex, resp. quaternionic) vector bundles, with structural group  $G = \mathbf{O}(n)$  (resp.  $\mathbf{U}(n)$ , resp.  $\mathbf{Sp}(n)$ ) for three suitable values of  $n$ . Then we have

$$(4) \quad w(\xi) = w(\xi') \cdot w(\xi'') \quad (G = \mathbf{O}(n)),$$

$$(5) \quad \tilde{p}(\xi) = \tilde{p}(\xi') \cdot \tilde{p}(\xi'') \quad (G = \mathbf{O}(n)),$$

$$(6) \quad c(\xi) = c(\xi') \cdot c(\xi'') \quad (G = \mathbf{U}(n)),$$

$$(7) \quad e(\xi) = e(\xi') \cdot e(\xi'') \quad (G = \mathbf{Sp}(n)).$$

(4) and (6) are classical; (5) and (7) follow from (6) and the definitions. We note that, in view of (5), we also have

$$(8) \quad p(\xi) \equiv p(\xi') \cdot p(\xi'') \quad \text{modulo 2-torsion.}$$

These formulae imply, in particular, that  $w$  or  $\tilde{p}$  (resp.  $c$ , resp.  $e$ ) is invariant under an extension relative to the standard inclusion  $\mathbf{O}(k) \subset \mathbf{O}(m)$  (resp.  $\mathbf{U}(k) \subset \mathbf{U}(m)$ , resp.  $\mathbf{Sp}(k) \subset \mathbf{Sp}(m)$ ), ( $m \geq k$ ).

## 10. Representations and characteristic classes.

10.1. *Integral forms as cohomology classes.* Let  $T$  be a torus,  $V$  its universal covering,  $\Gamma$  the unit lattice, and  $\Gamma^* = \text{Hom}(\Gamma, \mathbf{Z})$ . Thus  $\Gamma^* \cong H^1(T, \mathbf{Z})$ , and, for any commutative group,  $\Gamma^* \otimes A \cong H^1(T, A)$ . We shall make this identification and, in particular, identify  $H^1(T, \mathbf{R})$  with  $V^*$  and  $H^1(T, \mathbf{Z})$  with the integral linear forms on  $V$ . Also, the roots discussed in Chap. I will be considered in this way as elements of  $H^1(T, \mathbf{Z})$  or  $H^1(T, A)$ .

Let  $\xi$  be a principal  $T$ -bundle. Then  $\tau_\xi$  maps all of  $H^1(T, A)$  in  $H^2(B_\xi, A)$ . Unless this may lead to a confusion, we shall denote by the same symbol  $\omega \in \Gamma^* \otimes A$ , the corresponding element in  $H^1(T, A)$ , and  $-\tau_\xi(\omega) \in H^2(B_\xi, A)$ .

Let  $G$  be a compact connected Lie group and let  $T$  be a maximal torus of  $G$ . First assume  $G$  to be semi-simple and simply connected. Then the transgression in  $(G, G/T, T)$  is an isomorphism of  $H^1(T, \mathbf{Z})$  onto  $H^2(G/T, \mathbf{Z})$  since  $G/T$  is simply connected. Thus the previous conventions identify  $H^2(G/T, \mathbf{Z})$  and  $H^1(T, \mathbf{Z})$  with the weights of  $G$  (see 3.3). Let  $G^*$  be the quotient of  $G$  by a finite invariant subgroup and  $T^*$  the image of  $T$  under the natural map of  $G$  onto  $G^*$ . Then  $G/T$  is homeomorphic to  $G^*/T^*$ , as is well known (see e.g. [2], §26); however the transgression in  $(G^*, G^*/T^*, T^*)$  will be an isomorphism of  $H^1(T^*, \mathbf{Z})$  onto the subgroup of  $H^2(G/T, \mathbf{Z})$  corresponding to the weights which are integral on the unit

lattice of  $G^*$ . In the general case, let  $G_1$  be the greatest semi-simple subgroup of  $G$  (2.9); since maximal tori are maximal abelian subgroups,  $T_1 = G_1 \cap T$  is a maximal torus of  $G_1$ , and moreover (2.9),  $G/T$  may be identified with  $G_1/T_1$ . Since a toral subgroup of a torus is always a direct factor, the map  $H^1(T, \mathbb{Z}) \rightarrow H^1(T_1, \mathbb{Z})$ , induced by inclusion, is surjective, and the map:  $\mu = (G_1, G_1/T_1, T_1) \rightarrow \nu = (G, G/T, T)$ , defined by inclusion, shows then that  $\tau_\nu(H^1(T, \mathbb{Z})) = \tau_\mu(H^1(T_1, \mathbb{Z}))$ .

Let now  $G'$  be the quotient of  $G$  by a closed subgroup of the center,  $\pi: G \rightarrow G'$  the projection,  $U$  a connected subgroup of maximal rank in  $G$ ,  $U_1 = G_1 \cap U$ ,  $U' = \pi(U)$ . Then  $\text{rank } U_1 = \text{rank } G_1$ ,  $\text{rank } U' = \text{rank } G'$ , and it follows from 2.9 that  $G/U = G_1/U_1$ . Also, the argument of [2, § 26] referred to above shows that  $U$  is the full inverse image of  $U'$  in  $G$ , and, consequently, that  $G/U = G'/U'$ . Therefore, when we deal with coset spaces  $G/U$  ( $\text{rank } G = \text{rank } U$ ), there is no loss in generality in assuming that  $G$  is semi-simple and simply connected.

10.2. *The weights and the character of a homomorphism.* Let  $G, G'$  be two compact Lie groups,  $\lambda: G \rightarrow G'$  a homomorphism,  $T$  and  $T'$  toral subgroups of  $G$  and  $G'$  such that  $\lambda(T) \subset T'$ , and  $(x'_i)$  a base of  $H^1(T, \mathbb{Z})$ . Then  $\lambda$  induces homomorphisms of  $H^1(T', \mathbb{Z})$  and  $V'^*$  in  $H^1(T, \mathbb{Z})$  and  $V^*$ , both to be denoted by  $\lambda^*$ . The elements  $\omega_i = \lambda^*(x'_i)$ , viewed either as elements of  $H^1(T, \mathbb{Z})$  or as integral linear forms, will be called the  $(T, T')$ -weights of  $\lambda$ , or simply the weight of  $\lambda$  when  $T$  and  $T'$  are maximal.<sup>7</sup> The formal power series

$$ch(\lambda) = \sum \exp \omega_i$$

considered as an element of  $H^{**}(B_T, \mathbb{R})$  or of  $H^{**}(B_{\xi}, \mathbb{R})$ , where  $\xi$  is a principal  $T$ -bundle, will be called the *character* of  $\lambda$ .

Assume now  $T, T'$  to be maximal and  $G' = U(n)$ . Then for  $t \in T$ , the matrix  $\lambda(t)$  is diagonal with the coefficients  $\exp(2\pi i \omega_j)$ ; in other words, the  $\omega_j$  and the sum of the exponentials of the  $2\pi i \omega_j$  are, respectively, the weights and the character of the representation  $\lambda$  in the usual sense.

In the case  $G' = O(n)$ , i.e., of a real linear representation, we have analogously

$$\lambda(x) = \begin{bmatrix} D(2\pi\omega_1) & & 0 \\ & \ddots & \\ 0 & & D(2\pi\omega_m) \end{bmatrix} \quad \lambda(x) = \begin{bmatrix} D(2\pi\omega_1) & & 0 \\ & \ddots & \\ 0 & & D(2\pi\omega_m) \\ & & & 1 \end{bmatrix}$$

<sup>7</sup> More precisely, with respect to the basis  $(x'_i)$ , which is always supposed to be chosen as in § 9 when  $G'$  is a classical group.

for  $n = 2m$  and  $n = 2m + 1$  respectively, where  $D(\alpha)$  is a 2-dimensional rotation of angle  $\alpha$ ; the weights of  $\lambda$ , considered as a representation in  $U(n)$ , are then the forms  $\pm \omega_j$ , together with the zero form for odd  $n$ .

10.3. THEOREM. Let  $G, G'$  be two compact Lie groups,  $\lambda: G \rightarrow G'$  a homomorphism,  $T, T'$  maximal tori of  $G$  and  $G'$  such that  $\lambda(T) \subset T'$ , and  $(\omega_j)$  the weights of  $\lambda$ . Let  $\xi$  be a principal  $G$ -bundle,  $\eta$  its  $\lambda$ -extension,  $\rho$  the projection of  $E_\xi/T$  onto  $B_\xi$ . Then

(a) If  $G' = U(m)$ , then

$$\rho^*(c(\eta)) = \prod (1 + \omega_j); \quad \rho^{**}(ch(\eta)) = ch \lambda.$$

(b) If  $G' = SO(m)$  or  $O(m)$ , the Pontrjagin class  $p(\eta)$  satisfies

$$\rho^*(p(\eta)) = \rho^*(\tilde{p}(\eta)) = \prod (1 + (\omega_j)^2).$$

(c) If  $G' = SO(2m)$ , the Euler-Poincaré class  $W_{2m}(\eta)$  satisfies

$$\rho^*(W_{2m}(\eta)) = \prod \omega_j.$$

(a) We have a commutative diagram

$$(1) \quad \begin{array}{ccccc} E_\xi & \longrightarrow & E_\xi/T & \xrightarrow{\rho} & B_\xi \\ \phi \downarrow & & \downarrow \phi_1 & & \downarrow \\ E_\eta & \longrightarrow & E_\eta/T' & \xrightarrow{\rho'} & B_\eta \end{array}$$

where  $\phi$  is a  $\lambda$ -map. By (9.1), putting  $c'$  for  $c(\eta)$ , we have

$$\rho^*(c') = \prod (1 - \tau'(x'_i)),$$

where  $\tau'$  is the transgression in  $(E_\eta, E_\eta/T', T')$ ; and therefore

$$\rho^*(c') = \phi_1^* \cdot \rho^*(c') = \prod (1 - \phi_1^* \tau'(x'_i)).$$

Since  $\phi$  commutes with transgression, this gives

$$\rho^*(c') = \prod (1 - \tau \lambda^*(x'_j))$$

and our assertion follows from the definition of the weights and the notation convention of (10.1). The proofs for (b) and (c) are similar.

10.4. COROLLARY. Let  $G = U(n)$ ,  $G' = U(m)$ ,  $T$  the standard maximal torus of  $G$ ,  $\omega_j = \sum_i a_{ij} x_i$  the weights of  $\lambda$  expressed in terms of the canonical basis of  $H^1(T, \mathbb{Z})$ ,  $(i = 1, \dots, n; j = 1, \dots, m)$ . Then

$$c(\eta) = \prod (1 + \sum_i a_{ij} y_i)$$

where the  $y_i$ 's are formally defined by  $c(\xi) = \prod (1 + y_i)$ . The class  $c(\eta)$  is a polynomial with integral coefficients in the classes  $c_i(\xi)$ .

By (9.1), we have  $\rho^*(c(\xi)) = \prod (1 - \tau(x_i))$ , and our first assertion follows from 10.3 and the fact that  $\rho^*$  is injective when  $G$  is the unitary group. Moreover, the Weyl group  $W(U(n))$  operates in a natural way on the fibration  $(E_\xi/T, B_\xi, U(n)/T, \rho)$  and induces the identity on  $B_\xi$  (see [2], § 27). Therefore the image of  $\rho^*$ , and in particular,  $\rho^*c(\eta)$ , is made up of invariants of  $W(U(n))$ ; since the latter is the group of permutations of the  $x_i$ , or equivalently, of the  $\tau(x_i)$ , it follows that  $c(\eta)$  is a symmetric function in the  $y_i$ 's, whence our second assertion.

10.5. COROLLARY. Let  $G = O(n)$  or  $SO(n)$ ,  $G'$  be  $O(m)$  or  $SO(m)$ . Then  $p(\eta)$  reduced mod  $p$  ( $p \neq 2$ ), is a polynomial in the  $p_i(\xi)$ , and if  $G = SO(2m)$ , in  $W_{2m}(\xi)$ . If  $\lambda$  can be extended to a homomorphism of  $U(n)$  into  $U(m)$ , then  $\bar{p}(\eta)$  is a polynomial in the classes  $p_i(\xi)$ ,  $p_{i+1}(\xi)$ , and, in particular,  $p(\eta)$  reduced mod  $p$  ( $p \neq 2$ ), is a polynomial in the classes  $p_i(\xi)$ .

The first assertion is proved in the same way as 10.4, using 9.3, 9.5 and the properties of the invariants of  $W(G)$  recalled in 30.2. The second one follows from 10.4 by considering the Pontrjagin classes as the Chern classes of the complex extension.

10.6. Examples. (a) Let  $\xi$  be a complex vector bundle,  $\xi^*$  the dual bundle. Then, if  $c(\xi) = \prod (1 + x_i)$ , we have  $c(\xi^*) = \prod (1 - x_i)$ . In fact, the principal bundle  $\theta$  associated to  $\xi^*$  is the  $\lambda$ -extension of the principal bundle of  $\xi$ , where  $\lambda$  is the contragredient representation, whose weights are obviously the forms  $-x_i$ .

(b) Let  $G = U(n)$ ,  $j$  be a positive integer  $\leq n$ , and  $\lambda$  the natural representation of  $U(n)$  in the  $j$ -th exterior power  $\wedge^j \mathbf{C}^n$  of  $\mathbf{C}^n$ . Let  $(e_i)$  be the canonical base of  $\mathbf{C}^n$ . Then the products

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j} \quad (1 \leq i_1 < \cdots < i_j \leq n)$$

form a base of  $\wedge^j \mathbf{C}^n$ , and we have

$$\lambda(x)(e_{i_1} \wedge \cdots \wedge e_{i_j}) = \exp[2\pi i(x_{i_1} + \cdots + x_{i_j})](e_{i_1} \wedge \cdots \wedge e_{i_j});$$

i.e., the weights of  $\lambda$  are the sums

$$x_{i_1} + \cdots + x_{i_j}, \quad (1 \leq i_1 < \cdots < i_j \leq n).$$

Here  $\eta$  is the principal bundle associated to the bundle of contravariant  $p$ -

vectors in the complex vector bundle associated to  $\xi$ . This bundle has, therefore, as Chern class

$$c' = \prod_{1 \leq i_1 < \dots < i_j \leq n} (1 + x_{i_1} + \dots + x_{i_j}).$$

(c) In the same way, the Chern class of the bundle of contravariant symmetric tensors of degree  $j$  will be

$$\prod_{1 \leq i_1 \leq \dots \leq i_j \leq n} (1 + x_{i_1} + \dots + x_{i_j}).$$

(d) Let  $\xi_i$  ( $i=1, 2$ ), be two complex vector bundles over  $B$  and let

$$c_{(i)} = \prod_{j=1}^{n_i} (1 + x_j^{(i)})$$

be formal decompositions of their Chern polynomials. Then

$$c(\xi_1 \otimes \xi_2) = \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (1 + x_i^{(1)} + x_j^{(2)}).$$

To see this, we take as  $\xi$  the principal bundle with group  $U(n_1) \times U(n_2)$  associated to the sum  $\xi_1 \oplus \xi_2$ , whose Chern class is  $c_{(1)} \cdot c_{(2)}$  by the multiplication theorem (9.7), and as  $\lambda$  the representation of  $U(n_1) \times U(n_2)$  in  $U(n_1 \cdot n_2)$  defined by  $(g_1, g_2) \rightarrow g_1 \otimes g_2$ , considered as an automorphism of  $C^{n_1} \otimes C^{n_2}$ . The products  $e_i \otimes f_j$ , where  $(e_i)$  and  $(f_j)$  are the canonical bases of  $C^{n_1}$  and  $C^{n_2}$ , form a base of  $C^{n_1} \otimes C^{n_2}$ ; hence the weights of  $\lambda$  are the forms  $x_i^{(1)} + x_j^{(2)}$ , and our contention follows from (10.3) and from the fact that the  $\lambda$ -extension of  $\xi$  is the principal bundle of  $\xi_1 \otimes \xi_2$ .

(e) To compute the Pontrjagin classes of real vector bundles, it is often more convenient to look at the Chern classes of the complex extensions; as an illustration, we take the case where  $G = SO(2n)$  and  $\lambda$  is the representation in  $\wedge^2 \mathbf{R}^{2n}$ . Let  $\mu, \nu$  denote the complex extensions of  $\xi, \eta$  (as defined in 9.4), let  $\mathbf{T}, \mathbf{T}'$  be the standard maximal tori of  $SO(2n), U(2n)$ , and let  $(x_i), (x'_i)$  be the canonical bases of  $H^1(\mathbf{T}, \mathbf{Z})$  and  $H^1(\mathbf{T}', \mathbf{Z})$ . We have a commutative diagram

$$\begin{array}{ccccc} E_\xi & \longrightarrow & E_\xi/\mathbf{T} & \xrightarrow{\rho} & B_\xi \\ \phi \downarrow & & \phi_1 \downarrow & & \downarrow \text{Id} \\ E_\mu & \longrightarrow & E_\mu/\mathbf{T}' & \xrightarrow{\sigma} & B_{\xi'} \end{array}$$

and it follows from (9.4), (10.3) that

$$x_i = \lambda^*(x'_{2i-1}) = -\lambda^*(x'_{2i}), \quad (1 \leq i \leq n),$$

$$\rho^*(p(\xi)) = \prod_1^n (1 + x_i^2),$$

$$\sigma^*(c(\mu)) = \prod_1^{2n} (1 + x'_j).$$

Now,  $\nu$  is clearly the extension of  $\mu$  corresponding to the "complexification of  $\lambda$ ," i.e., to the natural representation of  $U(2n)$  in  $\wedge^2 \mathbb{C}^{2n}$ ; therefore, by example (b),

$$\sigma^*(c(\nu)) = \prod_{1 \leq i < j \leq 2n} (1 + x'_i + x'_j).$$

This gives

$$\phi^*_1 \cdot \sigma^*(c(\nu)) = \prod_{1 \leq i < j \leq n} (1 - (x_i + x_j)^2) \cdot (1 - (x_i - x_j)^2),$$

$$\rho^*(c(\nu)) = \prod_{1 \leq i < j \leq n} [(1 - x_i^2 - x_j^2)^2 - 4 \cdot x_i^2 \cdot x_j^2],$$

and finally

$$\rho^*(\tilde{p}(\eta)) = \rho^*(p(\eta)) = \prod_{1 \leq i < j \leq n} [(1 + x_i^2 + x_j^2)^2 - 4 \cdot x_i^2 \cdot x_j^2].$$

(f) It may be shown in the same way that if  $\xi_1, \xi_2$  are two real vector bundles over the same base space  $B$  with Pontrjagin classes reduced mod  $p$  ( $p \neq 2$ ), equal to

$$p(\xi_1) = \prod_1^m (1 + x_i^2), \quad p(\xi_2) = \prod_1^n (1 + y_j^2),$$

then

$$p(\xi_1 \otimes \xi_2) = \prod_{i=1}^m \prod_{j=1}^n (1 + (x_i + y_j)^2) \cdot (1 + (x_i - y_j)^2).$$

10.7. THEOREM. Let  $G$  be a compact connected Lie group,  $U$  a closed subgroup of  $G$ ,  $S$  a maximal torus of  $U$ , and  $(\pm b_j)$  ( $1 \leq j \leq k$ ), the roots of  $G$  with respect to  $S$  which are complementary to those of  $U$ . Let  $\xi$  be a principal  $G$ -bundle,  $\rho$  the projection of  $E_\xi/S$  onto  $E_\xi/U$ , and  $\eta$  the bundle along the fibres (7.4) of  $(E_\xi/U, B_\xi, G/U)$ . Then  $\rho^*(\tilde{p}(\eta)) = \prod (1 + b_j^2)$ ; if, moreover,  $U$  is connected and  $\dim G/U = m$  is even, then  $\rho^*(W_m(\eta)) = \pm \prod b_j$ .

By (7.5),  $\eta$  is the  $\iota$ -extension of  $(E_\xi, E_\xi/U, U)$ , where  $\iota$  is the isotropy representation; according to the definitions in (1.3) and (10.1), the  $b_j$ 's are the weights of  $\iota$  (up to a certain number of zero forms, but this does not alter our formulas), and (10.7) follows then from (10.3). The sign for the Euler-Poincaré class will be determined by the conventions made in 9.5, once the bundle along the fibres has been oriented.

10.8. We keep the previous notations. Let  $J$  be an invariant almost complex structure on  $G/U$  and  $\eta'$  be the complex vector bundle structure of  $\eta$  constructed by means of  $J$  (see 7.4).  $J$  is defined by a complex structure on the tangent space  $(G/U)_o$ , invariant under  $U$ ; this complex structure gives rise to a linear representation  $\iota_c$  of  $U$  in  $\mathbb{C}^{m/2}$  ( $m = \dim G/U$ ), which goes over to  $\iota$  by taking real and imaginary parts. The weights of  $\iota_c$  are some of the forms  $\pm b_j$ , and (10.3) also implies:

**THEOREM.** *We keep the notations of 10.7; assume, moreover, that  $G/U$  has an invariant almost complex structure  $J$ , and denote by  $\eta'$  the complex vector bundle structure of  $\eta$  associated to  $J$ . Then  $\rho^*(c(\eta')) = \prod_{j \in J} (1 + \epsilon_j b_j)$ , ( $\epsilon_j = \pm 1$ ), where  $\epsilon_j b_j$  runs through the weights of the complex isotropy representation  $\iota_c$  defining  $J$ .*

The weights of  $\iota_c$  will be discussed in detail for the case where  $\text{rank } G = \text{rank } U$  in Chapter IV.

10.9. In order to study the tangent bundle to  $G/U$  it is usually convenient to consider the bundle  $\hat{\theta}$  along the fibres of  $(B_U, B_G, G/U, \rho(U, G)) = \theta$  and restrict to a fibre of  $\theta$ , since this allows one to make use of known results about classifying spaces. We consider here, in particular, the case where  $U = T$  is a maximal torus and show the

**PROPOSITION.** *Let  $T$  be a maximal torus of  $G$ . Then the total Pontrjagin class  $\tilde{p}(\mu)$  of the tangent bundle  $\mu$  to  $G/T$  is 1.*

Let  $\theta = (B_T, B_G, G/T)$  and let  $i$  be the inclusion map of a fibre.  $H^*(G/T, \mathbb{Z})$  is torsion free ([5], or R. Bott, *Bull. Soc. Math. France* 84, (1956) 251-281), and therefore the subgroup  $S$  of invariants of  $W(G)$  in  $H^*(G/T, \mathbb{Z})$  is a free abelian group; since by a lemma of Leray (see [2], Lemma 27.1),  $H^*(G/T, \mathbb{R})$  is the space of the regular representation of  $W(G)$ , it follows that  $S = H^0(G/T, \mathbb{Z})$  and that the kernel of  $i^*$  contains the subgroup  $I_G^+$  of invariants of  $W(G)$  in  $H^*(B_T, \mathbb{Z})$  having strictly positive degrees.

By (10.7) we have

$$\tilde{p}(\hat{\theta}) = \prod (1 + b_j^2),$$

where the  $\pm b_j$ 's are the roots of  $G$ . Since  $W(G)$  permutes the  $b_j^2$ , it leaves  $\tilde{p}(\hat{\theta})$  invariant, whence

$$\tilde{p}(\mu) = i^*(\tilde{p}(\hat{\theta})) = 1.$$

**11. Representations and Stiefel-Whitney classes.** In this section,  $Q, Q'$  denote commutative groups of type  $(2, 2, \dots, 2)$ . The following discussion applies to any arbitrary compact Lie group, but has an interest only for groups in which maximal commutative subgroups of type  $(2, \dots, 2)$  are conjugate and play in cohomology mod 2 the role of maximal tori in real cohomology. We therefore assume tacitly that  $G, G'$  are products of copies of  $O(n), SO(n), U(n), SU(n), Sp(n), G_2$  (see [3]).

**11.1. Characters of  $Q$  as cohomology classes.**  $Q$  being discrete,  $H^*(B_Q, A)$  is the cohomology ring of  $Q$  in the sense of Hopf-Eilenberg-MacLane, and, in particular,  $H^1(B_Q, A) = \text{Hom}(Q, A)$ . Thus  $x \in \text{Hom}(Q, A)$  may be considered as a 1-dimensional cohomology element in  $B_Q$  or, via the characteristic map, in the base space of any principal bundle  $(E, B, Q)$ , which will be usually denoted by the same symbol ( $x \in H^1(B, A)$ ) in particular, the 2-roots introduced in § 5 will be considered as elements of  $H^1(B, \mathbb{Z}_2)$ . We note that if  $\lambda: Q \rightarrow Q'$  is a homomorphism, then

$$\rho(\lambda)^*: H^1(B_{Q'}, A) \rightarrow H^1(B_Q, A) \text{ and } \lambda': \text{Hom}(Q', A) \rightarrow \text{Hom}(Q, A)$$

are carried into one another by the previous identification.

**11.2. The 2-weights of a homomorphism.** Let  $\lambda: G \rightarrow G'$  be a homomorphism,  $Q, Q'$  maximal and such that  $\lambda(Q) \subset Q'$ , and  $(x_i), (x'_i)$  bases of  $\text{Hom}(Q, \mathbb{Z}_2)$  and  $\text{Hom}(Q', \mathbb{Z}_2)$ , considered as  $\mathbb{Z}_2$ -modules. Then  $\lambda^*: \text{Hom}(Q', \mathbb{Z}_2) \rightarrow \text{Hom}(Q, \mathbb{Z}_2)$  is characterized by elements  $\omega_j = \lambda^*(x'_j) = \sum a_{ij} x_i$ , to be called the 2-weights of  $\lambda$ . Here, also, we assume, in case of an orthogonal group, the basis to be chosen as in (9.2).

**11.3. THEOREM.** Let  $G$  be a compact Lie group,  $Q$  a maximal commutative subgroup of type  $(2, \dots, 2)$ ,  $\lambda: G \rightarrow O(n)$  a homomorphism,  $(\omega_j)$  its 2-weights,  $\xi$  a principal bundle,  $\xi'$  its  $\lambda$ -extension, and  $\rho$  the projection of  $E/Q$  onto  $B$ . Then  $\rho^*(w(\xi')) = \prod (1 + \omega_j)$ .

The proof is the same as for (10.3), except that instead of (1) § 10, we use the commutativity of the diagram

$$\begin{array}{ccc} E_{\xi}/Q & \xrightarrow{\phi^*} & E_{\xi'}/Q' \\ \downarrow \sigma & & \downarrow \sigma' \\ B_Q & \longrightarrow & B_{Q'}, \end{array}$$

where  $\sigma$  and  $\sigma'$  are characteristic maps, and the end remark of (11.1); therefore, we shall not reproduce it here.

*Remark.* For the groups mentioned at the beginning of § 11,  $\rho^*$  is injective [3].

11.4. COROLLARY. Assume, moreover, that  $G = \mathbf{O}(n)$ . Then  $w(\xi') = \prod (1 + \sum a_{ij}x_i)$ , where the  $x_i$  are the formal roots of  $w(\xi)$ . In particular,  $w(\xi')$  is a polynomial in the classes  $w_i(\xi)$ .

Same proof as for (10.4), except that instead of using the Weyl group, we take the quotient by  $Q$  of its normalizer in  $\mathbf{O}(n)$ ; its inner automorphisms also induce the group of permutations of the  $x_i$ 's.

*Examples.* Computations paralleling those of 10.6, (b), (c), (d) will lead to the same formulas for the Stiefel-Whitney classes of bundles of  $p$ -vectors, symmetric tensors, and for tensor products, the  $x_i$  and  $y_j$  standing now for 1-dimensional classes mod 2. Details are left to the reader.

11.5. THEOREM. Let  $G$  be a compact Lie group,  $U$  a closed subgroup, and  $Q$  a maximal commutative subgroup of type  $(2, \dots, 2)$  of  $U$ . Let  $\xi$  be a principal  $G$ -bundle,  $\rho$  the projection of  $E_\xi/Q$  on  $E_\xi/U$ , and  $\eta'$  the bundle along the fibres  $G/U$ . Then

$$\rho^*(w(\eta')) = \prod (1 + a_i),$$

where the  $a_i$ 's are the 2-roots of  $G$  with respect to  $Q$ , complementary to those of  $U$ .

The  $a_i$ 's are the 2-weights of the isotropy representation; hence (11.5) follows from (7.5) and (11.3).

Applications will be given in Chapter V.

## Chapter IV. Roots and Invariant Almost Complex Structures.

In this chapter,  $G$  is a compact, connected, semi-simple, Lie group,  $l$  its rank,  $U$  a proper closed connected subgroup of the same rank, and  $T$  a maximal torus of  $U$ . If  $\psi$  is a set of roots, we put  $-\psi = \{-a, a \in \psi\}$ .

12. Integrability of invariant almost complex structures. We recall here some known facts in a form convenient for the sequel.

12.1. Let  $V$  be a real  $2n$ -dimensional vector space, endowed with a complex structure defined by a linear transformation  $J$ , and let  $V^c$  be its complexification. Then

$$V^c = T^+ + T^-, \quad T^- = \overline{T^+}, \quad T^+ \cap T^- = (0),$$

where  $T^+$  (resp.  $T^-$ ) is the eigenspace of the extension  $J^c$  of  $J$  to  $V^c$  corresponding to the eigenvalue  $+i$  (resp.  $-i$ ) and where the bar denotes complex conjugation with respect to  $V$ . Conversely, given such a decomposition of  $V^c$ , we define  $J^c$  by  $J^c(x) = i \cdot x$  ( $x \in T^+$ ),  $J^c(x) = -ix$ , ( $x \in T^-$ ); then  $J^c$  leaves  $V$  invariant and induces there a complex structure such that  $x \rightarrow x + \bar{x}$  is a complex isomorphism ( $x \in T^+$ ). In particular, given a linear transformation  $A$  without real eigenvalues, we define  $T^+$  (resp.  $T^-$ ) as the sum of the eigenspaces of its semi-simple part corresponding to eigenvalues with positive (resp. negative) imaginary parts, and thus attach to  $A$  a complex structure on  $V$ .

12.2. The roots of  $G$  with respect to  $T$  define linear forms on the Lie algebra  $\mathfrak{t}$  of  $T$ , and it follows from (1.3) and standard facts about the adjoint representation that

$$\text{ad } x|_{\mathfrak{a}_i} = \begin{pmatrix} 0 & -2\pi a_i(x) \\ 2\pi a_i(x) & 0 \end{pmatrix}, \quad (x \in \mathfrak{t}),$$

$\text{ad } x$  being defined by  $(\text{ad } x)(y) = [x, y]$ , ( $x, y \in \mathfrak{g}$ ).

We have then, superscripts denoting complexification, that

$$\mathfrak{g}^c = \mathfrak{t}^c + \mathfrak{a}_1^c + \cdots + \mathfrak{a}_m^c, \quad \mathfrak{a}_i^c = \mathfrak{v}_{a_i} + \mathfrak{v}_{-a_i},$$

$$[x, e_{\pm a_j}] = \pm 2\pi i a_j(x) e_{\pm a_j}, \quad (e_{\pm a_j} \in \mathfrak{v}_{\pm a_j});$$

since any two roots are different from each other, the 1-dimensional eigenspaces  $\mathfrak{v}_{\pm a_j}$  are well determined by  $\mathfrak{t}$ . We recall that if  $\alpha, \beta$  are two roots, we have

$$\begin{aligned} [\mathfrak{v}_\alpha, \mathfrak{v}_\beta] &= 0 && \text{if } \alpha + \beta \text{ is not a root and not zero,} \\ (1) \quad [\mathfrak{v}_\alpha, \mathfrak{v}_\beta] &= \mathfrak{v}_{\alpha+\beta} && \text{if } \alpha + \beta \text{ is a root,} \\ [\mathfrak{v}_\alpha, \mathfrak{v}_{-\alpha}] &\subset \mathfrak{t}^c, [\mathfrak{v}_\alpha, \mathfrak{v}_{-\alpha}] \neq 0. \end{aligned}$$

12.3. Assume now that  $G/U$  has been endowed with an invariant almost complex structure and let  $\pm b_j$  ( $1 \leq j \leq k$ ) be the complementary roots. The almost complex structure is characterized by a linear transformation  $J$ , ( $J^2 = -\text{Id}$ ), of  $(G/U)_0$  which commutes with the linear isotropy group (1.1). Since  $b_i \neq b_j$  for  $i \neq j$ ,  $J$  must also leave the subspaces  $\mathfrak{b}_i$  invariant and it induces complex structures on them which characterize it completely. Now on each  $\mathfrak{b}_j$  there are two complex structures commuting with the isotropy representation of  $T$  in  $\mathfrak{b}_j$ , differing by the orientation they induce; to each  $\mathfrak{b}_j$  we attach a sign  $\epsilon_j$ , equal to  $+1$  (resp.  $-1$ ), according to whether the ordered pairs  $(e, \text{Ad } t(e))$  and  $(e, J(e))$  define the same orientation or not

( $e \in \mathfrak{b}_j$ ,  $e \neq 0$ ,  $t \in T$  such that  $0 < b_j(t) < \frac{1}{2}$ ). The  $\epsilon_j b_j$  will be called the roots of the almost complex structure, which they describe completely.

We extend  $J$  to a linear transformation  $\tilde{J}$  of  $\mathfrak{g}$  by putting it equal to zero on  $\mathfrak{u}$ , and to a linear transformation  $\tilde{J}^c$  of  $\mathfrak{g}^c$ ; it is readily seen that

$$\tilde{J}^c(e_{\epsilon_j b_j}) = i \cdot e_{\epsilon_j b_j}; \quad \tilde{J}^c(e_{-\epsilon_j b_j}) = -i \cdot e_{\epsilon_j b_j} (e_{\pm b_j} \in \mathfrak{v}_{\pm b_j}).$$

The space  $T^+$  of (12.1) may be identified with the space spanned by the  $e_{\epsilon_j b_j}$  which, by the foregoing, is invariant under  $\text{Ad}_G U$ . Since  $x \rightarrow x + \bar{x}$  is a complex isomorphism of  $T^+$  onto  $(G/U)_0$ , the previous identification carries the restriction of  $\text{Ad}_G U$  onto the complex isotropy representation  $\iota_c$  defined in § 10, and, therefore, the  $\epsilon_j b_j$  are the weights of  $\iota_c$ .

The almost complex structure is integrable, i.e. (since we are in the real analytic case), derives from an automatically invariant complex analytic structure, if and only if

$$\mathfrak{n} = \mathfrak{u}^c + \mathfrak{v}_{\epsilon_1 b_1} + \cdots + \mathfrak{v}_{\epsilon_k b_k}$$

is a Lie algebra [14, § 20]. In view of the properties of the bracket recalled above, this proves the first assertion of:

**12.4. LEMMA.** *Let  $\mathcal{B}$  be an invariant almost complex structure on  $G/U$ ,  $\psi$  the system of its roots, and  $\Sigma$  the system of roots of  $U$ . Then  $\mathcal{B}$  is integrable if and only if  $\Sigma \cup \psi$  is closed in the sense of § 4. In this case,  $\psi$  is closed and contained in a system of positive roots.*

As to the second assertion, we remark that by 4.10, we have  $\Sigma = \theta \cup -\theta$ , where  $\theta \cup \psi$  is a system of positive roots for some ordering. Since  $\Sigma \cup \psi$  and  $\theta \cup \psi$  are closed and since  $\psi \cap -\psi = \emptyset$ , it follows immediately that  $\psi$  is closed.

More precise statements about  $\psi$  will be given in 13.7.

### 13. Applications.

**13.1.** The following known facts will be used in this section. A compact connected Lie group  $K$  is semi-simple if and only if  $H^1(K, \mathbf{R}) = 0$ , and then  $H^2(K, \mathbf{R}) = 0$  (see, e.g., Chevalley-Eilenberg, Trans. Amer. Math. Soc., 63 (1948), 85-124). A simple spectral sequence argument then shows that, if  $K$  is compact and semi-simple and  $L$  is a closed connected subgroup, the transgression is an isomorphism of  $H^1(L, \mathbf{R})$  onto  $H^2(K/L, \mathbf{R})$ , and, in particular, that  $H^2(K/L, \mathbf{R}) = 0$  if and only if  $L$  is semi-simple, too.

**13.2.** *Coset spaces with second Betti number zero.* As a first application

of §§ 4 and 10, we prove anew a theorem of H. C. Wang [31, Theorem C] to the effect that a coset space  $G/U$  with  $\text{rank } G = \text{rank } U$  and second Betti number zero is not homogeneous complex.

Assume that  $G/U$  has an invariant almost complex structure with roots  $(\epsilon_j b_j)$ ,  $(1 \leq j \leq k)$ ; let  $c_1$  be its first Chern class and  $\rho$  be the projection of  $G/T$  onto  $G/U$ . By (10.8) and (12.3),

$$\rho^*(c_1) = -\tau(\epsilon_1 b_1 + \cdots + \epsilon_k b_k)$$

( $\tau$  transgression in  $(G, G/T, T)$ ). Since  $H^2(G/U, \mathbf{R}) = 0$ ,  $c_1$  must be a fortiori zero as a real cohomology class, and hence, by (13.1),  $\sum \epsilon_j b_j = 0$ . But then, by § 4, the system  $(\epsilon_j b_j)$  does not satisfy the condition of 12.4, and thus the almost complex structure is not integrable.

13.3. *Examples of (13.2).* Now let  $G$  be simple and  $U$  be a maximal connected subgroup of maximal rank. A complete list of such inclusions is given in [7]; to discuss it, we assume, moreover, the center of  $G$  to be reduced to the identity, which is no loss in generality. These inclusions may then be divided into three classes:

(a)  $U$  is the connected centralizer of an element of order 2, which generates its center.

(b)  $U$  is the centralizer of a one dimensional torus  $S$ , and  $S$  is the identity component of the center of  $U$ .

(c)  $U$  is the connected centralizer of an element  $z$  of order 3 or 5, which generates its center.

The coset spaces  $G/U$  corresponding to the classes (a), (b) are irreducible Riemannian and hermitian symmetric spaces respectively. In the class (c) we find seven spaces, namely  $G_2/A_2 = S_6$ ,  $F_4/A_2 \times A_2$ ,  $E_6/A_2 \times A_2 \times A_2$ ,  $E_7/A_2 \times A_5$ ,  $E_8/A_8$ ,  $E_8/A_2 \times E_6$  for  $z$  of order 3 and  $E_8/A_4 \times A_4$  for  $z$  of order 5.

$U$  being the connected centralizer of  $z$ , its algebra  $\mathfrak{u}$  is the set of fixed points under  $\text{Ad } z$ ; consequently,  $\text{Ad } z$  has no real eigenvalues on the complementary subspaces  $\mathfrak{h}_j$ , and we may attach to it a complex structure on  $(G/U)_0$ , as recalled in (12.1), which will be invariant under  $U$ , since the latter commutes with  $\text{Ad } z$ , and defines, consequently, an invariant almost complex structure on  $G/U$ . Here since  $U$  has a discrete center, it is semi-simple, and  $H^2(G/U, \mathbf{R}) = 0$  (see 13.1). Therefore, by (13.2), we have the

PROPOSITION. *The seven coset spaces of the class (c) above are homogeneous almost complex but not homogeneous complex.*

This generalizes a known result for  $S_6$  (Ehresmann Libermann, *C. R. Acad. Sci. Paris* 232 (1951), 1281 [14, § 10]).

13.4. PROPOSITION. Let  $G/U$  be homogeneous almost complex, and let  $\iota = \iota_1 + \cdots + \iota_s$  be a decomposition of the isotropy representation into real irreducible representations; then the  $\iota_i$  are unique, each one has the complex numbers as commuting field, and  $G/U$  admits exactly  $2^s$  invariant almost complex structures.

Let  $(G/U)_0 = W_1 + \cdots + W_s$  be a direct sum decomposition of  $(G/U)_0$  such that the restriction of  $\iota$  to  $W_i$  is  $\iota_i$ . Since these subspaces are invariant under  $T$ , they are direct sums of subspaces  $\mathfrak{h}_j$ , and the corresponding roots  $\pm b_j$  are the weights of  $\iota_i$ ; since any two roots are different, the complex irreducible components of the  $\iota_i$  will be pairwise inequivalent, from which follows the uniqueness of the  $\iota_i$  and of the  $W_i$ . Also, a straightforward application of Schur's lemma shows that any linear transformation commuting with  $\iota$  leaves the  $W_i$ 's invariant. Since, by assumption, there is at least one transformation without real eigenvalues commuting with  $\iota$ , we see that the commuting field of  $\iota_i$  is either the field of complex numbers  $\mathbf{C}$  or of quaternionic numbers  $\mathbf{K}$ ; in any case  $\iota_i$  is not complex irreducible and its extension to  $W_i \otimes \mathbf{C}$  decomposes into  $\gamma_i + \bar{\gamma}_i$ , where  $\gamma_i$  is complex irreducible and  $\bar{\gamma}_i$  is the complex conjugate representation of  $\gamma_i$ ; the weights of  $\bar{\gamma}_i$  are opposite in sign to the weights of  $\gamma_i$ . Since the roots of  $G$  are pairwise distinct (§ 2),  $\gamma_i$  is not equivalent to  $\bar{\gamma}_i$ , and it follows from Schur's lemma again that the commuting field of  $\iota_i$  is the field of complex numbers. Thus we have on each  $W_i$  exactly 2 invariant complex structures, from which our contention follows.

*Remark.* Let  $\sigma$  be an automorphism of  $G$  leaving  $T$  and  $U$  invariant,  $d\sigma$  the induced automorphism of  $\mathfrak{g}$ ; let  $\psi$  be the root system of an invariant almost complex structure  $\mathcal{B}$  on  $G/U$ , and  $\psi'$  the transform of  $\psi$  under  $d\sigma$ . From the formula (1) in § 1, it follows readily that the homeomorphism  $\sigma'$  of  $G/U$  defined by  $\sigma$  carries  $\mathcal{B}$  onto the invariant almost complex structure with roots  $\psi'$ . If, in particular,  $\sigma(x) = g \times g^{-1}$  with  $g \in N_T \cap U$ , then  $\sigma'$  reduces to the left translation defined by  $g$  and leaves  $\mathcal{B}$  invariant; hence the element of  $W(U)$  represented by  $g$  must leave  $\psi$  invariant.

13.5. *Centralizers of tori.* The following proposition is due to H. C. Wang [31]:

PROPOSITION.  $G/U$  (with  $\text{rank } U = \text{rank } G$ ;  $U$  connected) is homogeneous complex if and only if  $U$  is the centralizer of a torus in  $G$ .

*Proof.* Let  $U$  be the centralizer of a torus  $S$ , which we assume, as we may, to be in  $T$ , and let  $s \in S$  generate an everywhere dense subgroup of  $S$ . Then  $U$  is the centralizer of  $s$ ,  $u$  the space of vectors fixed under  $\text{Ad } s$ , and we have  $b_j(s) \neq 0(1)$  if and only if  $b_j$  is complementary. Let  $\epsilon_j = \text{sgn}(b_j(s))$  ( $1 \leq j \leq k$ ); since  $s$  centralizes  $U$ , the  $\epsilon_j b_j$  are the roots of an invariant almost complex structure; moreover, being characterized by  $\epsilon_j b_j(s) > 0$ , these roots satisfy the criterion of 12.4, and the structure is integrable.

Assume now, conversely, that  $G/U$  has been endowed with a homogeneous complex structure and let  $(\epsilon_j b_j)$  ( $1 \leq j \leq k$ ), be its roots. By (2.9), the group is locally isomorphic to the direct product of its largest semi-simple subgroup  $U'$  and of a torus  $S$ ; moreover, by 13.1 and 13.2,  $S \neq \{e\}$ . Now let  $W$  be the centralizer of  $S$ . We have  $W = S_1 \cdot W'$ , where  $S_1$  is a torus containing  $S$  and  $W'$  a semi-simple subgroup containing  $U'$  and  $W' \cap S_1$  is finite. The equalities

$$\text{rank } G = \text{rank } U' + \dim S = \text{rank } W' + \dim S_1$$

show then that  $S = S_1$ , that  $\text{rank } W' = \text{rank } U'$ , and that  $W/U$  is to be identified with  $W'/U'$ ; since  $W'$  and  $U'$  are semi-simple, we have  $H^2(W'/U', \mathbf{R}) = 0$  and  $W/U$  is not homogenous complex (13.1, 13.2).

Let  $J \subset [1, k]$  be such that the  $\pm b_j$ 's, with  $j \in J$ , are the complementary roots of  $U$  in  $W$ . The roots  $(\epsilon_j b_j)$  ( $j \in J$ ), define a complex structure on  $(W/U)_0$  which is invariant under the linear isotropy representation  $\iota'$  of  $U$  in  $(W/U)_0$  since  $\iota'$  is nothing but the restriction to an invariant subspace of the isotropy representation of  $U$  in  $(G/U)_0$ ; moreover, since the system  $(\epsilon_j b_j)$  ( $1 \leq j \leq k$ ), satisfies the condition of 12.4, so does  $(\epsilon_j b_j)$  ( $j \in J$ ). Therefore, if  $W \neq U$ , then we get on  $W/U$  an invariant integrable almost complex structure, in contradiction to what has already been proved. Thus  $U = W$  and  $U$  is the centralizer of the torus  $S$ .

13.6. For the sake of completeness, we recall the proof of the following well-known lemma.

**LEMMA.** *Let  $U$  be the centralizer of a torus in  $G$ ,  $S$  the connected center of  $U$  and  $k = \dim S$ . Then, for a suitable ordering  $\mathcal{B}$ , there are  $l-k$  simple roots  $a_i$  ( $1 \leq i \leq l-k$ ) vanishing on  $S$  and such that the roots of  $U$  are exactly the roots of  $G$  which are linear combinations of the  $a_i$  ( $1 \leq i \leq l-k$ ).*

The roots of  $U$  are those of  $G$  which vanish on  $S$ ; since the semi-simple part of  $U$  has rank  $l-k$ ,  $U$  has  $l-k$  independent roots. We consider in

the dual space  $V_T^*$  of the universal covering  $V_T$  of  $T$  the lexicographic order which is associated to a base whose first  $k$  elements span the covering  $V_S$  of  $S$ . It is then clear that if a sum of positive linear forms  $b_i$  with strictly positive coefficients vanishes on  $V_S$ , so does each  $b_i$ ; the lemma follows readily from this and from the fact that  $U$  has  $l-k$  linearly independent roots.

*Remark.* In the ordering  $\mathcal{S}$ , the complementary roots are linear combinations of the  $a_j$ 's with at least one of the  $k$  last ones appearing with a non-zero coefficient. Therefore, if the sum  $a+b$  of a root  $a$  of  $U$  and of a complementary root  $b$  is a root, then it must be a complementary root. Also, the set of positive complementary roots is closed.

13.7. *Number of invariant complex structures.* In this section, we assume  $G/U$  to be homogeneous complex;  $U$  is then the centralizer of a torus by 13.5, and we keep the notations of 13.6.

**PROPOSITION.** *Let  $\Theta$  be a system of positive roots of  $U$ . The roots of an invariant complex structure form a closed system  $\Psi$  such that  $\Theta \cup \Psi$  is a positive system of roots for  $G$ . Conversely, a closed set  $\Psi$  of complementary roots such that  $\Theta \cup \Psi$  is the set of positive roots of  $G$  for a suitable ordering is the system of roots of an invariant complex structure of  $G/U$ .*

Let  $\Psi$  be the root system of an invariant complex structure  $\mathcal{C}$ . Then (12.4)  $\Psi$  is closed and is contained in a system  $\Phi$  of positive roots of  $G$ .  $\Phi$  is necessarily of the form  $\Theta' \cup \Psi$ , where  $\Theta'$  is a system of positive roots for  $U$ . There exists, therefore,  $w \in W(U)$  which carries  $\Theta'$  onto  $\Theta$ ; since  $w$  leaves  $\Psi$  invariant (remark in 13.4), it carries  $\Theta' \cup \Psi$  onto  $\Theta \cup \Psi$ , and the latter is also a positive system.

Let now  $\Psi$  be a closed system of complementary roots such that  $\Theta \cup \Psi$  is the set of positive roots relative to an ordering  $\mathcal{S}'$ . The remark in 13.6 and the fact that  $\Psi$  is closed show that if  $a \in \Theta$  is a sum of two positive roots for  $\mathcal{S}'$ , then these two roots also belong to  $\Theta$ ; this means that the simple roots of  $\Theta$ , considered as a positive system for  $U$ , are also simple for  $\mathcal{S}'$ . Let then  $a_j$  ( $1 \leq j \leq l$ ) be the simple roots of  $\mathcal{S}'$ , with  $a_j \in \Theta$  for  $j \leq l-k$ ; the elements of  $\Theta$  (resp.  $\Psi$ ) are then linear combinations with positive coefficients of  $a_1, \dots, a_{l-k}$  (resp.  $a_1, \dots, a_l$ , where at least one  $a_j$  ( $j > l-k$ ) has a non-vanishing coefficient). This implies first that  $\Theta \cup -\Theta \cup \Psi$  is closed and second that there is an  $s \in S$  such that  $0 < b(s) < \frac{1}{2}$  for all  $b \in \Psi$ ; thus the map of  $(G/U)_0 \cong \mathfrak{g}/\mathfrak{u}$  onto itself, defined by  $\text{Ad } s$ , has no real eigenvalues and the complex structure attached to it by the rule of 12.1 has the root system  $\Psi$ . Since  $s$  commutes with  $U$ , this structure is invariant under the

isotropy representation, and hence, by (12.4), gives an invariant complex structure on  $G/U$ .

**13.8. PROPOSITION.** *Let  $G/U$  be homogeneous complex, let  $k$  be the dimension of the center of  $U$ , and let  $l$  be the rank of  $G$ . If  $k=1$  (resp.  $k=l$ , i. e.,  $U=T$ ), then the number of invariant complex structures is equal to two (resp. the order of  $W(G)$ ). Given two of them, there is a homeomorphism of  $G/U$  induced from an (resp. inner) automorphism of  $G$  leaving  $U$  invariant and carrying one onto the other.*

Let  $k=l$ . Then in the notations of 13.7,  $\Theta$  is empty and the invariant complex structures are in 1-1 correspondence with the different systems of positive roots of  $G$ , by 13.7 (or directly by 4.9 and 12.4). Since  $W(G)$  operates transitively on the set of systems of positive roots, it is obvious by the remark of 13.4 that the inner automorphisms of  $G$  defined by the elements of the normalizer of  $T$  induce homeomorphisms of  $G/T$  which permute transitively the invariant complex structures.

Now let  $k=1$ . We first take an ordering  $\mathfrak{J}$  having the properties mentioned in 13.6. Then the set of positive complementary roots  $\Psi$  defines an invariant complex structure by 13.7. Let  $\Psi'$  be the root system of another invariant complex structure. As in 13.7, we denote by  $\Theta$  the set of roots of  $U$  which are positive for  $\mathfrak{J}$  and by  $a_i$  the simple root of  $\mathfrak{J}$  not belonging to  $\Theta$ ;  $\Psi' \cup \Theta$  is the set of positive roots for some ordering  $\mathfrak{J}'$ , and the proof of Proposition 13.7 shows that  $a_1, \dots, a_{l-1}$  are also simple for  $\mathfrak{J}'$ . If  $a_l \in \Psi'$ , then  $\Psi' \cup \Theta$  contains all simple roots of  $\mathfrak{J}'$ , and hence  $\Psi = \Psi'$ . Let now  $-a_l \in \Psi'$  and let  $a'_l$  be the  $l$ -th simple root of  $\mathfrak{J}'$ ;  $-a_l$  is a linear combination with positive coefficients of  $a_1, \dots, a_{l-1}, a'_l$ , and therefore, if we express  $a'_l$  as a linear combination of  $a_1, \dots, a_l$ , then the root  $a_l$  must have coefficient  $-1$ . But then the elements of  $\Psi'$  which are combinations with positive coefficients of  $a_1, \dots, a_{l-1}, a'_l$ , where the last coefficient is  $\neq 0$ , must also have at least one negative coefficient when expressed as linear combinations of  $a_1, \dots, a_l$ ; this means that they are negative for  $\mathfrak{J}$ , and therefore that  $\Psi' = -\Psi$ . Thus we have only two invariant complex structures.

It is known (see [23] or Gantmacher, Rec. Math. Moscou 47 (1939), 101-144) that any automorphism of  $\mathfrak{t}$  permuting the roots extends to an automorphism of  $\mathfrak{g}$ . In particular, there is an automorphism  $\sigma$  carrying each root into its opposite; since we may assume here  $G$  to be simply connected (10.1),  $\sigma$  also defines an automorphism of  $G$  leaving  $T$  invariant; it maps  $\Psi$  onto  $-\Psi$  and leaves invariant the set of roots of  $U$ . Therefore  $\sigma$  leaves  $U$

invariant and defines a homeomorphism of  $G/U$  carrying the complex structure with roots  $\Psi$  onto the complex structure with roots  $-\Psi$ .

*Remarks.* 1) The argument which ends the preceding proof shows more generally the following: let  $\Psi, \Psi'$  be the root systems of two invariant complex structures  $\mathcal{L}, \mathcal{L}'$  on  $G/U$ . If there is an automorphism of  $V_T$  carrying  $\Psi$  onto  $\Psi'$  and leaving the root system of  $U$  invariant, then  $\mathcal{L}, \mathcal{L}'$  are equivalent under a differentiable homeomorphism of  $G/U$ , which is induced from an automorphism of  $G$  leaving  $U$  invariant.

2) If  $\Psi$  is the set of roots of an invariant complex structure  $\mathcal{L}$ , then  $-\Psi$  is clearly the root system of the "bar structure" or conjugate of  $\Psi$ , that is, of the structure in which the vectors of type  $(1, 0)$  are those of type  $(0, 1)$  for  $\mathcal{L}$ . Thus the last part of the above proof shows that on  $G/U$  an invariant complex structure and its conjugate are equivalent under an automorphism of  $G$ . In the case  $P_{n-1}(C) = U(n)/U(n-1) \times U(1)$ , the automorphism may be taken as complex conjugation and therefore has to be an outer automorphism for  $n \geq 3$ .

3) The case  $k=1$  in 13.8 includes the hermitian symmetric spaces for which our assertion has already been noticed by I. Satake, "A remark on bounded symmetric domains," Sci. Papers Coll. Ed. Gen. Univ. Tokyo 3 (1953), 131-144).

13.9. *Examples of inequivalent structures.* There are cases in which  $G/U$  carries at least two invariant complex structures which are not equivalent under a differentiable homeomorphism. For instance, take  $G = U(4)$ ,  $U = U(2) \times U(1) \times U(1)$ , embedded in the standard fashion. With respect to the standard maximal torus, the roots of  $U(4)$  are  $\pm(x_i - x_j)$ ,  $(1 \leq i < j \leq 4)$ , and those of  $U$  are  $\pm(x_1 - x_2)$ . Let  $\mathcal{S}_1$  (resp.  $\mathcal{S}_2$ ) be the ordering defined by  $x_4 > x_1 > x_2 > x_3 > 0$  (resp.  $x_1 > x_2 > x_3 > x_4 > 0$ ). Then by 13.7, the set  $\Psi_1$  (resp.  $\Psi_2$ ) formed by  $x_4 - x_1, x_4 - x_2, x_4 - x_3, x_1 - x_3, x_2 - x_3$  (resp.  $x_1 - x_3, x_1 - x_4, x_2 - x_3, x_2 - x_4, x_3 - x_4$ ) is the root system of an invariant complex structure  $\mathcal{L}_1$  (resp.  $\mathcal{L}_2$ ). The image in  $H^2(U(4)/T, \mathbb{Z})$  of the first Chern class of  $\mathcal{L}_1$  (resp.  $\mathcal{L}_2$ ) is, by 10.8, equal to  $3(x_4 - x_3)$  (resp.  $2x_1 + 2x_2 - x_3 - 3x_4$ ). Now  $U(4)/T = SU(4)/T'$  where  $T' = T \cap SU(4)$  is a maximal torus of  $SU(4)$ . The inclusion map of  $T'$  in  $T$  identifies  $H^1(T', \mathbb{Z})$  with the quotient of  $H^1(T, \mathbb{Z})$  by  $\mathbb{Z} \cdot (x_1 + x_2 + x_3 + x_4)$ ; since  $SU(4)$  is simply connected, the transgression is an isomorphism of  $H^1(T', \mathbb{Z})$  on  $H^2(SU(4)/T', \mathbb{Z})$ . It follows then that the first Chern class of  $\mathcal{L}_1$  (resp.  $\mathcal{L}_2$ ) is divisible (resp. not divisible) by 3. Hence  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are

not equivalent under a differentiable homeomorphism of  $G/U$ . Moreover it will be shown in Section 24.14 that the Chern numbers  $c_1^5$  of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are equal to 4860 and 4500 respectively.

The following observation leads to other examples:

**PROPOSITION.** *Assume that  $G/U$  carries two invariant homogeneous structures  $\mathcal{L}$ ,  $\mathcal{L}'$  with root systems  $\Psi$ ,  $\Psi'$ , and that  $G^c$  is the greatest connected group of automorphisms of  $\mathcal{L}$  and of  $\mathcal{L}'$ . Then  $\mathcal{L}$  and  $\mathcal{L}'$  are equivalent under a differentiable homeomorphism of  $G/U$  if and only if there is a linear transformation  $\alpha$  of  $\mathfrak{t}$  leaving the root system of  $U$  invariant and carrying  $\Psi$  onto  $\Psi'$ .*

The "if" part follows from Remark 1) in 13.8.

Let now  $\beta$  be a differentiable homeomorphism of  $G/U$  carrying  $\mathcal{L}$  onto  $\mathcal{L}'$ . By the assumption on  $G^c$ ,  $\beta$  defines an automorphism of  $G^c$ . Using homogeneity and the facts recalled in 14.3, it is then seen that  $\mathcal{L}$  and  $\mathcal{L}'$  are also equivalent under a homeomorphism  $\gamma$  which is induced from an automorphism of  $G^c$  leaving  $U^c$ ,  $T^c$  invariant. Since  $\gamma$  permutes the roots of  $\mathfrak{g}^c$  with respect to  $\mathfrak{t}^c$ , and since  $\mathfrak{t}$  is characterized as the subset of  $\mathfrak{t}^c$  on which the roots are real valued,  $\gamma$  leaves  $\mathfrak{t}$  invariant, and its restriction to  $\mathfrak{t}$  is the desired  $\alpha$ . Q. E. D.

According to Bott (unpublished),  $G^c$  satisfies our assumption, for instance, if  $G = E_7, E_8$ . Moreover,  $E_7, E_8$  have no outer automorphisms, hence the automorphisms of  $\mathfrak{t}$  keeping the root system of  $G$  invariant are just those of the Weyl group. Let now  $a, b$  be two different simple roots with respect to an ordering  $\mathcal{S}$  and let  $w \in W(G)$  be a transformation carrying  $b$  onto  $a$ . (This exists because the roots of  $E_7$  or  $E_8$  all have the same length and  $W(G)$  is known to be transitive on a set of roots of the same length.)

Let  $\Phi$  be the set of positive roots for  $\mathcal{S}$ , and  $\Psi = \Phi - a$ ,  $\Psi' = w(\Phi - b)$ . The symmetry to  $a=0$  carries  $\Phi$  onto  $-\mathfrak{a} \cup \Psi$  (see the proof of 3.1). Therefore, if there existed an  $\alpha$  carrying  $\pm a$  onto itself and  $\Psi$  onto  $\Psi'$ , there would also be a  $w' \in W(G)$  carrying  $a$  onto  $b$  and leaving  $\Phi$  invariant, but this contradicts the fact that  $W(G)$  is simply transitive on the Weyl chambers (2.7). Now  $\pm a$  is the set of roots of the centralizer  $U$  of the singular torus defined by  $a=0$ . Thus, by our criterion,  $\Psi$  and  $\Psi'$  are the root systems of two invariant complex structures on  $G/U$  which are not equivalent under a differentiable homeomorphism. A similar discussion would also show that the complex structures on  $U(4)/U(2) \times U(1) \times U(1)$  discussed above are not equivalent.

**14. Complex Lie groups, embeddings, representations, invariant differential forms.** We collect here some results to be used in the sequel. For more details about the facts mentioned without proofs in 14.3, 14.4 or about related questions, see [5], [7a], Goto "On algebraic homogeneous spaces," *Amer. Jour. Math.* 76 (1954), 811-818, J. Tits, "Sur certaines classes d'espaces homogènes de groupes de Lie," *Mém. Acad. Royale Belgique* 29 (1955), Chapitre III.

**14.1. Notations.**  $G$  is semi-simple and simply connected,  $\mathcal{S}$  is an ordering of the roots with respect to  $T$ , and  $a_1, \dots, a_l$  are the simple roots for  $\mathcal{S}$ . If  $I$  is a (possibly empty) proper subset of  $[1, l]$ , we denote by  $U_I$  the centralizer of the torus  $S_I$  defined by  $a_i(t) = 0$  ( $i \in I, t \in T$ ) and put  $M_I = G/U_I$ . We consider it to be endowed with the invariant complex structure  $\mathcal{B}_I$  defined by the set  $\Psi_I$  or  $\Psi$  of positive complementary roots (whose existence follows from 13.7).  $U_I'$  is the semi-simple part of  $U_I$ . Thus we have  $U_I = S_I \cdot U_I'$  with  $S_I \cap U_I'$  finite.

*Remark.* Let  $G$  be a compact connected Lie group and  $G_1, G', U, U_1, U'$  be as in 10.1. Then  $G/U = G'/U' = G_1/U_1$ . By a result of Hopf (see e.g. [23, Exp. XXI]), the centralizer of a toral subgroup in a compact connected Lie group is connected; hence, if one of  $U, U', U_1$  is centralizer of a torus, so are the other two. Thus the assumption  $G$  semi-simple and simply connected made in §14, which allows one to avoid some slight irrelevant technical complications, is no real restriction, and the results of this paragraph are valid, with little or no modification, in the general case. In particular, in 14.4 one has to consider then the representations of the group  $\tilde{G}$  mentioned in 2.9.

**14.2.** The natural map  $\nu_I: G/T \rightarrow G/U_I$  is the projection in the fibering  $(G/T, G/U_I, U_I/T)$ ; the spaces  $G/T, G/U_I, U_I/T$  have no torsion and have vanishing odd dimensional Betti numbers ([5] or R. Bott, *Bull. Soc. Math. France* 84 (1956), 251-81). Therefore [2, §4], for any commutative group  $A$  of coefficients, the fibre is totally non homologous to zero,  $\nu_I^*$  is injective,  $\nu_I^*(H^2(G/U_I, A))$  is the kernel of the map of  $H^2(G/T, A)$  into  $H^2(U_I/T, A)$  induced by inclusion. It follows that  $\nu_I^*(H^2(G/U_I, A))$  is a direct summand of  $H^2(G/T, A)$ ; also, since the transgression in  $(G, G/U_I, U_I)$  is an isomorphism of  $H^1(U_I, \mathbf{Z})$  onto  $H^2(G/U_I, \mathbf{Z})$ , the former group is free abelian.

**LEMMA.** Let  $A$  be a principal ideal ring. Then  $\nu_I^*$  is an isomorphism of  $H^2(G/U_I, A)$  onto the submodule of  $H^2(G/T, A)$  formed by the elements orthogonal to the  $a_i$ 's ( $i \in I$ ), that is, which is spanned by the fundamental weights  $\varpi_i$ 's ( $i \notin I$ ).

By the above and the universal coefficient formula, it is enough to prove the lemma for  $A = \mathbf{Z}$ . The fundamental weights  $\omega_i$  ( $1 \leq i \leq \text{rank } G$ ), form a basis of  $H^1(T, \mathbf{Z})$ , (see 3.4), hence of  $H^2(G/T, \mathbf{Z})$ , and the subgroup  $B_I$  spanned by the  $\omega_i$ 's ( $i \notin I$ ) is a direct summand whose rank equals the dimension of  $S_I$ .

The group  $T' = T \cap U_I'$  is a maximal torus of  $U_I'$ , whose covering in the universal covering  $V_T$  of  $T$  is spanned by the contravariant images of the  $a_i$  ( $i \in I$ ). Since  $U_I'$  is semi-simple and  $U_I$  is locally isomorphic to the product  $S_I \times U_I'$ , the map  $H_1(S_I, \mathbf{R}) \rightarrow H_1(U_I, \mathbf{R})$ , (resp.  $H_1(T', \mathbf{R}) \rightarrow H_1(U_I, \mathbf{R})$ ), induced by inclusion, is an isomorphism (resp. has zero image). Since  $H^1(U_I, \mathbf{Z})$  is free, it follows immediately that  $\alpha^*: H^1(U_I, \mathbf{Z}) \rightarrow H^1(T, \mathbf{Z})$  is injective, with its image contained in  $B_I$ , and of finite index in  $B_I$ , where  $\alpha$  is the inclusion of  $T$  in  $U_I$ .

The projection  $\nu_I$  defines a representation of the fibering  $(G, G/T, T)$  into  $(G, G/U_I, U_I)$ , whose restriction to a fibre is  $\alpha$ . Therefore, using transgression, we see that the image of  $\nu_I^*$  is a subgroup of finite index of  $B_I$ . But we have already shown that it is a direct summand, whence the lemma.

*Remark.* By transgression, we also see that  $\alpha^*$  identifies  $H^1(U_I, \mathbf{Z})$  with  $B_I$ .

14.3. *Complexification.*  $G^c$  denotes the complex Lie group containing  $G$ , with Lie algebra  $\mathfrak{g}^c$ , whose existence and uniqueness up to an isomorphism is well known. We use the notation of §§ 1, 12 and, moreover, put

$$\mathfrak{p}_I = \mathfrak{u}_I^c + \sum_{-b \in \Psi} \mathfrak{b}_b.$$

It is a subalgebra which generates a closed, connected, complex analytic subgroup  $P_I$  of  $G^c$ , equal to its normalizer, such that  $P_I \cap G = U_I$ ; it follows then that  $G$  is transitive on  $G^c/P_I$  and that there is a natural identification of  $G/U_I$  with  $G^c/P_I$  which carries  $\mathcal{B}_I$  onto the quotient complex structure, as defined in the theory of complex Lie groups (see e.g. [7a]). If, in particular,  $I$  is empty, then  $\mathfrak{b} = \mathfrak{p}_I$  is solvable and  $G^c/B = G/T$ , where  $B = P_I$ .

14.4. *Representations and embeddings.* Let  $\mathcal{J}^-$  be the ordering of the roots which is opposite to  $\mathcal{J}$ , that is, which has the negative roots of  $\mathcal{J}$  as positive roots. The highest weights of the irreducible representations of  $G$  with respect to  $\mathcal{J}^-$  are then the opposite of the highest weights in the order  $\mathcal{J}$ . If  $\Gamma$  has highest weight  $\omega$  for  $\mathcal{J}$  and  $\Gamma'$  highest weight  $-\omega$  for  $\mathcal{J}^-$ , then  $\Gamma'$  is the contragredient representation to  $\Gamma$ , and its weights are the opposite of those of  $\Gamma$ .

Let  $\Gamma$  be an irreducible representation of degree  $q+1$ ,

$$\varpi = c_1\varpi_1 + \cdots + c_l\varpi_l$$

its highest weight in the ordering  $\mathfrak{J}$ ,  $\check{\Gamma}$  the contragredient representation,  $\check{\Gamma}'$  and  $\Gamma'$  the associated representations by means of projective transformations in  $P_q(C)$ . Let  $V$  be a representation space for  $\check{\Gamma}$  and  $\pi$  be the projection of  $V-0$  on  $P_q(C)$ . There is in  $V$  exactly one 1-dimensional subspace  $W$  which is invariant under  $B$ , and we have

$$\check{\Gamma}(t)(x) = \exp[-2\pi i\varpi(t)] \cdot x, \quad (t \in T^c, x \in W).$$

$x' = \pi(W-0)$  is then the unique point of  $P_q(C)$  fixed under  $\check{\Gamma}'(B)$ .

We say that  $M_I$  is *associated* (resp. *strictly associated*) to  $\Gamma$  if  $c_i = 0$  for  $i \in I$  (resp. and, moreover, if  $c_j \neq 0$  for  $j \notin I$ ). If  $M_I$  is associated (resp. strictly associated) to  $\Gamma$ , then the map  $\phi: g \rightarrow \check{\Gamma}'(g) \cdot x'$  induces a holomorphic (resp. bijective and bi-holomorphic) map  $\beta_I$  of  $M_I$  onto a projective non-singular variety  $M_\Gamma$ . In particular, each irreducible representation yields a holomorphic map  $\beta_2$  of  $G/T$  into some projective space. Since a given  $M_I$  is strictly associated to infinitely many representations, it therefore admits projective embeddings.

The cone  $\pi^{-1}(M_\Gamma)$  over  $M_\Gamma$  is, in the obvious way, a  $\mathbf{C}^*$ -bundle  $\eta$  over  $M_\Gamma$ ; on the other hand, let  $N$  be the subgroup of  $B$  whose Lie algebra over  $\mathbf{C}$  is spanned by the  $\mathfrak{b}_a$  ( $a < 0$  for  $\mathfrak{J}$ ).  $N$  is the commutator subgroup of  $B$ , and  $B/N \cong T^c$ . The quotient  $G^c/N$  can be considered as the total space of a principal  $T^c$ -bundle  $\xi = (G^c/N, G^c/B, T^c)$ . If  $\gamma$  is the representation of  $T^c$  with character  $\exp(-2\pi i\varpi)$ , it is then easily seen that  $\beta_2$  induces a  $\gamma$ -homomorphism of  $\xi$  on  $\eta$ . Therefore, by 10.4,  $\varpi = -\beta_2^*(c_1(\eta))$ . It follows from this and from 14.2 that if  $M_I$  is associated to  $\Gamma$ , then  $-\varpi$  may be identified with an element of  $H^2(M_I, \mathbf{Z})$  which is the Chern class of the bundle over  $M_I$  induced from  $\eta$  by  $\beta_I: M_I \rightarrow M_\Gamma$ . But  $c_1(\eta)$  is  $-e^*$ , where  $e^*$  is the dual of the homology class containing a hyperplane section of  $M_I$  (see § 29). Thus  $\varpi$  is the dual of the homology class of a divisor on  $M_I$ , namely, the inverse image of a hyperplane section of  $M_\Gamma$ . It may be shown [7a] that the inverse images of the hyperplane sections of  $M_\Gamma$  form a complete linear system on  $M_I$ , and that this system is the only one on which the natural representation of  $G$  is  $\Gamma'$ .

14.5. *Positive classes.* Since we want to use some facts about complex Lie algebras, we now identify  $V_T$  with its tangent space  $\mathfrak{t}$  at  $e$ , and assume the invariant metric to be the restriction of  $-K$ , where  $K$  is the Killing

form. The roots of  $\mathfrak{g}^c$  with respect to  $\mathfrak{t}^c$  in the sense of infinitesimal theory are then the forms  $2\pi i a$ , where  $a$  runs through the roots as defined in this paper (or, more precisely, through their extensions to  $\mathfrak{t}^c$ , since they were originally defined on  $V_T$  or  $\mathfrak{t}$ ). If  $a$  is a linear form on  $\mathfrak{t}^c$ , we denote by  $h_a$  its contravariant representative with respect to  $-K$ ; if  $a$  is real valued on  $\mathfrak{t}$ , then  $h_a \in \mathfrak{t}$ . It is known (see, for instance, [23], Exp. 10, 11) that we can find  $e_a \in \mathfrak{v}_a$  with the following properties:

$$(1) \quad [e_a, e_{-a}] = 2\pi i h_a, \quad K(e_a, e_{-a}) = -1,$$

$\mathfrak{g}$  is spanned over the reals by  $\mathfrak{t}$  and the vectors  $e_a + e_{-a}$ ,  $i(e_a - e_{-a})$ , and the Killing form is the direct sum

$$K = K_1 - \sum_{a>0} x_a x_{-a},$$

where  $K_1$  is the restriction of  $K$  to  $\mathfrak{t}^c$  and the  $x_a, x_{-a}$  form the dual base to  $(e_a, e_{-a})$ .

Finally, we recall that if  $X, Y$  are left-invariant vector fields and  $\omega$  is a left-invariant 1-form on a Lie group  $H$ , then

$$(2) \quad d\omega(X, Y) = -\omega([X, Y]) \quad ([X, Y] = X \cdot Y - Y \cdot X)$$

(see e.g. Chevalley, *Theory of Lie groups I*, Princeton, Chap. V, § 4; it is there stated for real Lie groups, but the proof is also valid in the complex case).

Let us denote by  $\omega_a$  the left-invariant 1-form on  $G^c$  whose restriction to  $\mathfrak{g}^c$  is annihilated by  $\mathfrak{t}^c$  and which is such that  $\omega_a(e_b) = 1$  if  $a = b$  and is zero if  $a \neq b$ . A straightforward computation using (1), (2) and 12.2 yields then the

LEMMA. *Let  $\eta_b$  be the left-invariant 1-form on  $G^c$  whose restriction to  $\mathfrak{g}^c$  is zero on  $\sum \mathfrak{v}_a$  and which induces the linear form  $b$  on  $\mathfrak{t}^c$ . Then*

$$(3) \quad d\eta_b = -2\pi i \sum_{a>0} (b, a) \omega_a \wedge \omega_{-a}.$$

We are interested only in the case where  $b$  is real valued on  $\mathfrak{t}$ ; then  $(b, a)$  is also real valued. Assume now that  $b$  is orthogonal to the roots  $a_i$  ( $i \in I$ ), i.e., that  $h_b \in \mathfrak{s}_I$ . Then

$$(4) \quad d\eta_b = -2\pi i \sum_{a \in \Psi_I} (b, a) \omega_a \wedge \omega_{-a}.$$

The restriction  $\eta_b|_G$  of  $\eta_b$  to  $G$  is left-invariant. By (4),  $d\eta_b$  is zero on  $\mathfrak{u}$ , and hence, by invariance, it vanishes on  $U$ ; thus  $\eta_b|_U$  and, a fortiori,  $\eta_b|_T$  are closed. Since  $\eta_b|_T$  is clearly in the class  $b \in H^1(T, \mathbf{R})$ , it follows by 14.2

that  $\eta_b|_U$  represents the element of  $H^1(U, \mathbf{R})$  which we have identified with  $b$ . We want to show that  $d\eta_b|_G$  is the inverse image of a 2-form on  $M_I$ ; for this, it is necessary and sufficient that the 2-form defined by  $d\eta_b$  on  $\mathfrak{g}$  vanishes whenever one of the arguments is in  $\mathfrak{u}_I$  and is invariant under  $\text{Ad}_{\mathfrak{g}} U_I$  (see e.g. Chevalley-Eilenberg, Trans. A. M. S. 63 (1948), 85-124, Theorem 13.1). The first property is obvious from (4); as to the second, we may argue as follows: by (4), we have only to show that the restriction of  $d\eta_b$  to  $\sum_{b \in \Psi} \mathfrak{b}$  is invariant under the linear isotropy group  $\tilde{U}$ . From the properties of the Killing form recalled above, we see that if we take in  $\mathfrak{b}$  the real and imaginary parts of  $x_b$  as coordinates, then the Killing form is the negative unit form, and therefore, the isotropy group  $\tilde{U}$  is orthogonal; this means that  $d\eta_b$  is invariant under  $\tilde{U}$  if its matrix commutes with  $\tilde{U}$ . But it follows from (4) and 12.2 that this matrix is equal to the restriction of the matrix of  $\text{ad}_{\mathfrak{g}} h_b$ . Since  $h_b \in \mathfrak{s}_I$ , it centralizes  $\mathfrak{u}$ , and  $\text{ad}_{\mathfrak{g}} h_b$  does commute with  $\text{Ad}_{\mathfrak{g}} U$ .

By the foregoing and by the definition of transgression,  $d\eta_b|_G$  may be identified with a closed 2-form on  $G/U_I$  belonging to the image of  $b \in H^1(U, \mathbf{R})$  under transgression; in view of the conventions made in 10.1, this form represents the cohomology class which has been identified with  $-b$ . Moreover, by the definition of the complex structure  $\mathcal{B}_I$  on  $M_I$ , the  $v_a$  ( $a \in \Psi_I$ ), span the subspace of  $M_I \otimes \mathbf{C}$  which contains the differentials of local holomorphic functions, and we have  $\omega_{-a} = \bar{\omega}_a$  in the standard notations. Thus we have shown the following:

**14.6. PROPOSITION.** *We keep the notations of 14.1, 14.2. Let  $b$  be a linear form on  $V_T$  orthogonal to the simple roots  $a_i$  ( $i \in I$ ). Then the element of  $H^2(G/U_I, \mathbf{R})$  identified with  $b$  in 14.2 contains the invariant 2-form of type (1,1)*

$$(5) \quad \omega = 2\pi i \sum_{a \in \Psi_I} (b, a) \omega_a \wedge \bar{\omega}_a.$$

$\omega$  is the imaginary of the hermitian form

$$(6) \quad 4\pi \sum_{a \in \Psi} (b, a) \omega_a \cdot \bar{\omega}_a \text{ (symmetric product).}$$

A 2-dimensional complex cohomology class on a complex manifold  $M$  is *positive in the sense of Kodaira* if it contains the imaginary part of a positive non-degenerate hermitian metric, which is then necessarily kählerian. From (5), (6) and the remark in 13.6, we get:

**14.7. COROLLARY.**  *$b$  is positive in the sense of Kodaira if  $(b, a) > 0$  for  $a \in \Psi_I$ , that is, if  $(b, a_i) > 0$  for  $i \in I$ .*

This is in agreement with the fact (14.4) that when  $b$  is the highest weight of an irreducible representation  $\Gamma$  to which  $M_I$  is strictly associated, then it is dual to the class of a hyperplane section in the projective embedding provided by the representation  $\check{\Gamma}$ .

14.8. COROLLARY. *The first Chern class  $c_1$  of the tangent bundle to  $M_I$  is positive.*

$c_1$  is the sum of the positive complementary roots (10.7) and, by 13.6, these are the linear combinations of the simple roots in which at least one  $a_i$  with  $i \notin I$  has a strictly positive coefficient. Thus, if  $a \in \Psi_I$ , its transform by the symmetry to the plane  $a_i = 0$  ( $i \in I$ ) belongs to  $\Psi_I$ . Since  $(a_i, S_i a + a) = 0$ , it follows that  $(a_i, c_1) = 0$ . This is also a consequence of 14.2.

In view of (5), (6), we have to show that if  $a \in \Psi$ , then  $(a, c_1) > 0$ . Let  $b \in \Psi$  and assume that  $(a, b) < 0$ . Then (§ 2)

$$(7) \quad b, b + a, b + 2a, \dots, b + ka \quad (k = -2(a, b)(a, a)^{-1})$$

are roots of  $G$  and, in fact, belong to  $\Psi$ , since the latter is a closed system (13.7). We have  $(a, b + ka) = (a, a)k/2 > 0$  and

$$(a, b + b + a + \dots + b + ka) = (k + 1)(a, b) + (a, a)k \cdot (k + 1)/2 = 0.$$

From this we deduce readily that we may represent  $\Psi$  as a union of disjoint subsets  $\Psi_j$ , where  $\Psi_j$  consists either of one root  $b$  with  $(a, b) \geq 0$  or of a string of type (7), whose sum is orthogonal to  $a$ ; in the first category we have the set consisting of  $a$  itself, and hence, finally,  $(a, c_1) > 0$ .

14.9. Using some properties of the constants of structure of  $\mathfrak{g}^c$ , one can show that 14.6 gives all invariant 2-forms on  $M_I$ , as indicated in [5]; this implies that the condition of 14.7 is also necessary for  $b$  to be positive as will also follow from § 24.

14.10. We recall that for a kählerian compact manifold  $M$ , the  $d$ - and the  $\bar{\partial}$ -cohomology are identical, and that  $H^i(M, \mathbf{C})$  is a direct sum of subspaces  $H^{p,q}(M)$ , ( $p + q = i$ ), where  $H^{p,q}$  is the space of  $i$ -dimensional cohomology classes which can be represented by exterior differential forms of type  $(p, q)$ . This applies, in particular, to the projective variety  $G/U_I$ .

PROPOSITION. *In the previous notations, we have  $H^{2i+1}(M_I, \mathbf{C}) = 0$  and  $H^{2i}(M_I, \mathbf{C}) = H^{i,i}(M_I)$  for all  $i \geq 0$ .*

For the first assertion, see [2, Théorème 26.1].

As remarked in 14.2, the projection  $\nu_I$  of  $G/T$  onto  $G/U_I$  induces an

injective homomorphism of  $H^*(G/U_I, \mathbf{C})$  in  $H^*(G/T, \mathbf{C})$ . This map is also holomorphic with respect to the complex structures on  $G/T$  and  $G/U_I$  defined by the positive complementary roots, because, in the notations of 14.3, it can be identified with the projection of  $G^c/B$  onto  $G^c/P_I$ , both spaces being endowed with the natural quotient complex structures. Thus  $\nu_I^*$  identifies  $H^{p,q}(M_I)$  with a subspace of  $H^{p,q}(G/T, \mathbf{C})$ , and it suffices to prove our contention for  $G/T$ . Since  $H^*(G/T, \mathbf{C})$  is generated by the unit and its 2-dimensional classes [2, § 26], it is enough to show that  $H^2(G/T, \mathbf{C}) = H^{1,1}(G/T)$ , but this follows from 14.6.

## Chapter V. Special Cases.

### 15. Projective spaces.

15.1. *Complex projective spaces.* We wish to apply Theorem 10.8 to the case where  $G = \mathbf{U}(q)$  and  $U = \mathbf{U}(1) \times \mathbf{U}(q-1)$  and  $G/U = \mathbf{P}_{q-1}(\mathbf{C})$ , ( $q \geq 2$ ). The imbedding of  $U$  in  $G$  is the usual one; namely, as follows:  $\mathbf{U}(q)$  is the group of unitary matrices and  $\mathbf{U}(1) \times \mathbf{U}(q-1)$  is the group of  $q \times q$ -matrices of the form

$$\begin{pmatrix} a' & 0 \\ 0 & A'' \end{pmatrix},$$

where  $a' \in \mathbf{U}(1)$  and  $A'' \in \mathbf{U}(q-1)$ , which is a subgroup of maximal rank of  $G$ . Let  $\mathbf{T}$  be the standard maximal torus of diagonal unitary matrices

$$\begin{bmatrix} e^{2\pi i x_1} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & e^{2\pi i x_q} \end{bmatrix}.$$

$\mathbf{T}$  is contained in  $\mathbf{U}(1) \times \mathbf{U}(q-1)$  and plays the role of  $S$  in Theorem 10.7. The coordinates  $x_1, \dots, x_q$  are integral linear forms on  $V_{\mathbf{T}}$  (see 1.2 and 10.1), and the roots of  $\mathbf{U}(q)$  with respect to  $\mathbf{T}$  are  $\pm(x_j - x_k)$ , where  $1 \leq j < k \leq q$ . The roots of  $\mathbf{U}(1) \times \mathbf{U}(q-1)$  are  $\pm(x_j - x_k)$  with  $2 \leq j < k \leq q$ . Hence the roots of  $\mathbf{U}(q)$  complementary to  $\mathbf{U}(1) \times \mathbf{U}(q-1)$  are  $\pm(x_1 - x_j)$  with  $2 \leq j \leq q$ .

The usual invariant complex structure of  $\mathbf{P}_{q-1}(\mathbf{C})$  is given by regarding  $\mathbf{P}_{q-1}(\mathbf{C})$  as the space of the lines passing through the origin of  $\mathbf{C}^q$ . Let  $\mathbf{GL}(q, \mathbf{C})$  operate in the usual way on  $\mathbf{C}^q$  and thus on  $\mathbf{P}_{q-1}(\mathbf{C})$ . Let  $\mathbf{GL}(1, q-1; \mathbf{C})$  be the subgroup of those elements of  $\mathbf{GL}(q, \mathbf{C})$  which keep the point  $(1, 0, \dots, 0)$  of  $\mathbf{P}_{q-1}(\mathbf{C})$  fixed. Then

$$U(q)/(U(1) \times U(q-1)) = GL(q, C)/GL(1, q-1; C) = P_{q-1}(C).$$

Thus  $P_{q-1}(C)$  is represented as a quotient of complex Lie groups and is, therefore, endowed with an invariant structure which is just the usual one. The complex isotropy representation  $\iota_c$  of this complex structure has the weights  $x_j - x_1$  ( $j = 2, \dots, q$ ), which can be seen as follows: The element  $(e^{2\pi i x_1}, \dots, e^{2\pi i x_q})$  of  $T$ , when operating on the point  $(1, z_2, \dots, z_q)$  of  $P_{q-1}(C)$ , gives the point

$$(1, z_2 e^{2\pi i(x_2 - x_1)}, \dots, z_q e^{2\pi i(x_q - x_1)})$$

of  $P_{q-1}(C)$ , which proves the desired result. We may remark here that  $P_{q-1}(C)$  admits exactly two invariant complex structures and these are complex conjugate to each other (see 13.8).

Let  $\xi$  be a principal  $U(q)$ -bundle. We consider some associated fibre bundles and their projections according to the following diagram: Let  $\rho, \sigma$  be the natural projections

$$(1) \quad E_\xi/T \xrightarrow{\rho} E_\xi/(U(1) \times U(q-1)) \xrightarrow{\sigma} B_\xi$$

and  $\pi = \sigma \circ \rho$ . Let  $\eta$  be the real vector bundle along the fibres (7.4) of  $(E_\xi/(U(1) \times U(q-1)), B_\xi P_{q-1}(C))$  endowed with the complex structure  $\eta'$  coming from the usual invariant complex structure on  $P_{q-1}(C)$ ; i.e.,  $\eta'$  is defined by the complex isotropy representation  $\iota_c$  considered above. Then we have for the total Chern class of  $\eta'$

$$\rho^*c(\eta') = \prod_{j=1}^q (1 + x_j - x_1).$$

Now let  $c_i \in H^{2i}(B_\xi, \mathbf{Z})$  be the Chern classes of  $\xi$ . Then we have (9.1)

$$\pi^*(1 + c_1 + c_2 + \dots + c_q) = (1 + x_1)(1 + x_2) \dots (1 + x_q).$$

Considering  $z$  as an indeterminate over  $H^*(E_\xi/T, \mathbf{Z})$ , we have the equation

$$z^q + z^{q-1}\pi^*(c_1) + \dots + \pi^*(c_q) = (z + x_1)(z + x_2) \dots (z + x_q).$$

Replacing  $z$  by  $1 - x_1$  gives

$$\rho^*c(\eta') = \prod_{j=1}^q (1 + x_j - x_1) = \sum_{i=0}^q (1 - x_1)^{q-i} \pi^*(c_i).$$

$x_1, x_2, \dots, x_q$  are the first Chern classes of the  $q$  principal  $U(1)$ -bundles  $\xi_1, \dots, \xi_q$ , into which the principal bundle  $(E_\xi, E_\xi/T, T)$  splits. The principal bundle  $(E_\xi, E_\xi/(U(1) \times U(q-1)), U(1) \times U(q-1))$  splits into a principal  $U(1)$ -bundle  $\xi'$  and a principal  $U(q-1)$ -bundle  $\xi''$ . Obviously,

$\rho^*(\xi')$  is equivalent to  $\xi_1$  and  $\rho^*(\gamma_1) = x_1$ , where  $\gamma_1$  denotes the first Chern class of  $\xi'$ . Since  $\pi^* = \rho^* \circ \sigma^*$  and since  $\rho^*$  is injective for integral cohomology, we get

$$(2) \quad c(\eta') = \sum_{i=0}^q (1 - \gamma_1)^{q-i} \sigma^*(c_i).$$

Since  $c_q(\eta') = 0$ , we have

$$(3) \quad \sum_{i=0}^q (-\gamma_1)^{q-i} \sigma^*(c_i) = 0.$$

Once  $\gamma_1$  is defined, the Chern classes of  $\xi$  are characterized by (3), a fact due to Hirsch (compare [11]).

To calculate the Chern class of  $\mathbf{P}_{q-1}(\mathbf{C})$ , we now specialize to the case where  $E_\xi = \mathbf{U}(q)$  and where  $B_\xi$  is a single point. Then  $\eta'$  is the tangent bundle of  $\mathbf{P}_{q-1}(\mathbf{C})$ . Since  $c_i = c_i(\xi) = 0$  for  $i > 0$ , we get

$$c(\eta') = c(\mathbf{P}_{q-1}(\mathbf{C})) = (1 - \gamma_1)^q.$$

Now we observe that  $\xi'$  corresponds to the Hopf bundle; i.e.,  $(\mathbf{C}^q - \{0\}, \mathbf{P}_{q-1}(\mathbf{C}), \mathbf{C}^*)$  is the extension of  $\xi'$  with respect to the natural imbedding of  $\mathbf{U}(1)$  in  $\mathbf{C}^*$ . Thus  $-\gamma_1 = e^*$ , where  $e^* \in H^2(\mathbf{P}_{q-1}(\mathbf{C}), \mathbf{Z})$  is dual to the hyperplane of  $\mathbf{P}_{q-1}(\mathbf{C})$  (see §29). Therefore

$$c(\mathbf{P}_{q-1}(\mathbf{C})) = (1 + e^*)^q.$$

15.2. *Complex projective bundles.* In Sections 15.2 and 15.3, all cohomology groups are taken with real coefficients. The projective unitary group is defined by

$$\mathbf{PU}(q) = \mathbf{U}(q)/\mathbf{D},$$

where  $\mathbf{D}$  is the 1-dimensional torus of scalar matrices of  $\mathbf{U}(q)$ . Let  $\mathbf{T}^q$  be the maximal torus of diagonal matrices of  $\mathbf{U}(q)$ . Then  $\tilde{\mathbf{T}}^{q-1} = \mathbf{T}^q/\mathbf{D}$  is a maximal torus of  $\mathbf{PU}(q)$ , and we have the commutative diagram

$$(4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{T}^q & \longrightarrow & \tilde{\mathbf{T}}^{q-1} \longrightarrow 0 \\ & & \downarrow \text{Id} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{U}(q) & \xrightarrow{\alpha} & \mathbf{PU}(q) \longrightarrow 0 \end{array}$$

which induces a commutative diagram for the classifying spaces (6.6)

$$(5) \quad \begin{array}{ccccc} B_{\mathbf{D}} & \longrightarrow & B_{\mathbf{T}^q} & \longrightarrow & B_{\tilde{\mathbf{T}}^{q-1}} \\ \downarrow \text{Id} & & \downarrow \pi & & \downarrow \\ B_{\mathbf{D}} & \longrightarrow & B_{\mathbf{U}(q)} & \xrightarrow{\rho(\alpha)} & B_{\mathbf{PU}(q)} \end{array}$$

where the two horizontal lines come from the fibre bundles  $(B_{T^q}, B_{\tilde{T}^{q-1}}, B_D)$  and  $(B_{U(q)}, B_{PU(q)}, B_D)$ , see [2]. Then, in the commutative diagram

$$(6) \quad \begin{array}{ccc} H^*(B_{T^q}) & \longleftarrow & H^*(B_{\tilde{T}^{q-1}}) \\ \uparrow \pi^* & & \uparrow \rho(\alpha)^* \\ H^*(B_{U(q)}) & \longleftarrow & H^*(B_{PU(q)}) \end{array}$$

which is induced by (5), all arrows indicate injections. If we denote by  $\xi$  the universal principal  $U(q)$ -bundle and by  $\tilde{\xi}$  the universal principal  $PU(q)$ -bundle, then  $\rho(\alpha)^*\tilde{\xi}$  is (up to equivalence) the  $\alpha$ -extension of  $\xi$ . Let  $x_1, \dots, x_q$  be the first Chern classes (with real coefficients) of the  $q$  principal  $U(1)$ -bundles into which  $\pi^*\xi$  splits. Then

$$H^*(B_{T^q}) = \mathbf{R}[x_1, \dots, x_q].$$

Using the Weyl group of  $PU(q)$  (which is isomorphic to that of  $U(q)$ ), it follows easily from diagram (6) that  $\pi^*\rho(\alpha)^*H^*(B_{PU(q)})$  is the subring of those polynomials in  $\mathbf{R}[x_1, \dots, x_q]$  which are symmetric in  $x_1, \dots, x_q$  and invariant under the substitution  $t: x_i \rightarrow x_i + b$ , where  $b$  is an indeterminate. Roughly speaking, for a  $PU(q)$ -bundle the Chern classes  $c_j$  (i.e., the elementary symmetric functions in the  $x_i$ ) make no sense, but the polynomials in the  $c_j$  invariant under  $t$  do.

Now let  $(L, X, \mathbf{P}_{q-1}(\mathbf{C}), \sigma)$  be a bundle with  $PU(q)$  as a structural group. It is known that  $\sigma^*$  maps  $H^*(X)$  isomorphically in  $H^*(L)$  (real cohomology) and that, for every element  $\gamma \in H^2(L)$  whose restriction to the fibre equals the generator  $e^*$  of  $H^2(\mathbf{P}_{q-1}(\mathbf{C}))$ , there is (8.4) a relation

$$(7) \quad \gamma^q - \sigma^*(d_1)\gamma^{q-1} + \sigma^*(d_2)\gamma^{q-2} - \dots + (-1)^q \sigma^*(d_q) = 0$$

with uniquely determined elements  $d_i \in H^{2i}(X)$  depending only on  $\gamma$ . Let  $\eta'$  be the complex vector bundle along the fibres of  $L$ . We recall that  $\mathbf{P}_{q-1}(\mathbf{C}) = PU(q)/((U(1) \times U(q-1))/D)$  and that the complex structure  $\eta'$  comes from the complex isotropy representation  $\iota_c$  considered in 15.1 ( $\iota_c$  is trivial on  $D$ ).

15.3. THEOREM. *The Chern class (with real coefficients) of the complex vector bundle  $\eta'$  along the fibres of a fibre bundle  $(L, X, \mathbf{P}_{q-1}(\mathbf{C}), \sigma)$  with  $PU(q)$  as structural group is given by the formula*

$$(8) \quad c(\eta') = \sum_{i=0}^q (1-\gamma)^{q-i} \sigma^*(d_i), \quad (d_0 = 1),$$

where  $\gamma \in H^2(L)$  is an arbitrary element whose restriction to the fibre gives the generator  $e^*$  and where the  $d_i \in H^2(X)$  are defined by the relation (7).

For the proof, we introduce an indeterminate  $z$ . The polynomial

$$F(z) = \sum_{i=0}^q z^{q-i} \sigma^*(d_i), \quad (\text{see } (7)),$$

is then the unique element of  $\sigma^*H^*(X)[z]$  which is a polynomial of degree  $q$  in  $z$ , has the unit 1 as the coefficient of  $z^q$ , and for which  $F(-\gamma)$  vanishes. If  $\tilde{\gamma} = \gamma + \sigma^*(b)$ , where  $b \in H^2(X)$ , then  $\tilde{F}(z) = F(z + \sigma^*(b))$  is the analogous unique polynomial with  $\tilde{F}(-\tilde{\gamma}) = 0$ . Since  $F(1-\gamma) = F(1-\tilde{\gamma})$ , the right side of (8) is independent of the choice of  $\gamma$ . Taking this into account, (2) and (3) of 15.1 yield our theorem for bundles whose structural groups can be  $\alpha$ -reduced to  $U(q)$ . Furthermore, we see that it is enough to prove the theorem for the universal principal  $PU(q)$ -bundle  $\tilde{\xi}$ . Since the theorem is true for  $\rho(\alpha)^*\tilde{\xi}$  (see (5)), it follows easily in full generality.

Theorem 15.3 was announced in the first note of [16].

15.4. *Real projective spaces.* In Section 15.4, all cohomology groups are taken with  $\mathbf{Z}_2$  as coefficients. Let  $\xi$  be a principal  $O(q)$ -bundle and  $\eta$  the vector bundle along the fibres of  $(E_\xi/(O(1) \times O(q-1)), B_\xi, P_{q-1}(\mathbf{R}))$ . Let  $Q$  be the group of all diagonal matrices of  $O(q)$ . Consider the maps

$$E_\xi/Q \xrightarrow{\rho} E_\xi/(O(1) \times O(q-1)) \xrightarrow{\sigma} B_\xi,$$

and let  $w_i$  ( $1 \leq i \leq q$ ) be the Stiefel-Whitney classes of  $\xi$ . Then we have

$$(9) \quad w(\eta) = \sum_{i=0}^q (1 - \gamma_1)^{q-i} \sigma^*(w_i),$$

where  $\gamma_1$  is the 1-dimensional characteristic class of the  $O(1)$ -bundle over  $E_\xi/(O(1) \times O(q-1))$ . The class  $\gamma_1$  induces on each fibre the generator of its cohomology ring. The proof uses 5.3 and 11.5 but is otherwise formally identical to that given in 15.1 (except that the  $x_i$ 's are now 1-dimensional classes), and is therefore left to the reader. (9) implies the well-known fact that the Stiefel-Whitney class of  $P_{q-1}(\mathbf{R})$  equals  $(1+a)^q$ , where  $a$  is the generator of  $H^*(P_{q-1}(\mathbf{R}))$ .

15.5. *The Pontrjagin classes of the quaternionic projective spaces.* The treatment of the quaternionic projective spaces

$$P_{q-1}(\mathbf{K}) = Sp(q)/(Sp(1) \times Sp(q-1)), \quad q \geq 2,$$

is similar to the discussion in 15.1.

$Sp(q)$  is the group of all unitary quaternionic  $q \times q$ -matrices.  $Sp(q)$  contains  $U(q)$ , and  $U(q)$  contains the maximal torus  $T$  of 15.1 which is

also maximal in  $\mathbf{Sp}(q)$ . When applying Theorem 10.7, we let  $\xi$  be the universal principal  $\mathbf{Sp}(q)$ -bundle. Putting

$$G = \mathbf{Sp}(q) \text{ and } U = \mathbf{Sp}(1) \times \mathbf{Sp}(q-1),$$

we have the diagram

$$(10) \quad B_T = E_\xi / T \xrightarrow{\rho} B_U \xrightarrow{\sigma} B_G,$$

and the integral cohomology ring of  $P_{q-1}(\mathbf{K})$  has to be identified with  $H^{**}(B_U, \mathbf{Z})$  (see 6.1) modulo the ideal  $(I_G^+)^*$ . As in 15.1, we have the elements  $x_1, \dots, x_q \in H^1(T, \mathbf{Z})$  which, via the negative transgression, are to be regarded as elements of  $H^2(B_T, \mathbf{Z})$ .

By means of (10), the cohomology ring  $H^{**}(B_G, \mathbf{Z})$  will be considered as a subring of  $H^{**}(B_U, \mathbf{Z})$ , and  $H^{**}(B_U, \mathbf{Z})$  as a subring of  $H^{**}(B_T, \mathbf{Z})$ . The latter ring will be identified with  $\mathbf{Z}\{x_1, \dots, x_q\}$ . Thus  $H^*(P_{q-1}(\mathbf{K}), \mathbf{Z})$  is the quotient of

$$(11) \quad \mathbf{Z}\{x_1^2\} \otimes S\{x_2^2, \dots, x_q^2\}$$

modulo the ideal  $(I_G^+)^*$  which is generated by the symmetric power series in  $x_1^2, \dots, x_q^2$  without constant terms. (We use here essentially, that  $G$  and  $U$  have no torsion, see [2].)

We restrict ourselves to the calculation of the Pontrjagin classes of  $P_{q-1}(\mathbf{K})$ , i.e., of its tangent bundle. The more general case of the bundle along the fibres is left to the reader.

We do the calculations following a schema which will also be used in other cases.

$$\text{Roots of } \mathbf{Sp}(q): \quad \pm x_i \pm x_j, \pm 2x_k \quad (1 \leq i < j \leq q)$$

$$\text{Roots of } \mathbf{Sp}(1) \times \mathbf{Sp}(q-1): \quad \pm x_i \pm x_j, \pm 2x_k \quad (2 \leq i < j \leq q)$$

Complementary roots:

$$\pm(x_1 - x_2, x_1 - x_3, \dots, x_1 - x_q; x_1 + x_2, x_1 + x_3, \dots, x_1 + x_q).$$

We have

$$(1 + x_2^2)(1 + x_3^2) \cdots (1 + x_q^2) = (1 + x_1^2)^{-1} \pmod{(I_G^+)^*}.$$

This shows that the  $r$ -th elementary symmetric function in the  $x_j^2$  ( $2 \leq j \leq q$ ) equals  $(-1)^r x_1^{2r} \pmod{(I_G^+)^*}$  and that  $x_1^2$  represents an element  $u \in H^4(P_{q-1}(\mathbf{K}), \mathbf{Z})$  which generates  $H^*(P_{q-1}(\mathbf{K}), \mathbf{Z})$ ; in particular,  $u$  is a generator of the infinite cyclic group  $H^4(P_{q-1}(\mathbf{K}), \mathbf{Z})$ .

Introducing an indeterminate  $z$ , we have

$$\prod_{j=1}^q (z + x_j)(z - x_j) = z^{2q} \pmod{(I_G^+)^*}.$$

Setting  $z = 1 + x_1$ , this yields

$$(12) \quad \prod_{j=2}^q (1 + x_1 + x_j)(1 + x_1 - x_j) = (1 + x_1)^{2q}(1 + 2x_1)^{-1} \pmod{(I_G^+)^*}.$$

Theorem 10.7 shows that the (integral) Pontrjagin class of  $P_{q-1}(K)$  is represented by the element

$$\prod_{j=2}^q (1 + (x_1 + x_j)^2)(1 + (x_1 - x_j)^2).$$

Taking into account that we are dealing with graded rings and that  $x_1^2$  represents the generator  $u$  of  $H^*(P_{q-1}(K), \mathbb{Z})$ , we obtain from (12)

$$(13) \quad p(P_{q-1}(K)) = (1 + u)^{2q}(1 + 4u)^{-1}.$$

15.6. *Application.* Formula (13) implies, in particular, for the first (i.e., four-dimensional) Pontrjagin class of  $P_{q-1}(K)$  that

$$p_1 = (2q - 4)u.$$

$p_1$  is different from 0 for  $q - 1 > 1$ .

Since  $\phi^*(p_1) = p_1$  for all diffeomorphisms  $\phi$  of  $P_{q-1}(K)$  onto itself, we see that, for  $q - 1 > 1$ , there does not exist a diffeomorphism  $\phi$  with  $\phi^*(u) = -u$ ; i.e., for all  $\phi$ , we have  $\phi^*(u) = u$ . In particular, all diffeomorphisms preserve orientation ( $\phi^*(u^{q-1}) = u^{q-1}$ ).

This fact on the orientation is obvious for  $q - 1 \equiv 0 \pmod{2}$  and arbitrary homeomorphisms of  $P_{q-1}(K)$  onto itself, since then

$$\phi^*(u) = \pm u \text{ implies } \phi^*(u^{q-1}) = u^{q-1}.$$

15.7. *The Stiefel-Whitney class of the quaternionic projective spaces.* We keep essentially the preceding notations and denote by  $Q$  the subgroup of elements of order 2 in  $T$ . For suitable (1-dimensional) generators  $u_i$  of  $H^*(B_Q, \mathbb{Z}_2)$ , we have [4, § 11]

$$\rho_2^*(Q, T)(x_i) = u_i^2 \quad (i = 1, \dots, q),$$

where here the  $x_i$  are elements of  $H^2(B_T, \mathbb{Z}_2)$ , namely, the reductions mod 2 of the  $x_i$  of 15.5. Thus the images of  $\rho_2^*(Q, Sp(q))$  and  $\rho_2^*(Q, Sp(1) \times Sp(q-1))$  are, respectively,  $S(u_1^4, \dots, u_q^4)$  and  $\mathbb{Z}_2[u_1^4] \otimes S(u_2^4, \dots, u_q^4)$ , and  $H^*(P_{q-1}(K), \mathbb{Z}_2)$  may be identified with the quotient of the latter ring by the ideal  $I$  generated by the symmetric functions in the  $u_i^4$  of strictly positive degrees; in particular,  $u_1^4$  represents the generator  $\bar{u}$  of  $H^*(P_{q-1}(K), \mathbb{Z}_2)$ .

By (5.5), the complementary 2-roots are  $(u_1 - u_i)$ ,  $(i = 2, \dots, p)$ ,

each counted with multiplicity 4, and therefore, 11.5 gives for the Stiefel-Whitney class of  $P_{q-1}(K)$ :

$$w = \prod_{i=2}^q (1 + u_1 - u_i)^4 = \prod_{i=1}^q (1 + u_1^4 - u_i^4) \bmod I$$

which, modulo  $I$ , is equal to  $(1 + u_1^4)^q$ , and hence, finally,

$$w(P_{q-1}(K)) = (1 + \tilde{u})^q,$$

where  $\tilde{u}$  is the generator of  $H^*(P_{q-1}(K), \mathbb{Z}_2)$ .

The characteristic classes of  $P_{q-1}(K)$  were calculated in [17] by a different method.

## 16. Hermitian symmetric spaces.

16.1. We consider here first the homogeneous spaces of the form  $G/U$ , where  $U$  is the centralizer of a 1-dimensional torus  $S$ , has a 1-dimensional center, and  $G$  is semi-simple. As was shown in 13.8, such a space admits exactly two invariant complex structures; they are conjugate to each other and equivalent under an automorphism of  $G$ .

We may assume that  $S$  is defined by  $a_1 = \cdots = a_{l-1} = 0$ , where  $a_1, \dots, a_l$  are the simple roots with respect to some ordering  $\mathcal{S}$ . Then (see 13.6, 13.8), the roots of  $U$  are the linear combinations of the  $a_i$ 's with  $1 \leq i \leq l-1$ , and the set  $\Psi$  of positive complementary roots is closed and is the root system of one of the two invariant complex structures on  $G/U$ , to be denoted by  $\mathcal{C}$ . Moreover, a root  $b$  is in  $\Psi$  if and only if, when expressed as a linear combination of the simple roots, the term containing  $a_l$  has a strictly positive coefficient. (The space  $G/U$  is irreducible hermitian symmetric if and only if  $G$  is simple and  $U$  is maximal connected. In this case, the coefficients of  $a_l$  in the complementary roots are all equal to one and the sum of two elements of  $\Psi$  is never a root of  $G$ .)

By 14.2,  $H^2(G/U, \mathbb{Z})$  is infinite cyclic and has a generator  $g$  such that  $\nu^*(g) = \varpi_l$ , where  $\nu$  is the natural projection of  $G/T$  on  $G/U$  and  $\varpi_l$  is the  $l$ -th fundamental weight. If  $c_1(G/U)$  denotes the first Chern class of  $G/U$  with respect to  $\mathcal{C}$ , we have then, necessarily, that

$$c_1(G/U) = \lambda(G/U) \cdot g \quad (\lambda(G/U) \in \mathbb{Z}).$$

By 14.7 and 14.8, both  $g$  and  $c_1(G/U)$  are positive classes, and hence  $\lambda(G/U) > 0$ . By 10.8,  $\nu^*(c_1)$  is equal to the sum of the positive complementary roots, and hence

$$\lambda(G/U) = 2(b, a_1)/(a_1, a_1), \quad (b = \sum_{a \in \Psi} a).$$

Since the invariant complex structures on  $G/U$  are equivalent,  $\lambda(G/U)$  does not depend on the choice of the invariant complex structure. Among the spaces considered here are the compact irreducible hermitian symmetric spaces which are divided into six classes, see, e.g., A. Borel, *Bull. Soc. Math. France* 80 (1952), 167-182):

- I.  $U(m+n)/(U(m) \times U(n))$
- II.  $SO(2n)/U(n)$
- III.  $Sp(n)/U(n)$
- IV.  $SO(n+2)/(SO(2) \times SO(n)), \quad (n > 2)$
- V.  $E_6/Spin(10) \times T^1$
- VI.  $E_7/E_6 \times T^1$ .

By the preceding formula, we get for  $\lambda(G/U)$  the following values

I.  $m+n$ , II.  $2n-2$ , III.  $n+1$ , IV.  $n$ , V. 12, VI. 18.

In the following sections 16.2 to 16.5, we study the non-exceptional types I.-IV. and give formulas for their Chern classes. We also obtain in these cases the values of  $\lambda(G/U)$  in a different way. To describe the complex structure on  $G/U$ , we choose an ordering having the properties of 13.6. The maximal torus is always chosen in the standard way, i.e., in the cases I, II, III, it is the maximal torus of  $U(m+n)$  or  $U(n)$  respectively, used in 15.1.

16.2. The Grassmannian  $W(m, n) = U(m+n)/(U(m) \times U(n))$ .

As a system of positive roots of  $U(m+n)$ , we take

$$\{-x_i + x_j \mid 1 \leq i < j \leq m+n\}, \quad (\text{see 15.1}).$$

$H^*(W(m, n), \mathbb{Z})$  has to be identified with the quotient of

$$(1) \quad S\{x_1, \dots, x_m\} \otimes S\{x_{m+1}, \dots, x_{m+n}\}$$

by the ideal  $I$  generated by the symmetric power series in  $x_1, \dots, x_{m+n}$  without constant term. The (total) Chern class of  $W(m, n)$  is given by

$$c(W(m, n)) = \prod_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq m+n}} (1 - x_i + x_j) \quad \text{mod } I.$$

In the tensor product (1), we have

$$\prod_{1 \leq i \leq m} (1 + x_i)^{-1} = \prod_{m+1 \leq j \leq m+n} (1 + x_j) \quad \text{mod } I.$$

The  $r$ -th elementary symmetric function in the  $x_i$  ( $1 \leq i \leq m$ ) represents an element  $\sigma_r$  of  $H^*(\mathcal{W}(m, n), \mathbf{Z})$ . The preceding equation shows that the  $\sigma_r$  ( $1 \leq r \leq m$ ) generate  $H^*(\mathcal{W}(m, n), \mathbf{Z})$ . (Recall that we are dealing with graded rings.) The element  $\sigma_1$  generates the infinite cyclic group  $H^2(\mathcal{W}(m, n), \mathbf{Z})$ . Using an indeterminate  $z$ , we have

$$(2) \quad \prod_{m+1 \leq j \leq m+n} (z + x_j) = z^{m+n} \prod_{1 \leq i \leq m} (z + x_i)^{-1} \quad \text{mod } I.$$

By replacing in the preceding formula  $z$  by  $1 - x_s$  ( $1 \leq s \leq m$ ), respectively, we obtain  $m$  equations; multiplying all of these together yields

$$(3) \quad c(\mathcal{W}(m, n)) = \prod_{i=1}^m (1 - x_i)^{m+n} \prod_{1 \leq i \leq j \leq m} (1 - (x_i - x_j)^2)^{-1} \quad \text{mod } I.$$

We recall that  $\sigma_r$  is the  $r$ -th Chern class of the canonical principal  $\mathbf{U}(m)$ -bundle over  $\mathcal{W}(m, n)$ . Formula (3) expresses  $c(\mathcal{W}(m, n))$  by the  $\sigma_r$ ; for example,

$$c_1(\mathcal{W}(m, n)) = -(m + n)\sigma_1,$$

$$c_2(\mathcal{W}(m, n)) = (C_2^{m+n} + m - 1)\sigma_1^2 + (n - m)\sigma_2.$$

The formula for the first Chern class gives us the value of  $\lambda(\mathcal{W}(m, n))$  and shows that  $-\sigma_1$  is a positive generator of  $H^2(\mathcal{W}(m, n), \mathbf{Z})$  in the sense of 16.1. For  $m = 1$ , the Grassmannian  $\mathcal{W}(1, n)$  is the complex projective space  $\mathbf{P}_n(\mathbf{C})$  discussed in 15.1.

16.3. The space  $\mathbf{F}_n = \mathbf{SO}(2n)/\mathbf{U}(n)$ , ( $n \geq 2$ ). As a system of positive roots of  $\mathbf{SO}(2n)$  we take  $\{\pm x_i + x_j \mid 1 \leq i < j \leq n\}$ . We regard the  $x_i$  as elements of  $H^2(B_{\mathbf{F}}, K_p)$ . If  $p \neq 2$ , then  $H^*(\mathbf{F}_n, K_p)$  may be identified with the quotient of  $S\{x_1, \dots, x_n\}$  by the ideal  $I$  generated by the symmetric power series without constant terms in  $x_1^2, \dots, x_n^2$  and by the element  $x_1 x_2 \cdots x_n$ . All power series under consideration have coefficients in  $K_p$ . The total Chern class of  $\mathbf{F}_n$  is given by

$$(4) \quad c(\mathbf{F}_n) = \prod_{1 \leq i < j} (1 + x_i + x_j) \quad \text{mod } I.$$

The preceding formula expresses the Chern classes of  $\mathbf{F}_n$  as polynomials in the elementary symmetric functions of the  $x_i$ . The coefficients are integral. Let  $\sigma_r \in H^{2r}(\mathbf{F}_n, \mathbf{Z})$  be the  $r$ -th Chern class of the canonical  $\mathbf{U}(n)$ -principal

bundle over  $F_n$ . If we reduce  $\sigma_r$  to coefficients  $K_p$  ( $p \neq 2$ ), we get the  $r$ -th elementary symmetric function in the  $x_i \pmod I$ . Since  $F_n$  has no torsion, (4) gives a formula for integral cohomology if we replace the  $r$ -th elementary symmetric function in the  $x_i$  by  $\sigma_r$ . For example, we get

$$c_1(F_n) = (n-1)\sigma_1.$$

$\sigma_1$  is not a generator of the infinite cyclic group  $H^2(F_n, \mathbf{Z})$ , but  $\sigma_1$  equals  $2\tilde{g}$ , where  $\tilde{g}$  is a generator. This is true for  $n=2$ , since  $F_2$  is the complex projective line for which  $c_1(F_2)$  is twice a generator. It follows then for all  $n$  by induction, using the natural imbedding of  $F_n$  in  $F_{n+1}$  and the fibre bundle  $(F_{n+1}, S_{2n}, F_n)$ , see [26, § 41.18]. Thus we have

$$c_1(F_n) = (2n-2)\tilde{g};$$

moreover,  $\tilde{g}$  is a positive generator of  $H^2(F_n, \mathbf{Z})$  and  $\lambda(F_n) = 2n-2$  (see 16.1).

16.4. The space  $G_n = Sp(n)/U(n)$ . As a system of positive roots of  $Sp(n)$  we take

$$\{\pm x_i + x_j (1 \leq i < j \leq n), 2x_i (1 \leq i \leq n)\}.$$

The integral cohomology ring of  $G_n$  has to be identified with the quotient of  $S\{x_1, \dots, x_n\}$  by the ideal  $I$  generated by the symmetric power series without constant terms in  $x_1^2, \dots, x_n^2$ . The (total) Chern class of  $G_n$  is given by

$$c(G_n) = \prod_{1 \leq i \leq j \leq n} (1 + x_i + x_j) \pmod I.$$

This formula expresses the Chern classes of  $G_n$  by the Chern classes  $\sigma_r$  of the canonical  $U(n)$ -bundle over  $G_n$ . The element  $x_1 + \dots + x_n$  represents  $\sigma_1$ , and  $\sigma_1$  is a generator of the infinite cyclic group  $H^2(G_n, \mathbf{Z})$ ; we have

$$c_1(G_n) = (n+1)\sigma_1.$$

Thus  $\sigma_1$  is a positive generator and  $\lambda(G_n) = n+1$  (see (16.1)).

16.5. The complex quadric  $Q_n = SO(n+2)/(SO(2) \times SO(n))$ . We distinguish the two cases (a)  $n$  is even and (b)  $n$  is odd.

$$(a) \quad n+2 = 2k.$$

We have the natural imbedding of  $U(k)$  in  $SO(2k)$  and take for the maximal torus  $T$  of  $SO(2k)$  the maximal torus of  $U(k)$  considered in 15.1. As a system of positive roots of  $SO(2k)$ , we choose

$$\{x_i \pm x_j \mid 1 \leq i < j \leq k\}.$$

The  $x_i$  are to be regarded as elements of  $H^2(B_T, K_p)$ . If  $p \neq 2$ , then  $H^*(Q_n, K_p)$  may be identified with the quotient of the ring  $V$  generated by  $S\{x_2^2, \dots, x_k^2\}$  and by the elements  $x_2 x_3 \cdots x_k$ ,  $x_1$  in  $K_p\{x_1, x_2, \dots, x_k\}$  by the ideal  $I$  generated by the symmetric power series without constant terms in  $x_1^2, \dots, x_k^2$  and by  $x_1 x_2 \cdots x_k$ ; i.e.,

$$H^*(Q_n, K_p) = V/I.$$

Using an indeterminate  $z$ , we have

$$(5) \quad \prod_{i=1}^k (z - x_i)(z + x_i) = z^{2k} \quad \text{mod } I.$$

We have also

$$(6) \quad (1 + x_2^2)(1 + x_3^2) \cdots (1 + x_k^2) = (1 + x_1^2)^{-1} \quad \text{mod } I.$$

From (6), we see easily that the elements

$$1, x_1, x_1^2, \dots, x_1^n, \text{ and } x_2 x_3 \cdots x_k \in V$$

constitute an additive base of  $H^*(Q_n, K_p)$ . The Chern class of  $Q_n$  is given by

$$c(Q_n) = \prod_{j=2}^k (1 + x_1 - x_j)(1 + x_1 + x_j) \quad \text{mod } I.$$

Replacing in (5) the indeterminate  $z$  by  $1 + x_1$ , yields

$$(7a) \quad c(Q_n) = (1 + x_1)^{n+2} (1 + 2x_1)^{-1} \quad \text{mod } I.$$

$$(b) \quad n + 1 = 2k.$$

We have the imbedding of  $U(k) = U(k) \times 1$  in  $SO(2k + 1)$  and take for the maximal torus  $T$  of  $SO(2k + 1)$  the maximal torus of  $U(k)$  considered in 15.1. As a system of positive roots of  $SO(2k + 1)$ , we choose

$$\{x_i \pm x_j \ (1 \leq i < j \leq k); x_i \ (1 \leq i \leq k)\}.$$

If  $p \neq 2$ , then  $H^*(Q_n, K_p)$  may be identified with the quotient of

$$K_p\{x_1\} \otimes S\{x_2^2, \dots, x_k^2\}$$

by the ideal  $I$  generated by the symmetric power series without constant terms in  $x_1^2, \dots, x_k^2$ . As in the case (a), we see that

$$1, x_1, x_1^2, \dots, x_1^n$$

constitute an additive base of  $H^*(Q_n, K_p)$ . The Chern class of  $Q_n$  is given by

$$c(Q_n) = (1 + x_1) \prod_{j=2}^k (1 + x_1 - x_j)(1 + x_1 + x_j), \quad \text{mod } I.$$

As in (a), we get

$$(7b) \quad c(Q_n) = (1 + x_1)(1 + x_1)^{2k}(1 + 2x_1)^{-1} = (1 + x_1)^{n+2}(1 + 2x_1)^{-1} \pmod{I}.$$

Now we combine again the two cases (a) and (b). Let  $\tilde{g}$  be the Euler class of the canonical principal  $SO(2)$ -bundle over  $Q_n = SO(n+2)/(SO(2) \times SO(n))$ ,

$$\tilde{g} \in H^2(Q_n, \mathbf{Z}).$$

If we apply the coefficient homomorphism  $\mathbf{Z} \rightarrow K_p$ , the element  $\tilde{g}$  goes over into  $x_1 \pmod{I}$ . Since  $Q_n$  has no torsion, we get from (7a) and (7b)

$$(8) \quad c(Q_n) = (1 + \tilde{g})^{n+2}(1 + 2\tilde{g})^{-1}.$$

For the Pontrjagin class, we obtain

$$(9) \quad p(Q_n) = (1 + \tilde{g}^2)^{n+2}(1 + 4\tilde{g}^2)^{-1}.$$

The Euler number  $E(Q_n)$  equals  $n+2$  in the case (a), resp.  $n+1$  in the case (b). An easy calculation shows that (8) implies

$$2c_n(Q_n) = E(Q_n)\tilde{g}^n.$$

Therefore  $\tilde{g}^n$  is twice a generator of  $H^n(Q_n, \mathbf{Z})$  and it follows that  $\tilde{g}$  is a generator of  $H^2(Q_n, \mathbf{Z})$  for  $n > 2$ . Formula (8) shows that  $\tilde{g}$  is the positive generator  $g$  of 16.1 and that, for  $n > 2$ ,  $\lambda(Q_n)$  equals  $n$ . For  $n=1$  the quadric  $Q_n$  is the complex projective line; for  $n=2$  it is reducible, namely the product of two projective lines. For  $n \neq 2$ ,  $Q_n$  is irreducible.

(8) can, of course, be derived by other methods (see, e.g., F. Hirzebruch, Proc. Intern. Congress Math. 1954, Vol. III, pp. 457-473).

16.6. The homogeneous space  $Q_n = SO(n+2)/(SO(2) \times SO(n))$  can be regarded as the space of oriented planes through the origin of  $\mathbf{R}^{n+2}$ . In this section, we assume  $n > 2$ . If one attaches to each oriented plane the same plane with the opposite orientation, one gets a one-one real analytic map  $\sigma$  of  $Q_n$  onto itself.  $\sigma$  has no fixed points and is involutive ( $\sigma\sigma = \text{Id}$ ). Identifying the points  $u$  and  $\sigma(u)$  of  $Q_n$  gives a manifold

$$\tilde{Q}_n = Q_n/\sigma, \quad \pi: Q_n \rightarrow \tilde{Q}_n.$$

Here  $\pi$  denotes the covering map.  $Q_n$  is simply connected and is a twofold covering of  $\tilde{Q}_n$ . The manifold  $\tilde{Q}_n$  is the space of all (non-oriented) planes through the origin of  $\mathbf{R}^{n+2}$ , or the space of all projective lines in  $\mathbf{P}_{n+1}(\mathbf{R})$ . For  $p \neq 2$ , it is known that  $\pi^*$  maps  $H^*(\tilde{Q}_n, K_p)$  isomorphically onto the

subring of those elements of  $H^*(Q_n, K_p)$  which are invariant under  $\sigma^*$ . For the Pontrjagin class of  $\tilde{Q}_n$ , we have

$$(10) \quad \pi^*(p(\tilde{Q}_n)) = p(Q_n) = (1 + g^2)^{n+2}(1 + 4g^2)^{-1},$$

where  $g$  is a generator of  $H^2(Q_n, \mathbb{Z})$ . By (10), the Pontrjagin class of  $\tilde{Q}_n$  with coefficients reduced to  $K_p$  ( $p \neq 2$ ) is completely given.

Let  $a$  be the following element of  $SO(n+2)$ :  $a$  is a diagonal matrix which has the entry  $-1$  at the first and third places in the diagonal and otherwise  $+1$ . Then  $a$  is in the normalizer of  $SO(2) \times SO(n)$ . The operation of  $a$  by right translation on  $Q_n$  is just  $\sigma$ . (We may remark here that  $\sigma$  is a map which carries one of the homogeneous complex structures into the other.) On the other hand,  $a$  induces the operation of the Weyl group which maps  $x_1, x_2$  in  $-x_1, -x_2$  and keeps the other  $x_i$  fixed. Thus we know how  $\sigma^*$  operates on the additive bases of  $H^*(Q_n, K_p)$  given in (a), resp. (b). In either case, the ring of invariants of  $\sigma^*$  is generated by  $x_1^2$ . If  $n$  is odd, then  $\tilde{Q}_n$  is non-orientable. For  $n = 2m$ , we see that  $\tilde{Q}_n$  is orientable and that  $H^*(P_m(K), K_p)$  is isomorphic to  $H^*(\tilde{Q}_n, K_p)$ . The isomorphism

$$\alpha_p: H^*(P_m(K), K_p) \rightarrow H^*(\tilde{Q}_n, K_p)$$

can be chosen such that

$$(\pi^* \circ \alpha_p)(u_p) = x_1^2 \pmod{I} = g^2 \text{ (reduced to } K_p),$$

where  $u_p$  is the reduction to coefficients  $K_p$  of the generator  $u \in H^4(P_m(K), \mathbb{Z})$  used in 15.5. Under the isomorphism  $\alpha_p$ , the Pontrjagin class (coefficients  $K_p$ ) of  $P_m(K)$  is carried over into that of  $\tilde{Q}_n$  (see 15.5 and (10)). The value of  $\alpha_0(u_0^m)$  on the fundamental cycle of  $\tilde{Q}_n$  equals  $\pm 1$ , since  $g^n$  takes the value  $\pm 2$  on the fundamental cycle of  $Q_n$ . Therefore, when using proper orientations, the Pontrjagin numbers of  $P_m(K)$  equal those of  $\tilde{Q}_n$ .

The proof in [17] of the fact that  $P_m(K)$ , for  $m \neq 2, 3$ , does not admit an almost complex structure, works also for  $\tilde{Q}_{2m}$  and thus shows that  $\tilde{Q}_{2m}$  ( $m \neq 2, 3$ ) does not admit an almost complex structure (compatible with its usual differentiable structure).

16.7. *Remark.* Let us consider  $Q_n$  as imbedded in  $P_{n+1}(\mathbb{C})$  by the equation

$$z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0.$$

The conjugation map

$$z = (z_0, \dots, z_{n+1}) \rightarrow \bar{z} = (z_0, \dots, z_{n+1})$$

induces a map  $\kappa$  of  $Q_n$  onto itself which has no fixed point and which is involutive. If  $z \in Q_n$ , then the line passing through  $z, \kappa z$  is a real line. If we attach to the point  $z$  this real line, then we get a homeomorphism between  $Q_n/\kappa$  and the space of the projective lines in  $P_{n+1}(R)$ . The map  $\kappa$  corresponds to  $\sigma$ .

### 17. The Stiefel-Whitney class of $G_2/SO(4)$ .

17.1. We recall first some known properties of  $G_2$  and of the Cayley numbers. The Cayley-Graves algebra of octonions over the real numbers will be denoted by  $\mathfrak{Q}$ . It is spanned by 1 and seven purely imaginary elements  $e_i$  ( $i \in \mathbf{Z}_7$ ) satisfying

$$(1) \quad e_i \cdot e_{i+1} = e_{i+3}; e_{i+1} \cdot e_{i+3} = e_i; e_{i+3} \cdot e_i = e_{i+1} \quad (i \in \mathbf{Z}_7),$$

and, of course,  $e_i \cdot e_i = -1$ . Thus  $e_i, e_{i+1}, e_{i+3}$  generate a subalgebra isomorphic to the field of quaternionic numbers.

$G_2$  may be defined as the group of automorphism of  $\mathfrak{Q}$ . It is a compact, connected, and simply connected 14-dimensional Lie group of rank 2 and with center reduced to the identity; it leaves invariant the subspace  $\mathfrak{M}$  of  $\mathfrak{Q}$  spanned by the  $e_i$ 's and thus may be identified with a subgroup of  $SO(7)$ , whose Lie algebra is the following:

Let  $G_{ij}$  ( $1 \leq i, j \leq 7$ ) be the endomorphisms of  $\mathfrak{M}$  defined by  $G_{ii} = 0$  and

$$G_{ij}(e_j) = e_i; G_{ij}(e_i) = -e_j; G_{ij}(e_k) = 0 \quad (i \neq j; k \neq i, j).$$

The  $G_{ij}$  ( $i < j$ ) form a basis of the Lie algebra of  $SO(7)$ . We have

$$(2) \quad \begin{aligned} G_{ij} + G_{ji} &= 0, [G_{ij}, G_{jk}] = G_{ik} \quad (i \neq j; j \neq k), \\ [G_{ij}, G_{kl}] &= 0 \quad (i, j, k, l \text{ pairwise distinct}). \end{aligned}$$

Using (1) and (2), it is readily seen that the Lie algebra  $\mathfrak{g}_2$  of  $G_2$ , that is, the Lie algebra of derivations of  $\mathfrak{Q}$ , is the direct sum of the seven 2-dimensional commutative subalgebras<sup>\*</sup>

$$(3) \quad v_i = \{aG_{i+1, i+3} + bG_{i+2, i+6} + cG_{i+4, i+5}; a + b + c = 0\}$$

and that we have

$$(4) \quad [v_i, v_{i+1}] = v_{i+3}; [v_{i+1}, v_{i+3}] = v_i; [v_{i+3}, v_i] = v_{i+1} \quad (i \in \mathbf{Z}_7).$$

<sup>\*</sup> For more details, see H. Freudenthal, "Oktavengeometrie," mimeographed Notes, University of Utrecht. Freudenthal's  $e_1, \dots, e_7$  are replaced here by  $e_1, e_4, e_2, e_7, -e_3, e_5, -e_6$  respectively. On page 17, line 14 from the bottom, read  $G_{76}$  instead of  $G_{67}$ .

The subspace  $u = v_3 + v_4 + v_6$  is therefore a subalgebra of rank 2 and dimension 6, and hence is isomorphic to the Lie algebra of  $SO(4)$ , as can also be easily checked directly. Thus the subgroup  $U$  of  $G_2$  generated by  $u$  is a compact group locally isomorphic to  $SO(4)$ , that is, having the product  $Sp(1) \times Sp(1)$  as universal covering. It is, in fact, globally isomorphic to  $SO(4)$ , as can be derived for instance from [7]. In fact, it is proved in [7] that  $G_2$  contains exactly one class (relative to inner automorphisms) of subgroups locally isomorphic to  $SO(4)$ ; one of them, say  $U_1$ , is the centralizer of a vertex  $z$  of order two of a fundamental simplex, and  $z$  generates the center of  $U_1$  ([7], Remarque II, p. 220). Looking at the diagram of  $G_2$ , one sees that the two invariant 3-dimensional subgroups of  $U_1$  are globally isomorphic to  $Sp(1)$ ; since  $U_1$  has a center of order two, it is then isomorphic to  $SO(4)$ .

17.2. The subgroup  $Q$ . The relations (1) imply that the linear transformation  $S_i$  ( $i \in \mathbf{Z}_7$ ) of  $\mathfrak{M}$  which keeps  $e_{i+1}$ ,  $e_{i+5}$ ,  $e_{i+6}$  fixed and changes the signs of the other  $e_j$ 's is an automorphism of  $\mathfrak{L}$ . The seven elements  $S_i$  and the identity form a commutative subgroup  $Q$  of  $G_2$  of type  $(2, 2, 2)$ . Moreover,  $G_2$  contains no commutative subgroup of type  $(2, 2, 2, 2)$  (see A. Borel-J.-P. Serre, *Comm. Math. Helv.* 27 (1953), 128-129 or 17.5).

PROPOSITION. We keep the previous notations and denote by  $x_i$  the element of  $\text{Hom}(Q, \mathbf{Z}_2)$  defined by  $x_i(S_j) = \delta_{ij}$  ( $1 \leq i, j \leq 3$ ). Then  $Q \subset U$ , and the 2-roots of  $U$  (resp.  $G_2$ ) with respect to  $Q$  are  $x_1 + x_2$ ,  $x_1 + x_3$ ,  $x_2 + x_3$  (resp. together with  $x_1, x_2, x_3, x_1 + x_2 + x_3$ ). Each has multiplicity 2 and is the character of  $Q$  in one of the  $v_i$ 's.

It follows from the definition of  $U$  that this group leaves invariant the subspaces  $\mathfrak{C}$ ,  $\mathfrak{D}$  of  $\mathfrak{M}$  spanned, respectively, by  $e_3, e_4, e_6$  and  $e_1, e_2, e_5, e_7$  and that the restriction to  $\mathfrak{D}$  of the standard maximal abelian subgroup  $Q'$  of type  $(2, 2, 2)$  of  $U$  consists of the diagonal matrices of determinant  $+1$ . On the other hand, it is readily seen that this is also the restriction of  $Q$  to  $\mathfrak{D}$ ; since by (1) an automorphism of  $\mathfrak{L}$  leaving  $e_1, e_2, e_5, e_7$  fixed must be the identity, we have  $Q' = Q$  and  $Q \subset U$ . The other assertions follow from the fact that the inner automorphism  $\text{Ad} S_i$  defined by  $S_i$  is the identity on  $v_{i+1}, v_{i+5}, v_{i+6}$  and is  $-\text{Id}$  on the other  $v_j$ 's.

17.3. The cohomology ring mod 2 of  $G_2/SO(4)$ . The following facts are proved in [3]:  $H^*(B_{SO(4)}, \mathbf{Z}_2)$  and  $H^*(B_{G_2}, \mathbf{Z}_2)$  are rings of polynomials in three variables of degrees 2, 3, 4 and 4, 6, 7 respectively; the homomorphisms  $\rho_2^*(Q, SO(4))$  and  $\rho_2^*(Q, G_2)$  are injective. The ring  $H^*(G_2/SO(4), \mathbf{Z}_2)$

is the quotient of  $H^*(B_{SO(4)}, \mathbf{Z}_2)$  by the ideal generated by the elements of strictly positive degrees in the image of  $\rho_2^*(SO(4), \mathbf{G}_2)$ ; the Poincaré polynomial mod 2 of  $\mathbf{G}_2/SO(4)$  is

$$P_2(\mathbf{G}_2/SO(4), t) = (1 - t^4)(1 - t^6)(1 - t^7)/(1 - t^2)(1 - t^3)(1 - t^4),$$

and hence

$$P_2(\mathbf{G}_2/SO(4), t) = 1 + t^2 + t^3 + t^4 + t^5 + t^6 + t^8.$$

**PROPOSITION.** We keep the previous notations and denote by  $\sigma_i$  the  $i$ -th elementary symmetric function in the  $x_i$ 's. Then the image of  $\rho_2^*(Q, U)$  equals  $\mathbf{Z}_2[u_2, u_3, u_4]$  with  $u_2 = \sigma_2 + \sigma_1^2$ ,  $u_3 = \sigma_3 + \sigma_1\sigma_2$ ,  $u_4 = \sigma_1\sigma_3$ , and the image of  $\rho_2^*(Q, \mathbf{G}_2)$  equals  $\mathbf{Z}_2[g_4, g_6, g_7]$  with  $g_4 = u_2^2 + u_4$ ,  $g_6 = u_3^2 + u_2u_4$  and  $g_7 = u_4u_3$ . Consequently,  $H^*(\mathbf{G}_2/SO(4), \mathbf{Z}_2)$  is generated by two elements  $u_2, u_3$  of degrees 2, 3 with the relations  $u_2^3 = u_3^2$  and  $u_3u_2^2 = 0$ .

We identify  $U$  with  $SO(4)$  by means of the representation in  $\mathfrak{D}$ . Let  $Q_1$  be the subgroup of diagonal matrices of  $O(4)$  and  $\mu$  the inclusion of  $Q$  in  $Q_1$ . Then, for an obvious choice of a basis  $(y_1, y_2, y_3, y_4)$  of  $\text{Hom}(Q_1, \mathbf{Z}_2)$ , the image of  $\rho_2^*(Q_1, O(4))$  is the ring of symmetric functions in the  $y_i$  (cf. [3]), and the homomorphism  $\mu': \text{Hom}(Q_1, \mathbf{Z}_2) \rightarrow \text{Hom}(Q, \mathbf{Z}_2)$  induced by  $\mu$  is given by

$$\mu'(y_i) = x_i \quad (i = 1, 2, 3), \quad \mu'(y_4) = x_1 + x_2 + x_3.$$

Therefore,  $\mu'$  annihilates  $y_1 + y_2 + y_3 + y_4$  and maps the  $i$ -th elementary symmetric function in the  $y_j$ 's onto  $u_i$ , for  $i = 2, 3, 4$ . Since  $\rho_2^*(Q, O(4))$  and  $\rho_2^*(Q, SO(4))$  have the same image (see [3]), this proves our first assertion.

The image of  $\rho_2^*(Q, \mathbf{G}_2)$  is a subring of  $\mathbf{Z}_2[u_2, u_3, u_4]$  generated by elements of degrees 4, 6, 7 and its elements are invariant under the action of the normalizer of  $Q$  in  $\mathbf{G}_2$ , operating in the usual way [3]; therefore, in order to prove the second assertion, it suffices to exhibit an automorphism  $\alpha$  of  $H^*(B_Q, \mathbf{Z}_2) = \mathbf{Z}_2[x_1, x_2, x_3]$  induced by an inner automorphism of  $\mathbf{G}_2$ , leaving  $Q$  invariant, and for which  $g_i$  is the only non-zero invariant of degree  $i$  ( $i = 4, 6, 7$ ).

Let  $S$  be the linear transformation of  $\mathfrak{M}$  which sends  $e_1, \dots, e_7$  onto  $e_5, e_7, -e_3, e_4, e_1, -e_6, e_2$  respectively. It follows from (1) that  $S \in \mathbf{G}_2$ . Moreover, it is seen without difficulty that

$$S \cdot S_1 \cdot S = S_1, \quad S \cdot S_2 \cdot S = S_1 \cdot S_3 = S_4, \quad S \cdot S_3 \cdot S = S_1 \cdot S_2 = S_6$$

and, therefore, the automorphism  $\alpha$  of  $H^*(B_Q, \mathbf{Z}_2)$  induced by  $\text{Ad } S$  satisfies:

$$(5) \quad \alpha(x_1) = x_1, \quad \alpha(x_2) = x_1 + x_3, \quad \alpha(x_3) = x_1 + x_2,$$

$$(6) \quad \alpha(\sigma_1) = \sigma_1, \quad \alpha(\sigma_2) = x_1^2 + x_2x_3, \quad \alpha(\sigma_3) = x_1(x_1^2 + \sigma_2),$$

$$\alpha(u_2) = x_2^2 + x_3^2 + x_2x_3,$$

$$(7) \quad \alpha(u_3) = x_2x_3(x_2 + x_3),$$

$$\alpha(u_4) = x_1(x_1^2 + \sigma_2)\sigma_1.$$

An element  $h \in H^4(B_U, \mathbf{Z}_2)$  may be written in the form  $h = a \cdot u_4 + b \cdot u_2^2$  ( $a, b \in \mathbf{Z}_2$ ), and hence

$$\alpha(h) = ax_1(x_1^2 + \sigma_2)\sigma_1 + bx_2^2x_3^2 + b(x_2^4 + x_3^4).$$

The coefficients of  $x_1^4$  in  $h$  and  $\alpha(h)$  are  $b$  and  $a$  respectively; therefore, if  $h = \alpha(h)$  with  $h \neq 0$  we must have  $a = b = 1$  and  $h = g_4$ . That  $g_4$  is in fact invariant under  $\alpha$  can be checked directly, but this is not necessary since we know a priori that  $H^4(B_{G_2}, \mathbf{Z}_2)$  has dimension one.

The proofs of the invariance of  $g_6, g_7$  under  $\alpha$  are quite analogous: an element of degree six may be written

$$h = au_2^3 + bu_3^2 + cu_2g_4 \quad (a, b, c \in \mathbf{Z}_2).$$

Using (7), one sees that the coefficients of  $x_1^6$  in  $h$  and  $\alpha(h)$  are  $a + c$  and zero respectively, while those of  $x_1^4 \cdot x_2^2$  are  $a + b + c$  and  $c$ . Thus  $\alpha(h) = h$  and  $h \neq 0$  imply  $a = b = c = 1$  and  $h = g_6$ . Finally, starting with a general element

$$h = u_3(au_2^2 + bg_4) \quad (a, b \in \mathbf{Z}_2)$$

of degree seven, we see by looking at the coefficients of  $x_1^6 \cdot x_2$  that  $h = \alpha(h)$  and  $h \neq 0$  imply  $a = b = 1$ , that is  $h = g_7$ .

The last assertion follows then from the results recalled at the beginning of 17.3.

17.4. PROPOSITION. *The Stiefel-Whitney classes of  $G_2/\mathbf{SO}(4)$  are non-zero only in the dimensions 0, 4, 6, 8.*

By 11.5 and 17.2, the image under  $\rho_2^*(Q, U)$  of the total Stiefel-Whitney class of the bundle along the fibres of  $(B_U, B_{G_2}, G/U)$  is

$$w' = (1 + \sigma_1)^2 \cdot \prod_{i=1}^3 (1 + x_i)^2;$$

therefore

$$w' = (1 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2)(1 + \sigma_1^2) = 1 + u_2^2 + u_3^2 + u_4^2,$$

and 17.4 follows now from 17.3.

17.5. *Remarks.* 1) The 2-roots of  $G_2$  have been computed with respect to a particular subgroup of type  $(2, 2, 2)$ . In fact, these subgroups are conjugate by inner automorphisms, and these 2-roots are therefore invariants of  $G_2$ . To see this, one uses the fact that given three orthonormal purely imaginary Cayley numbers  $u, v, w$  with  $w$  also orthogonal to  $u \cdot v$ , there exists exactly one automorphism of  $\mathfrak{Q}$  which maps  $e_1, e_2, e_3$  onto  $u, v, w$ , respectively (see N. Jacobson, *Duke Math. Jour.* 5 (1939), 776-783). This implies easily that any commutative subgroup of type  $(2, 2, 2)$  of  $G_2$  can be put in the diagonal form by means of an automorphism of  $G_2$ . Moreover, one deduces from (1) that  $Q$  contains all diagonal matrices of  $G_2$ ; hence,  $G_2$  does not contain commutative subgroups of type  $(2, 2, 2, 2)$ .

(2) It follows a posteriori from the proof of 17.3 that  $\rho_{20}(Q, G_2)$  maps  $H^*(B_{G_2}, \mathbb{Z}_2)$  isomorphically onto the ring of invariants of the normalizer of  $Q$  in  $G_2$ . Thus, the analogy between the role of  $Q$  in cohomology mod 2 and that of a maximal torus in real cohomology, which is the basis of [3], is also very complete for  $G_2$ .

## 18. Some manifolds with Poincaré polynomial $1 + t^4 + t^8$ .

18.1. The quaternionic plane  $P_2(K)$  is an 8-dimensional manifold with real Poincaré polynomial  $1 + t^4 + t^8$ . We know (15.5) that the (integral) Pontrjagin class of  $P_2(K)$  is given by

$$p = (1 + u)^6(1 + 4u)^{-1}.$$

Thus  $p_1 = 2u$  and  $p_2 = 7u^2$ . The Pontrjagin numbers of  $P_2(K)$  are

$$p_1^2[P_2(K)] = 4 \quad \text{and} \quad p_2[P_2(K)] = 7.$$

Here we use the orientation defined by  $u^2[P_2(K)] = 1$ .

18.2. The manifolds  $G_2/SO(4)$  (see §17) and  $\tilde{Q}_4$  (see 16.6) have the real Poincaré polynomial  $1 + t^4 + t^8$ . Their Pontrjagin numbers are (for suitable orientations) the same as those of the quaternionic plane. For  $\tilde{Q}_4$ , this was proved already in 16.6. In the next section, it will be shown for  $G_2/SO(4)$ . We do not know whether all differentiable manifolds with

real Poincaré polynomial  $1 + t^4 + t^8$  have the same Pontrjagin numbers. By the index theorem ([19], p. 85), we know for such a manifold  $X$  that (for suitable orientation)

$$(1) \quad (7p_2 - p_1^2)[X] = 45,$$

and therefore, it is sufficient to calculate one Pontrjagin number. But, as an example, we shall make the computations without the use of the index theorem in the case of  $G_2/SO(4)$ .

*Remark.* Milnor has constructed an 8-dimensional combinatorial manifold with real Poincaré polynomial  $1 + t^4 + t^8$  whose Pontrjagin numbers (in the sense of Thom) satisfy (1), but are rational, non-integral numbers. This manifold of Milnor does not admit a differentiable structure compatible with its combinatorial structure.

18.3. By the Hirsch formula, the manifold  $G_2/SO(4)$  has the real Poincaré polynomial  $(1 - t^4)(1 - t^{12})(1 - t^4)^{-1}(1 - t^4)^{-1} = 1 + t^4 + t^8$ . We calculate the Pontrjagin class of  $G_2/SO(4)$  by the schema used in 15.5. All cohomology groups are taken with real coefficients.

*Roots of  $G_2$ :*

$$\pm x_1, \pm x_2, \pm(x_1 - x_2), \pm(x_1 - 2x_2), \pm(x_1 - 3x_2), \pm(2x_1 - 3x_2)$$

with respect to a convenient base  $x_1, x_2 \in H^2(G_2/T)$  for a maximal torus  $T$  of  $G_2$ .

Following de Siebenthal [25a] we take  $\phi_1 = x_2$  and  $\phi_2 = x_1 - 3x_2$  as simple roots of  $G_2$ . The dominant root is then  $3\phi_1 + 2\phi_2 = 2x_1 - 3x_2$ . By [1, p. 218], we know that there is an imbedding of  $SO(4)$  in  $G_2$  for which  $\pm\phi_1$  and  $\pm(3\phi_1 + 2\phi_2)$  are the roots of  $SO(4)$ .

*Complementary roots:*  $\pm x_1, \pm(x_1 - x_2), \pm(x_1 - 2x_2), \pm(x_1 - 3x_2)$ .

We put  $y_1 = 2x_1 - 3x_2$  and  $y_2 = x_2$ .

*Invariants of the Weyl group of  $G_2$ :* Since  $H^*(B_{G_2})$  is the polynomial ring over  $\mathbf{R}$  in two indeterminates of degrees 4 and 12, we have only one invariant in dimension 4. Since the dimension of  $G_2/SO(4)$  equals 8, this is the only invariant we need. An invariant of dimension 4 is always given by the sum of the squares of all roots, which, up to a factor, is, in our case,

$$4(x_1^2 - 3x_1x_2 + 3x_2^2) = (y_1^2 + 3y_2^2).$$

It is convenient to express the complementary roots as linear combinations of  $y_1, y_2$ . This gives

$$\begin{aligned}x_1 &= \frac{1}{2}(y_1 + 3y_2), \\x_1 - x_2 &= \frac{1}{2}(y_1 + y_2), \\x_1 - 2x_2 &= \frac{1}{2}(y_1 - y_2), \\x_1 - 3x_2 &= \frac{1}{2}(y_1 - 3y_2).\end{aligned}$$

Euler class  $W$  of  $G_2/SO(4)$ :

$$\begin{aligned}\pm 16W &= (y_1^2 - y_2^2)(y_1^2 - 9y_2^2) = 4y_2^2 \cdot 12y_2^2, \\ \pm W &= 3y_2^4,\end{aligned}$$

the computations being made modulo the invariants. Since the Euler number of  $G_2/SO(4)$  equals 3, we get

$$(2) \quad y_2^4[G_2/SO(4)] = 1$$

after choosing the orientation of  $G_2/SO(4)$  conveniently.

The Pontrjagin class  $p$  of  $G_2/SO(4)$ :

$$\begin{aligned}1 + 4p_1 + 16p_2 &= (1 + (y_1 + 3y_2)^2)(1 + (y_1 - 3y_2)^2)(1 + (y_1 - y_2)^2)(1 + (y_1 + y_2)^2) \\ &= (1 + 2(y_1^2 + 9y_2^2) + (y_1^2 - 9y_2^2)^2)(1 + 2(y_1^2 + y_2^2) + (y_1^2 - y_2^2)^2) \\ &= (1 + 12y_2^2 + 144y_2^4)(1 - 4y_2^2 + 16y_2^4).\end{aligned}$$

$$1 + p_1 + p_2 = (1 + 3y_2^2 + 9y_2^4)(1 - y_2^2 + y_2^4).$$

$$p_1 = 2y_2^2, \quad p_2 = 7y_2^4.$$

(All calculations modulo the invariants.)

By (2), we get for the Pontrjagin numbers (with respect to the orientation defined by (2))

$$p_1^2[G_2/SO(4)] = 4, \quad p_2[G_2/SO(4)] = 7.$$

## 19. The Cayley plane.

19.1. The center of the simply connected representative of the local structure  $F_4$  consists only of the unit element, as is well known. The structure  $F_4$  has, therefore, one and only one representative which we also denote by  $F_4$ . According to [7], the group  $F_4$  contains exactly one class (relative to inner

automorphisms) of subgroups with local structure  $B_4$ . They are, in fact, globally isomorphic to  $\mathbf{Spin}(9)$ . The homogeneous space  $F_4/\mathbf{Spin}(9)$  has dimension 16 and may be identified with the Cayley plane  $W$ , the projective plane over the Cayley-numbers. The Cayley plane  $W$ , considered as the homogeneous space  $F_4/\mathbf{Spin}(9)$ , was studied, for instance, in [1], see also [2], § 29 and Freudenthal, loc. cit.,<sup>8</sup> § 17. The integral cohomology of  $W$  is given by

$$H^0 = H^8 = H^{16} = \mathbb{Z}, \quad H^i = 0 \text{ otherwise.}$$

19.2. Let  $T$  be the standard maximal torus of  $\mathbf{SO}(9)$  with the base  $x_1, x_2, x_3, x_4 \in H^1(T, \mathbb{Z})$  (see 16.5(b)). Then the roots of  $\mathbf{SO}(9)$  are

$$(1) \quad \pm x_i \pm x_j \quad (1 \leq i < j \leq 4); \quad \pm x_1, \pm x_2, \pm x_3, \pm x_4.$$

We have the projection

$$\pi: \mathbf{Spin}(9) \rightarrow \mathbf{SO}(9).$$

$\pi^{-1}(T) = T'$  is a maximal torus of  $\mathbf{Spin}(9)$ . The restriction of  $\pi$  to  $T'$  induces an isomorphism of  $H^1(T, \mathbb{R})$  onto  $H^1(T', \mathbb{R})$ . Thus  $x_1, x_2, x_3, x_4$  may be regarded as elements of  $H^1(T', \mathbb{R})$ . They constitute a base of  $H^1(T', \mathbb{R})$ . By [7, Théorème 4], we can choose an embedding of  $\mathbf{Spin}(9)$  in  $F_4$  such that the roots of  $F_4$  with respect to  $T'$  (considered as elements of  $H^1(T', \mathbb{R})$ ) are those given in (1) together with the following (see, e.g., [25a]):

$$(2) \quad \frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4),$$

which are the roots of  $F_4$  complementary to  $\mathbf{Spin}(9)$ . We now regard  $x_1, x_2, x_3, x_4$  as elements of  $H^2(B_{T'}, \mathbb{R})$ . We introduce the elementary symmetric functions  $a_1, \dots, a_4$  in the  $x_i^2$

$$(3) \quad 1 + a_1 + a_2 + a_3 + a_4 = \prod_{i=1}^4 (1 + x_i^2).$$

The polynomials

$$(4) \quad a_1, -6a_3 + a_1a_2, 12a_4 + a_2^2 - \frac{1}{2}a_1^2a_2$$

are invariants of the Weyl group of  $F_4$ .

*Proof.* Since the  $a_i$  are invariants of the Weyl group of  $\mathbf{Spin}(9)$ , it suffices to check that the polynomials (4) are invariant under the reflection to the plane

$$x_1 + x_2 + x_3 + x_4 = 0,$$

with respect to the usual Euclidean metric.

19.3. The calculation of the Pontrjagin class of  $W$  goes in the same way as for  $G_2/SO(4)$  (see §18). We indicate briefly the various steps of the computation. We put

$$r_i = \frac{1}{2}(x_1 \pm x_2 \pm x_3 \pm x_4), \quad (i = 1, 2, \dots, 8).$$

For the Pontrjagin class with real coefficients, we get, modulo the invariants (4),

$$p(W) = \prod_{i=1}^8 (1 + r_i^2) = 1 - a_2 - 13a_4, \text{ i.e.,}$$

$$(5) \quad p_1 = p_3 = 0, \quad p_2 = -a_2, \quad p_4 = -13a_4,$$

$$(6) \quad p_2^2 = a_2^2 = -12a_4.$$

Let  $u$  be a generator of the infinite cyclic group  $H^s(W, \mathbb{Z})$ . Then the Euler class of  $W$  equals  $\pm 3u^2$ . On the other hand, we have, after reducing to real coefficients and modulo the invariants (4),

$$\pm 3u^2 = \prod_{i=1}^8 r_i = -a_4 = a_2^2/12.$$

The preceding equation and (6) yield, since  $p_2$  is a real multiple of  $u$ ,

$$(7) \quad p_2^2 = +36u^2, \quad p_2 = \pm 6u,$$

$$(8) \quad p_4 = 39u^2.$$

Since  $W$  is without torsion, we conclude that (7) and (8) are also true in integral cohomology. We choose the generator  $u$  such that  $p_2 = 6u$ .

19.4. THEOREM. *There exists a generator  $u$  of the infinite cyclic group  $H^s(W, \mathbb{Z})$  such that the integral Pontrjagin classes of  $W$  are given by*

$$p_2(W) = 6u, \quad p_4(W) = 39u^2.$$

Choosing that orientation of  $W$  which is defined by  $u^2$ , the non-vanishing Pontrjagin numbers of  $W$  are

$$(9) \quad p_2^2[W] = 36, \quad p_4[W] = 39.$$

19.5. The manifold  $W$ , oriented as in 19.4, has the index  $\tau(W) = 1$ . By the index theorem ([19], Satz 8.2.2), we have

$$(10) \quad (381p_4 - 19p_2^2)[W] = 3^4 \cdot 5^2 \cdot 7.$$

We shall see later in this paper by some general arguments that the  $A$ -genus ([19], 1.6) of  $W$  vanishes. This, together with (10), gives a system of two

linear equations for the Pontrjagin numbers from which (9) can also be obtained.

19.6. From Theorem 19.4, we can easily draw the following consequences.

(a) Let  $\mathcal{P}_5^1$  be the Steenrod reduced power

$$\mathcal{P}_5^1: H^k(X, \mathbf{Z}_5) \rightarrow H^{k+8}(X, \mathbf{Z}_5).$$

For the generator  $u$  of 19.4 we have, by [15] (coefficients reduced to  $\mathbf{Z}_5$ ),

$$(11) \quad \mathcal{P}_5^1 u = \frac{1}{5} (7p_2 - p_1^2)u = -2p_2 u = -2u^2.$$

(11) implies that, for each homeomorphism  $\phi$  of  $W$  onto  $W$ , we have  $\phi^* u = u$ .

(b) The manifold  $W$  with its usual differentiable structure does not admit an almost complex structure.

*Proof.*

$$c^2 = (1 + c_4 + c_8)^2 = 1 + 6u + 39u^2$$

would imply

$$c_8 = 15u^2,$$

but, for an almost complex structure, we would have

$$c_8 = \pm 3u^2.$$

(c) The (total) Stiefel-Whitney class  $w$  of  $W$  is

$$w = 1 + u + u^2 \text{ (coefficients reduced to } \mathbf{Z}_2 \text{)}.$$

*Proof.* We have (coefficients reduced to  $\mathbf{Z}_2$ ), see 9.2, 9.3 and Appendix II,

$$w_8^2 = p_4, \quad w_{16} = 3u^2 \text{ (Euler class).}$$

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# THE CRITERION FOR UNIT MULTIPLICITY AND A GENERALIZATION OF HENSEL'S LEMMA.\*<sup>1</sup>

By WEI-LIANG CHOW.

*To Artin on his 60th birthday*

The well-known criterion for unit multiplicity in algebraic geometry states that if two cycles  $\mathfrak{X}^r$  and  $\mathfrak{Y}^s$  in a variety  $V^n$  ( $n = r + s$ ) are transversal to each other at a simple point  $\alpha$  in  $V$ , then  $\mathfrak{X}$  and  $\mathfrak{Y}$  intersect properly at  $\alpha$  with the multiplicity 1. This is the version of the criterion as formulated in Weil's *Foundations of Algebraic Geometry*,<sup>2</sup> whereby we have restricted ourselves for simplicity to the case  $r + s = n$ . It is possible to give this criterion another formulation which is somewhat more elementary in the sense that it involves the multiplicity of a specialization rather than that of an intersection. It can be stated as follows: If two cycles  $\mathfrak{X}^r$  and  $\mathfrak{Y}^s$  are specializations of two subvarieties  $\mathfrak{X}'$  and  $\mathfrak{Y}'$  respectively in a variety  $V^n$  ( $n = r + s$ ) over a field of definition  $k$  for  $V$ , if  $\mathfrak{X}' \cap \mathfrak{Y}'$  consists of a finite set of (distinct) points  $a^{(i)}$ , and if  $\mathfrak{X}$  and  $\mathfrak{Y}$  are transversal to each other at a simple point  $\alpha$  in  $V$ , then the point  $\alpha$  occurs exactly once in every specialization of the set  $a^{(i)}$  over the specialization  $(\mathfrak{X}', \mathfrak{Y}') \rightarrow (\mathfrak{X}, \mathfrak{Y})$  over  $k$ . This version of the criterion for unit multiplicity appeared in the earlier intersection theory of van der Waerden,<sup>3</sup> and played an essential role there in the development of the concept of the intersection-multiplicity itself. However, if one examines very closely the treatment of van der Waerden, one observes that only a part of this criterion, namely the assertion that the specialization-multiplicity is *at most* one, was given a simple proof before the introduction of the concept of the intersection-multiplicity, while the other part, which asserts that this specialization-multiplicity is *at least* one, was proved only later after the concept of the intersection-multiplicity had

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<sup>2</sup> See Chapter VI, Theorem 6.

<sup>3</sup> B. L. van der Waerden, "Zur algebraischen Geometrie V. Ein Kriterium fuer die Einfachheit von Schnittpunkten," *Math. Annalen*, 110, 128-133, 1934.

already been introduced.<sup>4</sup> In fact, this second part concerning the positivity of this particular specialization-multiplicity was proved by van der Waerden in a rather complicated and indirect way, at first only for cycles in a projective space and later carried over to the general case. As we have recently developed a new theory of intersection which preserves the basic ideas of the original theory of van der Waerden, although in an extensively generalized form, we become naturally interested in finding a proof of the criterion for unit multiplicity which is completely free from any use of the notion of the intersection-multiplicity. It turns out that we are able not only to find a simple proof of this nature, but also at the same time, by virtue of the abstract local form in which we have recast this criterion itself, to obtain a more precise result which contains as a special case a certain generalization of the Hensel's Lemma conjectured by Weil some years ago. Although the subject forms an integral part of our intersection theory, which will be published elsewhere at a later date, we feel that this connection with the Hensel's Lemma justifies its separate publication here as a short note. In keeping with the foundational nature of the subject, we shall make our treatment as elementary as possible, using only a few elementary properties in the theory of local rings. Finally, we shall show in the last section that this connection with Hensel's Lemma can be extended to a more general result in the intersection-theory, namely the invariance of intersection-multiplicity under specialization; it turns out that this invariance can be easily deduced from the Associativity Formula, as generalized in a recent paper<sup>5</sup> of Nagata. Here we shall of course need the more sophisticated results in the theory of local rings.

1. Let  $K$  be a field, and let  $v$  be a real discrete valuation of  $K$ ; let  $K_v$  be the valuation ring of  $v$  in  $K$ , let  $\pi$  be a generator of its maximal prime ideal, and let  $\bar{K}$  be the residue field of  $K_v$  modulo the maximal prime ideal. We denote by  $S^m$  and  $\bar{S}^m$  the projective spaces over the fields  $K$  and  $\bar{K}$  respectively, and we shall be interested in the relation between the varieties and cycles contained in them. Let  $\alpha$  be a rational point over  $\bar{K}$  in  $\bar{S}^m$ . We choose an affine coordinate system in  $S^m$  so that in the corresponding affine coordinate system in  $\bar{S}^m$  the point  $\alpha$  is at the origin; we introduce

<sup>4</sup> The first part was proved in the paper cited above; the second part is contained implicitly in B. L. van der Waerden, "Zur algebraischen Geometrie, XIV. Schnittpunktszahlen von algebraischen Mannigfaltigkeiten," *Math. Annalen*, 115, 619-642, 1938, see § 4.

<sup>5</sup> M. Nagata, "The Theory of Multiplicity in General Local Rings," *Proceedings of the International Symposium on Algebraic Number Theory*, Tokyo-Nikko 1955, pp. 191-226.

the indeterminates  $X_1, \dots, X_m$  corresponding to the affine coordinates so that we have  $\alpha = (0, \dots, 0)$  and consider the ring  $K[X] = K[X_1, \dots, X_m]$  as well as the rings  $K_v[X] = K_v[X_1, \dots, X_m]$  and  $\bar{K}[X] = \bar{K}[X_1, \dots, X_m]$ . The canonical homomorphism  $E$  of  $K_v$  onto  $\bar{K}$  can be extended to a homomorphism of the polynomial ring  $K_v[X]$  onto the polynomial ring  $\bar{K}[X]$ , and for any  $f(X)$  in  $K_v[X]$  we shall denote its image under  $E$  in  $\bar{K}[X]$  by  $\bar{f}(X)$ . Let  $\mathfrak{Q}(S/\alpha)$  be the set of all quotients  $f(X)/g(X)$  of two elements  $f(X)$  and  $g(X)$  in  $K_v[X]$  such that  $\bar{g}(\alpha) \neq 0$ ; it is easily seen that  $\mathfrak{Q}(S/\alpha)$  is a local ring and that its maximal prime ideal  $p(S/\alpha)$  is generated by the  $m+1$  elements  $\pi, X_1, \dots, X_m$ . Similarly, let  $\mathfrak{Q}(\bar{S}/\alpha)$  be the set of all quotients  $\phi(X)/\theta(X)$  of two elements  $\phi(X)$  and  $\theta(X)$  in  $\bar{K}[X]$  such that  $\theta(\alpha) \neq 0$ ; it is well known that  $\mathfrak{Q}(\bar{S}/\alpha)$  is a regular local ring of dimension  $m$  and that its maximal prime ideal  $p(\bar{S}/\alpha)$  is generated by the  $m$  elements  $X_1, \dots, X_m$ . It is easily seen that the homomorphism  $E$  of  $K_v[X]$  onto  $\bar{K}[X]$  can be extended to a homomorphism of  $\mathfrak{Q}(S/\alpha)$  onto  $\mathfrak{Q}(\bar{S}/\alpha)$  and that the kernel of the so extended homomorphism  $E$  is the ideal in  $\mathfrak{Q}(S/\alpha)$  generated by the element  $\pi$ . This shows, in particular, that  $\mathfrak{Q}(S/\alpha)$  has a dimension at least  $m+1$ , and since  $p(S/\alpha)$  is generated by the  $m+1$  elements  $\pi, X_1, \dots, X_m$ , it follows that  $\mathfrak{Q}(S/\alpha)$  is a regular local ring of dimension  $m+1$ .

Let  $\mathfrak{X}^r$  be a positive cycle in  $S^m$ , rational over  $K$ , and let  $F(U) = F(\alpha U, {}_1U, \dots, {}_rU)$  be the associated form of  $\mathfrak{X}$ , where each  ${}_iU$  is a system of  $m+1$  indeterminates; by a proper choice of the proportionality factor, we can obtain that the coefficients in  $F(U)$  are all in  $K_v$ , but not all in the maximal prime ideal, and we observe that in this way the coefficients in  $F(U)$  are uniquely determined up to a common factor which is a unit in  $K_v$ . If we denote by  $\bar{F}(U)$  the form obtained from  $F(U)$  by replacing every coefficient by its image in  $\bar{K}$ , then  $\bar{F}(U)$  is the associated form of a positive  $r$ -cycle  $\bar{\mathfrak{X}}$  in  $\bar{S}^m$ ; since the form  $\bar{F}(U)$  is uniquely determined up to a factor in  $\bar{K}$  by the form  $F(U)$ , the cycle  $\bar{\mathfrak{X}}$  is uniquely determined by the cycle  $\mathfrak{X}$ . We shall call the cycle  $\bar{\mathfrak{X}}$  the *specialization of the cycle  $\mathfrak{X}$  over the valuation  $v$* . We shall now express this situation in terms of local ideals at a given point in  $\bar{\mathfrak{X}}$ . Let  $\alpha$  be a point in  $\bar{\mathfrak{X}}$  (i.e., a point in the support of  $\bar{\mathfrak{X}}$ ), rational over  $\bar{K}$ , and assume that an affine coordinate system in  $S^m$  has been chosen as indicated above. Let  $\mathfrak{P}$  be the ideal in  $K[X]$  associated with the support of the positive cycle  $\mathfrak{X}$ , i.e.,  $\mathfrak{P}$  consists of all polynomials in  $K[X]$  which vanish at every point in the support of  $\mathfrak{X}$ ; we set  $\mathfrak{P}_v = K_v[X] \cap \mathfrak{P}$  and  $\mathfrak{P}_\alpha = \mathfrak{Q}(S/\alpha)\mathfrak{P}_v$ . The fact that  $\mathfrak{X}$  has the dimension  $r$  can be expressed by the property that every minimal prime divisor of  $\mathfrak{P}$  has the rank  $m-r$ ;

we recall that the rank of a prime ideal is defined as the maximum length of a descending chain of prime ideals starting with, but not including, the given ideal itself, and the rank of any ideal is defined as the minimum of the ranks of its minimal prime divisors. Since every element in  $\mathfrak{P}$  can be made into an element in  $\mathfrak{P}_v$  by multiplying with a suitable power of  $\pi$ , it follows that every minimal prime divisor of  $\mathfrak{P}_v$  must have the same rank  $m-r$ ; since  $\mathfrak{X}$  contains the point  $\alpha$ , the ideal  $p(\mathcal{S}/\alpha)$  contains  $\mathfrak{P}_\alpha$ , and it follows then from well-known properties of quotient rings that every minimal prime divisor of  $\mathfrak{P}_\alpha$  must have the same rank  $m-r$  as  $\mathfrak{P}_v$ . Let  $\bar{\mathfrak{P}}_v$  be the image ideal of  $\mathfrak{P}_v$  under the homomorphism  $E$ ; then  $\bar{\mathfrak{P}}_v$  is contained in the ideal associated with the support of  $\bar{\mathfrak{X}}$ , and hence every minimal prime divisor of  $\bar{\mathfrak{P}}_v$  must have a rank at most  $m-r$ . If we denote by  $\bar{\mathfrak{P}}_\alpha$  the image ideal of  $\mathfrak{P}_\alpha$  under the homomorphism  $E$ , then it is easily seen that  $\bar{\mathfrak{P}}_\alpha = \mathcal{Q}(\bar{\mathcal{S}}/\alpha)\bar{\mathfrak{P}}_v$ ; hence  $\bar{\mathfrak{P}}_\alpha$  must have a rank at most  $m-r$ .

We shall say that the positive cycle  $\bar{\mathfrak{X}}$  is *simple* at the point  $\alpha$  if exactly one component variety  $\bar{\mathfrak{X}}_\alpha$  in  $\bar{\mathfrak{X}}$ , with the coefficient 1, contains the point  $\alpha$ , and if this component is simple at  $\alpha$ .

LEMMA 1. *If the cycle  $\bar{\mathfrak{X}}$  is simple at the point  $\alpha$ , then  $\bar{\mathfrak{P}}_\alpha$  has a basis consisting of  $m-r$  elements which can be extended to a minimal basis for  $p(\bar{\mathcal{S}}/\alpha)$ ; and when such is the case, then  $\bar{\mathfrak{P}}_\alpha$  is the prime ideal in  $\mathcal{Q}(\bar{\mathcal{S}}/\alpha)$  associated with the variety  $\bar{\mathfrak{X}}_\alpha$ .*

*Proof.* We assume that  $\bar{\mathfrak{X}} = \bar{\mathfrak{X}}_\alpha + \bar{\mathfrak{X}}_\beta$ , where  $\bar{\mathfrak{X}}_\alpha$  is the variety having a simple point at  $\alpha$  and  $\bar{\mathfrak{X}}_\beta$  is a positive cycle not containing  $\alpha$  at all, and let  $\bar{F}(U) = \Phi_1(U)\Phi_2(U)$  be the corresponding factorization of the associated form; since  $\bar{\mathfrak{X}}_\alpha$  and hence also  $\bar{\mathfrak{X}}_\beta$  are rational over  $\bar{K}$ , the forms  $\Phi_1(U)$  and  $\Phi_2(U)$  can also be taken to be rational over  $\bar{K}$ . Since  $\bar{\mathfrak{X}}_\alpha$  is simple at the origin  $\alpha$ , we can choose the affine coordinate system so that the equations  $X_{r+1} = \cdots = X_m = 0$  define the tangential space of  $\bar{\mathfrak{X}}_\alpha$  at  $\alpha$ . We set  ${}_i u_0 = X_i$ ,  ${}_i u_i = 1$ ,  ${}_i u_j = 0$  ( $j \neq 0, i$ ), for  $i = 1, \cdots, m$ , and consider the  $m-r$  polynomials  $\Psi_k(X) = \Phi_1({}_k u, {}_1 u, \cdots, {}_r u)$ ,  $k = r+1, \cdots, m$ ; if we assume for the moment that  $\bar{K}$  is infinite, then for a suitable choice of the affine coordinate system in  $\bar{\mathcal{S}}^m$  we can obtain that the determinant  $|\partial \Psi_k(X)/\partial X_j|$ ,  $j, k = r+1, \cdots, m$ , does not vanish at  $\alpha$ . We set  $\phi_k(X) = \bar{F}({}_k u, {}_1 u, \cdots, {}_r u) = \Phi_1({}_k u, {}_1 u, \cdots, {}_r u)\Phi_2({}_k u, {}_1 u, \cdots, {}_r u)$ ; since the second factor on the right does not vanish at all at  $\alpha$ , it follows that the determinant  $|\partial \phi_k(X)/\partial X_j|$ ,  $j, k = r+1, \cdots, m$ , also does not vanish at  $\alpha$ . This shows that the elements  $X_1, \cdots, X_r, \phi_{r+1}(X), \cdots, \phi_m(X)$  form a minimal basis for  $p(\bar{\mathcal{S}}/\alpha)$ . If we now set  $f_k(X) = F({}_k u, {}_1 u, \cdots, {}_r u)$ ,

$k=r+1, \dots, m$ , then it is easily seen that the polynomials  $f_k(X)$  all vanish at every point of  $\mathfrak{X}$  and hence are contained in  $\mathfrak{P}_v$  (and hence also in  $\mathfrak{P}_a$ ); since we have evidently  $\phi_k(X) = \bar{f}_k(X)$  for every  $k$ , this shows that the polynomials  $\phi_k(X)$  are all in  $\mathfrak{P}_a$ . Since the polynomials  $\phi_k(X)$  generate in  $\mathfrak{Q}(\bar{S}/\alpha)$  a prime ideal which has the rank  $m-r$  and is contained in  $\mathfrak{P}_a$ , and since  $\mathfrak{P}_a$  has a rank at most  $m-r$ , it follows that  $\mathfrak{P}_a$  must coincide with this prime ideal and hence has the polynomials  $\phi_k(X)$  as a basis. Finally, since the prime ideal in  $\mathfrak{Q}(\bar{S}/\alpha)$  associated with  $\bar{\mathfrak{X}}_a$  has the rank  $m-r$  and contains  $\mathfrak{P}_a$ , it must coincide with  $\mathfrak{P}_a$ .

In case  $\bar{K}$  is finite, we take the ring  $K_v[w]$  obtained from  $K_v$  by the adjunction of an auxiliary variable  $w$  over  $K$ , and consider the quotient ring  $K_{v,w}$  of  $K_v[w]$  with respect to the set  $K_v[w] - K_v[w]\pi$ ; if we denote by  $\bar{w}$  the  $K_{v,w}\pi$ -residue of  $w$ , then  $\bar{w}$  is a variable over  $\bar{K}$ , and  $\bar{K}(\bar{w})$  is the residue field of  $K_{v,w}$ . If we replace  $K_v$  by  $K_{v,w}$  and apply our lemma, and if we denote by  $\mathfrak{Q}(\bar{S}/\alpha, \bar{K}(\bar{w}))$  the local ring defined in a similar way as  $\mathfrak{Q}(\bar{S}/\alpha)$  with  $\bar{K}(\bar{w})$  replacing  $\bar{K}$ , then we conclude that the ideal  $\mathfrak{Q}(\bar{S}/\alpha, \bar{K}(\bar{w}))\bar{\mathfrak{P}}_a$  is the prime ideal in  $\mathfrak{Q}(\bar{S}/\alpha, \bar{K}(\bar{w}))$  associated with  $\bar{\mathfrak{X}}_a$ ; since  $\bar{\mathfrak{P}}_a$  is the contraction of  $\mathfrak{Q}(\bar{S}/\alpha, \bar{K}(\bar{w}))\bar{\mathfrak{P}}_a$  in  $\mathfrak{Q}(\bar{S}/\alpha)$ , it follows that the ideal  $\bar{\mathfrak{P}}_a$  is the prime ideal in  $\mathfrak{Q}(\bar{S}/\alpha)$  associated with  $\bar{\mathfrak{X}}_a$ .

In the next two sections we shall be concerned only with those points in  $S^m$  (and in  $\bar{S}^m$ ) which are finite with respect to the given affine coordinate system, so that we shall be really dealing with affine varieties and cycles. For the sake of convenience we shall keep the same notations as used before for the entities in the projective space; there will be no danger of confusion, as the context will make it clear what is meant in each case.

2. Consider now a variety  $V^n$  in  $S^m$ , defined over  $K$ , and assume that the specialization  $\bar{V}$  of  $V$  over  $v$  is simple at the point  $\alpha$ . Let  $\mathfrak{D}$  be the prime ideal in  $K[X]$  associated with  $V$ , and set  $\mathfrak{D}_v = K_v[X] \cap \mathfrak{D}$ ,  $\mathfrak{D}_\alpha = \mathfrak{Q}(S/\alpha)\mathfrak{D}_v$ ,  $R = \mathfrak{Q}(S/\alpha)/\mathfrak{D}_\alpha$ , and  $\mathfrak{m} = \mathfrak{p}(S/\alpha)/\mathfrak{D}_\alpha$ ; then  $R$  is a local ring with  $\mathfrak{m}$  as the maximal prime ideal, and since  $\mathfrak{D}_\alpha$  does not contain any element in  $K_v$ , we can embed  $K_v$  canonically as a subring in  $R$ . Let  $\bar{\mathfrak{D}}_\alpha$  be the image ideal of  $\mathfrak{D}_\alpha$  under the homomorphism  $E$ , and set  $\bar{R} = \mathfrak{Q}(\bar{S}/\alpha)/\bar{\mathfrak{D}}_\alpha$ ; then  $\bar{R}$  is a local ring, and since  $\bar{\mathfrak{D}}_\alpha$  does not contain any element in  $\bar{K}$ , we can embed  $\bar{K}$  canonically as a subring in  $\bar{R}$ . It is clear that  $E$  induces a homomorphism of  $R$  onto  $\bar{R}$ , which we shall denote by  $E_R$ ; and if we denote by  $G$  and  $\bar{G}$  the canonical homomorphisms of  $\mathfrak{Q}(S/\alpha)$  onto  $R$  and of  $\mathfrak{Q}(\bar{S}/\alpha)$  onto  $\bar{R}$  respectively, then we have the relation  $E_R G = \bar{G} E$ . We set  $x_i = G(X_i)$  and  $\bar{x}_i = \bar{G}(X_i)$  for  $i=1, \dots, m$ ; then the point  $x = (x_1, \dots, x_m)$  in  $S^m$  is a generic point of  $V$  over  $K$  and the point  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$  is a generic point

over  $\bar{K}$  of the component  $\bar{V}_\alpha$  of  $\bar{V}$  containing  $\alpha$ , and it is clear that  $\bar{x}$  is the image of  $x$  under the homomorphism  $E_R$ . According to Lemma 1 above, applied to  $V$  as a prime cycle in  $S^m$ , the ideal  $\bar{\mathfrak{D}}_\alpha$  is the prime ideal in  $\mathfrak{Q}(\bar{S}/\alpha)$  associated with  $\bar{V}_\alpha$ ; since  $\bar{V}_\alpha$  is simple at  $\alpha$ , this means that  $\bar{R}$  is a regular local ring of dimension  $n$ . If now  $y_1, \dots, y_n$  are elements in  $R$  such that their images  $\bar{y}_1, \dots, \bar{y}_n$  under  $E_R$  form a minimal basis for the maximal prime ideal  $\bar{\mathfrak{m}}$  in  $\bar{R}$ , then it is easily seen that the  $n+1$  elements  $\pi, y_1, \dots, y_n$  will form a basis for the maximal prime ideal  $\mathfrak{m}$  in  $R$ ; since  $R$  has the dimension at least  $n+1$ , this shows that this basis is minimal and that  $R$  is a regular local ring of dimension  $n+1$ .

We consider the (affine) coordinate ring  $K[x]$  of  $V$ , and the subring  $K_v[x]$  contained in it; it is clear that  $K[x]$  is the quotient ring of  $K_v[x]$  with respect to the multiplicatively closed set of all non-zero elements in  $K_v$ . Let  $\mathfrak{i}$  be an ideal in  $K_v[x]$ , and let  $\mathfrak{p}_i, i=1, \dots, c$ , be the minimal prime divisors of  $\mathfrak{i}$ ; then, for each  $\mathfrak{p}_i$  which does not contain any element in  $K_v$  (i.e.  $K[x]\mathfrak{p}_i \neq K[x]$ ), the ideal  $K[x]\mathfrak{p}_i$  is a minimal prime divisor of the ideal  $K[x]\mathfrak{i}$ , and in this way we obtain all the minimal prime divisors of  $K[x]\mathfrak{i}$ . It is well-known that if the ideal  $\mathfrak{i}$  is the contraction of an ideal in  $K[x]$ , then none of the ideals  $\mathfrak{p}_i$  can contain any element in  $K_v$  and  $K[x]\mathfrak{i}$  is the only ideal in  $K[x]$  which contracts to  $\mathfrak{i}$  in  $K_v[x]$ . If we denote by  $W$  and  $W_i$  the subsets in  $V$  (or rather in the part of  $V$  which is finite with respect to the given affine coordinate system) defined by the ideals  $\mathfrak{i}$  and  $\mathfrak{p}_i$  respectively (or by  $K[x]\mathfrak{i}$  and  $K[x]\mathfrak{p}_i$  respectively), then each  $W_i$  is an affine  $K$ -variety and we have the equation  $W = \bigcup_i W_i$ ; it is clear that if  $\mathfrak{i}$  is the contraction of an ideal in  $K[x]$ , then none of the sets  $W_i$  is empty.

Consider now the coordinate ring  $\bar{K}[\bar{x}]$  of the variety  $\bar{V}_\alpha$ , and denote by  $E'$  the homomorphism of  $K_v[x]$  onto  $\bar{K}[\bar{x}]$  which is the restriction to  $K_v[x]$  of  $E_R$  ( $E'$  is also an extension of the canonical homomorphism  $E$  of  $K_v$  onto  $\bar{K}$ ); we observe that the kernel of  $E'$  is the ideal  $K_v[x]\pi$ . We denote by  $\bar{\mathfrak{i}}$  and  $\bar{\mathfrak{p}}_i$  the image ideals of  $\mathfrak{i}$  and  $\mathfrak{p}_i$  respectively under  $E'$ ; we have evidently the relation  $\bar{\mathfrak{i}} \subset \bar{\mathfrak{p}}_i$ . On the other hand, if  $f_i$  are elements in  $\mathfrak{p}_i$  such that their images  $\bar{f}_i$  under  $E'$  all coincide with one element  $\phi$  in  $\bigcap_i \bar{\mathfrak{p}}_i$ , then the element  $\prod_i f_i$  is in the radical of  $\mathfrak{i}$ ; since the image of the radical of  $\mathfrak{i}$  under  $E'$  is contained in the radical of  $\bar{\mathfrak{i}}$ , this means that a power of the element  $\phi$  is in the radical of  $\bar{\mathfrak{i}}$ , and hence  $\phi$  is itself in the radical of  $\bar{\mathfrak{i}}$ ; this shows that  $\bigcap_i \bar{\mathfrak{p}}_i$  is contained in the radical of  $\bar{\mathfrak{i}}$ . If we denote by  $\bar{W}_i$  the subsets in  $\bar{V}_\alpha$  defined by the ideals  $\bar{\mathfrak{p}}_i$ , then it follows from what we have just said that the union  $\bigcup_i \bar{W}_i$  is precisely the subset in  $\bar{V}_\alpha$  defined by the

ideal  $\bar{i}$ . Now, for each  $\mathfrak{p}_i$  which does not contain any element in  $K_v$  (and hence  $W_i$  is not empty), the set  $\bar{W}_i$  is by definition the specialization of the set  $W_i$  over  $E$  (or over the valuation  $v$ ); hence the set  $\bar{W} = \bigcup_i \bar{W}_i$ , where the symbol  $\bigcup'$  indicates that the union is taken over all  $i$  such that  $\mathfrak{p}_i$  does not contain any non-zero element in  $K_v$ , is the specialization of  $W$  over  $E$ .

Now, the point  $\alpha$  in  $\bar{V}_\alpha$  is the only zero of the ideal  $\bar{K}[\bar{x}] \cap \bar{m}$ , and  $\bar{R}$  is the quotient ring of  $\bar{K}[\bar{x}]$  with respect to  $\bar{K}[\bar{x}] - (\bar{K}[\bar{x}] \cap \bar{m})$ ; it follows that  $\alpha$  is contained in  $\bar{W}$  if and only if for at least one  $i$  the ideal  $\bar{R}\bar{\mathfrak{p}}_i$  is not the unit ideal and the ideal  $\mathfrak{p}_i$  does not contain any non-zero element in  $K_v$ . Since  $\bar{R}\bar{\mathfrak{p}}_i$  is clearly the image of  $R\mathfrak{p}_i$  under  $E_R$ , it follows that  $\alpha$  is contained in  $\bar{W}$  if and only if for at least one  $i$  the ideal  $R\mathfrak{p}_i$  does not contain any non-zero element in  $K_v$ , and it is easily seen that this is so if and only if the ideal  $R\bar{i}$  does not contain any non-zero element in  $K_v$ .

We shall say that the point  $\alpha$  is an *isolated* zero of the ideal  $\bar{i}$  if the ideal  $\bar{R}\bar{i}$  is primary for the ideal  $\bar{m}$ , and the isolated zero  $\alpha$  of  $\bar{i}$  is said to be a *simple* zero of  $\bar{i}$  if  $\bar{R}\bar{i} = \bar{m}$ .

**THEOREM 1.** *If the point  $\alpha$  is an isolated zero of  $\bar{i}$  and if  $R\bar{i}$  has a rank at most  $n$ , then  $\alpha$  is contained in  $\bar{W}$ ; furthermore, if  $K_v$  is complete and if  $\alpha$  is a simple zero of  $\bar{i}$ , then there is exactly one point  $a$  in  $W$  which specializes over  $E$  to  $\alpha$ , and this point  $a$  is rational over  $K_v$  and is a simple zero of  $i$ .*

*Proof.* To prove that  $\alpha$  is contained in  $\bar{W}$ , we have to show that the ideal  $R\bar{i}$  does not contain any non-zero element in  $K_v$ ; we assume that  $\bar{R}\bar{i}$  is primary for  $\bar{m}$  and the  $R\bar{i}$  contains a non-zero element in  $K_v$ , and we shall show that  $R\bar{i}$  must then have the rank  $n+1$ . In fact, for every  $i$ , the prime ideal  $R\mathfrak{p}_i \cap K_v$  in  $K_v$  is in this case not the zero ideal and hence must coincide with the ideal  $K_v\pi$ ; it follows that every ideal  $R\mathfrak{p}_i$  contains the kernel  $R\pi$  of  $E_R$  and hence must coincide with the inverse image under  $E_R$  of the ideal  $\bar{R}\bar{\mathfrak{p}}_i$ . Since  $\bar{R}\bar{i}$  is primary for  $\bar{m}$ , every ideal  $\bar{R}\bar{\mathfrak{p}}_i$  is primary for  $\bar{m}$ ; it follows that every ideal  $R\mathfrak{p}_i$  is primary for  $\mathfrak{m}$  and hence has the rank  $n+1$ , and this implies that  $R\bar{i}$  has also the rank  $n+1$ .

If  $\bar{R}\bar{i} = \bar{m}$ , then we have the relation  $R(i, \pi) = \mathfrak{m}$ . Consider now the residue ring  $T = R/R\bar{i}$ , which is a local ring with the maximal prime ideal  $\mathfrak{m}/R\bar{i}$ ; since  $R\bar{i}$  does not contain any non-zero element in  $K_v$ , we can embed  $K_v$  canonically in  $T$  as a subring. It is clear that  $T$  and  $K_v$  have the same residue field  $K$ , and the relation  $\mathfrak{m}/R\bar{i} = R(i, \pi)/R\bar{i}$  shows that the maximal prime ideal of the so embedded ring  $K_v$  generates in  $T$  the maximal prime ideal of  $T$ . If  $K_v$  is complete, then it follows from a well-known result that

$T$  coincides with its subring  $K_v$ . Since  $T = K_v$  is an integral domain,  $R_i$  is a prime ideal, and hence we have  $c = 1$  and  $R_i = Rp_1$ ; since the quotient ring of  $K_v[x]$  with respect to the set  $K_v[x] - p_1$  coincides with the quotient ring of  $R$  with respect to the set  $R - Rp_1 = R - R_i$ , we can identify  $T$  with the residue ring  $K_v[x]/p_1$ , so that  $K_v[x]/p_1$  also coincides with  $K_v$ . There exists therefore a homomorphism  $H$  of  $K_v[x]$  onto  $K_v$  which leaves every element in  $K_v$  invariant and whose kernel is the ideal  $p_1$ ; if we set  $a_i = H(x_i)$ ,  $i = 1, \dots, m$ , then the point  $a = (a_1, \dots, a_m)$  in  $V$  is clearly the only zero of  $p_1$ , and since the point  $a$  has a unique image  $\bar{a}$  under  $E$  and has the point  $\alpha$  as a specialization over  $E$ , we must have the equality  $\bar{a} = \alpha$ . Thus the rational point  $a$  is the only point in  $W$  which can specialize over  $E$  to the point  $\alpha$ , and it can specialize over  $E$  only to  $\alpha$ . Finally, it is clear that  $K[x]p_1$  is the defining prime ideal of the point  $a$  in  $K[x]$ , and that the quotient ring of  $K[x]$  with respect to the set  $K[x] - K[x]p_1$  coincides with the quotient ring of  $K_v[x]$  with respect to the set  $K_v[x] - p_1$ , which in turn coincides with the quotient ring of  $R$  with respect to the set  $R - Rp_1$ ; it follows then from the relation  $R_i = Rp_1$  that the point  $a$  is a simple zero of  $i$ . This concludes the proof of our theorem.

As a corollary of the above theorem we obtain a result which can be considered as a generalization of the well-known Hensel's Lemma:<sup>6</sup>

**COROLLARY.** *Let  $K$  be a field which is complete with respect to a real discrete valuation  $v$ , let  $V$  be a variety in  $S^m$  defined over  $K$ , and let  $\bar{V}$  be its specialization over  $v$ ; if  $\alpha$  is a point in  $\bar{V}$  such that  $\alpha$  is rational over  $\bar{K}$  and  $\bar{V}$  is simple at  $\alpha$ , then there exists a point  $a$  in  $V$  such that  $a$  is rational over  $K_v$ ,  $V$  is simple at  $a$ , and  $\bar{a} = \alpha$ .*

In fact, if  $y_1, \dots, y_n$  are elements in  $R$  such that their images  $\bar{y}_1, \dots, \bar{y}_n$  under  $E_R$  form a basis for  $\bar{m}$ , then we can take for  $i$  the ideal in  $K_v[x]$  generated by the elements  $y_1, \dots, y_n$ , which has a rank at most  $n$ ; our corollary then follows immediately from Theorem 1.

**3.** Let  $\mathbf{x} = (x^{(1)}, \dots, x^{(d)})$  be a system of  $d$  points in  $W$ , which for the sake of convenience we shall assume to be distinct. A system of  $d$  (not necessarily distinct) points  $\boldsymbol{\beta} = (\beta^{(1)}, \dots, \beta^{(d)})$  in  $\bar{V}_\alpha$  is said to be a specialization of the system  $\mathbf{x}$  over  $E$  if the mapping  $x^{(i)} \rightarrow \beta^{(i)}$  defines an extension of  $E$  to a homomorphism  $E_0$  of the ring  $K_v[\mathbf{x}]$  onto the ring  $\bar{K}[\boldsymbol{\beta}]$ . The number of times the point  $\alpha$  appears in the system  $\boldsymbol{\beta}$  is then called the

<sup>6</sup> Such a generalization of Hensel's Lemma was conjectured by A. Weil some years ago; a special case of it, namely, for the Jacobian variety of a curve, has been recently proved by A. P. Mattuck in an unpublished manuscript.

multiplicity of  $\alpha$  in the specialization  $\beta$  of  $\alpha$  over  $E$ . It is clear that in order that this multiplicity can be positive, it is necessary that  $\alpha$  is not only a zero of the ideal  $\bar{i}$ , but also contained in the set  $\bar{W}$ ; as we have seen above, this condition is equivalent to the condition that the ideal  $Ri$  does not contain any element in  $K_v$ .

**THEOREM 2.** *If the point  $\alpha$  is a simple isolated zero of  $\bar{i}$ , then the multiplicity of  $\alpha$  in any specialization over  $E$  of any system of points in  $W$  is at most equal to 1.*

*Proof.* If  $K_v$  is complete, then this theorem follows immediately from Theorem 1; for then the point  $a$  is the only point in  $W$  which specializes over  $E$  to  $\alpha$ , and hence  $\alpha$  appears in  $\beta$  if and only if  $a$  appears in  $\alpha$  (observe that, since the points in  $\alpha$  are assumed to be distinct,  $a$  can only appear once in  $\alpha$ ). In order to prove our theorem it is sufficient to show that the homomorphism  $E_0$  can be extended to a homomorphism of  $K_v^*[\alpha]$  onto  $\bar{K}[\beta]$ , where  $K_v^*$  denotes the completion of  $K_v$ . Since  $\bar{K}[\beta]$  is an integral domain, the kernel  $r_0$  of  $E_0$  is a prime ideal, and  $E_0$  can be extended to a homomorphism  $E_1$  of the quotient ring  $N_1$  of  $K_v[\alpha]$  with respect to the set  $K_v[\alpha] - r_0$  onto the field  $\bar{K}(\beta)$ ; since the contraction of  $r_0$  in  $K_v$  is evidently the kernel  $K_v\pi$  of  $E$ , it follows that  $K_v = N_1 \cap K$ . Let  $N_1$  be embedded in a complete local domain  $N_2$  which dominates  $N_1$  (i.e.  $N_2$  contains  $N_1$  and its maximal prime ideal contract in  $N_1$  to the maximal prime ideal of  $N_1$ ) and whose residue field coincides with the residue field  $\bar{K}(\beta)$  of  $N_1$ ; for example, we can take  $N_2$  to be the residue ring of the completion of  $N_1$  modulo a minimal prime divisor of its zero ideal. The canonical homomorphism  $E_2$  of  $N_2$  onto its residue field  $\bar{K}(\beta)$  is then an extension of  $E_1$ . Since  $N_2$  contains  $K_v$  and is complete, we can embed the completion  $K_v^*$  of  $K_v$  in  $N_2$ , and  $E_2$  evidently induces a homomorphism of  $K_v^*[\alpha]$  onto  $\bar{K}[\beta]$  which is an extension of  $E_0$ .

We consider now the positive cycle  $\mathfrak{X}$  introduced in Section 1, and assume that the variety  $V$  contains  $\mathfrak{X}$  so that  $\mathfrak{P}$  contains  $\mathfrak{Q}$  and hence also  $\mathfrak{P}_\alpha$  contains  $\mathfrak{Q}_\alpha$ . Let  $\mathfrak{p}$  be the image ideal of  $\mathfrak{P}_\alpha$  under the homomorphism  $G$ , and let  $\bar{\mathfrak{p}}$  be the image of  $\bar{\mathfrak{P}}_\alpha$  under the homomorphism  $\bar{G}$ ; then both ideals  $\mathfrak{p}$  and  $\bar{\mathfrak{p}}$  will have ranks at most  $n - r$ , and  $\bar{\mathfrak{p}}$  is the image of  $\mathfrak{p}$  under the homomorphism  $E_R$ . Since  $\bar{V}_\alpha$  contains  $\bar{\mathfrak{X}}_\alpha$ , the tangential space of  $\bar{V}_\alpha$  at  $\alpha$  contains the tangential space of  $\mathfrak{X}_\alpha$  at  $\alpha$ ; by a suitable choice of the affine coordinate system, we can assume that the equations  $X_{n+1} = \cdots = X_m = 0$  defines the tangential space of  $\bar{V}_\alpha$  at  $\alpha$ . Then, as in the proof of Lemma 1, the ideal  $\bar{\mathfrak{S}}_\alpha$  has a basis  $\omega_{n+1}(X), \cdots, \omega_m(X)$ , in  $\bar{K}[X]$ , such that  $X_1, \cdots, X_n, \omega_{n+1}(X), \cdots, \omega_m(X)$  form a minimal basis for

$p(\bar{S}/\alpha)$ ; using the determinant criterion for the minimal basis, one sees readily that the basis  $\phi_{r+1}(X), \dots, \phi_m(X)$  of  $\mathfrak{P}_\alpha$  can be so chosen that  $\phi_i(X) = \omega_i(X)$  for  $i = n+1, \dots, m$ . Going over to the images in  $\bar{K}$ , under the homomorphism  $\bar{G}$  and observing that  $\bar{K}$  remains invariant under  $\bar{G}$ , this means that the ideal  $\bar{\mathfrak{p}}$  has a basis  $\phi_{r+1}(\bar{x}), \dots, \phi_n(\bar{x})$  which can be extended to a minimal basis  $\bar{x}_1, \dots, \bar{x}_r, \phi_{r+1}(\bar{x}), \dots, \phi_n(\bar{x})$  for  $\bar{\mathfrak{m}}$ . Let  $f_k(X)$ ,  $k = r+1, \dots, m$ , be elements in  $\mathfrak{P}_\alpha$  such that  $\bar{f}_k(X) = \phi_k(X)$  for every  $k$  and that  $f_{n+1}(X), \dots, f_m(X)$  are in  $\mathfrak{Q}_\alpha$ ; then  $f_{n+1}(x) = \dots = f_m(x) = 0$  and the elements  $f_{r+1}(x), \dots, f_n(x)$  are in  $\mathfrak{p}$ , with  $\phi_{r+1}(\bar{x}), \dots, \phi_n(\bar{x})$  as their respective images under  $E_R$ . Since the elements  $\pi, x_1, \dots, x_r, f_{r+1}(x), \dots, f_n(x)$  clearly generate the ideal  $\mathfrak{m}$ , they must form a minimal basis for  $\mathfrak{m}$ ; this shows that the elements  $f_{r+1}(x), \dots, f_n(x)$  generate a prime ideal of rank  $n-r$  in  $R$ , and since  $\mathfrak{p}$  has a rank at most  $n-r$ , this prime ideal must coincide with  $\mathfrak{p}$ . We observe also that since the elements  $f_{r+1}(X), \dots, f_m(X)$  can be extended to a minimal basis for  $p(S/\alpha)$ , by adding the elements  $\pi, X_1, \dots, X_r$ , they generate a prime ideal of rank  $m-r$  and hence must form a basis for  $\mathfrak{P}_\alpha$ .

Now, let  $\mathfrak{Y}^s$  ( $s = n-r$ ) be another positive cycle contained in the variety  $V$ , and assume that  $\mathfrak{Y}$  is rational over  $K$  and that the specialization  $\bar{\mathfrak{Y}}$  of  $\mathfrak{Y}$  over  $v$  is also simple at  $\alpha$ ; we denote by  $\mathfrak{Q}, \mathfrak{Q}_v, \mathfrak{Q}_\alpha$ , and  $\mathfrak{q}$  the ideals having the same meanings for the cycle  $\mathfrak{Y}$  as the ideals  $\mathfrak{P}, \mathfrak{P}_v, \mathfrak{P}_\alpha$ , and  $\mathfrak{p}$  respectively for the cycle  $\mathfrak{X}$ . Then there exists a basis  $\theta_{s+1}(X), \dots, \theta_m(X)$  in  $\bar{K}[X]$  for  $\bar{\mathfrak{Q}}_\alpha$  such that  $\theta_i(X) = \omega_i(X)$  for  $i = n+1, \dots, m$ , and such that  $\theta_{s+1}(\bar{x}), \dots, \theta_n(\bar{x})$  form a basis for  $\bar{\mathfrak{q}}$  which can be extended to a minimal basis for  $\bar{\mathfrak{m}}$ . The condition that  $\bar{\mathfrak{X}}$  and  $\bar{\mathfrak{Y}}$  are transversal to each other at  $\alpha$  in  $\bar{V}$  is equivalent to the condition that the elements  $\phi_{r+1}(\bar{x}), \dots, \phi_n(\bar{x}), \theta_{s+1}(\bar{x}), \dots, \theta_n(\bar{x})$  form a minimal basis for  $\bar{\mathfrak{m}}$ . If this is the case, and if we take elements  $g_{s+1}(X), \dots, g_n(X)$  in  $\mathfrak{Q}_\alpha$  such that  $\bar{g}_j(X) = \theta_j(X)$  for every  $j$ , then  $g_{s+1}(x), \dots, g_n(x)$  form a basis for  $\mathfrak{q}$ , and the elements  $\pi, f_{r+1}(x), \dots, f_n(x), g_{s+1}(x), \dots, g_n(x)$  form a minimal basis for  $\mathfrak{m}$ ; this shows that the ideal  $(\mathfrak{p}, \mathfrak{q})$  has the rank  $n$  (and is in fact a prime ideal). If we now apply Theorems 1 and 2 by taking for  $i$  the restriction of the ideal  $(\mathfrak{p}, \mathfrak{q})$  in  $K_v[x]$ , we obtain then the following theorem, which is the well-known criterion for unit multiplicity in a somewhat more precise version.

**THEOREM 3.** *Let  $V^n$  be a variety in  $S^n$ , defined over  $K$ , and let  $\mathfrak{X}^r$  and  $\mathfrak{Y}^s$  ( $r+s=n$ ) be two positive cycles in  $V$ , rational over  $K$ ; let  $v$  be a real discrete valuation of  $K$ , and let  $\bar{V}$ ,  $\bar{\mathfrak{X}}$ , and  $\bar{\mathfrak{Y}}$  be the specializations of  $V$ ,  $\mathfrak{X}$ , and  $\mathfrak{Y}$  respectively over  $v$ . If  $\alpha$  is a rational point over  $\bar{K}$  in  $\bar{V}$  such that  $\bar{V}$  is simple at  $\alpha$  and that  $\bar{\mathfrak{X}}$  and  $\bar{\mathfrak{Y}}$  are transversal to each other at  $\alpha$ ,*

then  $\alpha$  has the multiplicity 1 in the specialization over  $v$  of every system of points in  $|\mathfrak{X}| \cap |\mathfrak{Y}|$  which includes all the isolated points in  $|\mathfrak{X}| \cap |\mathfrak{Y}|$ . If furthermore  $K$  is complete with respect to  $v$ , then there is exactly one point  $a$  in  $|\mathfrak{X}| \cap |\mathfrak{Y}|$  which specializes to  $\alpha$  over  $v$ ; and this point  $a$  is rational over  $K_v$  (hence  $\bar{a} = \alpha$ ).  $V$  is simple at  $a$ , and  $\mathfrak{X}$  and  $\mathfrak{Y}$  are transversal to each other at  $a$ .

4. In the above treatment of the criterion for unit multiplicity, we have kept our arguments on as elementary a level as possible, in consonant with the foundational nature of our purpose. In fact, the greater part of our argument consists of a precise local formulation of the various geometric concepts involved in this criterion; the actual proof of the criterion itself is very short and is contained essentially in the proof of Theorem 1. If we now take a more sophisticated point of view and assume a knowledge of the intersection-theory, then the criterion for unit multiplicity, in the present version as an assertion on the specialization-multiplicity, is really only a special case of a more general theorem on the invariance of intersection-multiplicity under specialization, and there naturally arises the question whether, corresponding to Theorem 3, we can also prove a more precise version of this invariance theorem. In the following we shall show that this can be done, by using the Chevalley-Samuel-Nagata theory of multiplicity in a local ring. We shall need some of the results in the paper of Nagata cited before.<sup>5</sup>

As in Section 1, we consider the positive  $r$ -cycle  $\mathfrak{X}$  in  $S^n$ , except that we shall now no longer assume that  $\mathfrak{X}$  is simple at the point  $\alpha$ ; for the sake of simplicity, we shall assume that  $\mathfrak{X}$  is a prime rational cycle over  $K_v$  (so that  $\mathfrak{P}_v$  and hence  $\mathfrak{P}_\alpha$  are prime ideals). Let  $\Omega_0$  be a prime rational  $r$ -cycle over  $\bar{K}$  in  $\bar{S}^n$  which contains the point  $\alpha$  and whose support is contained in the support of  $\mathfrak{X}$ , and let  $\Phi_0(U)$  be the associated form of  $\Omega_0$ ; we set  $\tilde{\mathfrak{X}} = \Omega_1 + \Omega_2$ , where  $\Omega_1$  is a positive  $r$ -cycle with the same support as  $\Omega_0$  and  $\Omega_2$  is a positive cycle relatively prime to  $\Omega_0$ , and let  $\tilde{F}(U) = \Phi_1(U)\Phi_2(U)$  be the corresponding factorization of  $\tilde{F}(U)$ . Let  $d_0$  and  $d_1$  be the degrees of  $\Phi_0(U)$  and  $\Phi_1(U)$  respectively; the number  $d_1/d$ , which will presently be shown to be an integer, is the coefficient of  $\Omega_0$  as a component in  $\tilde{\mathfrak{X}}$  and will be denoted by  $\mu(\Omega_0, \tilde{\mathfrak{X}})$ . Let  $\bar{\mathfrak{M}}$  be the prime ideal in  $\mathfrak{Q}(\bar{S}/\alpha)$  associated with  $\Omega_0$ , and let  $\mathfrak{M} = E^{-1}(\bar{\mathfrak{M}})$  be the corresponding prime ideal in  $\mathfrak{Q}(S/\alpha)$ ; then  $\mathfrak{M}$  is a minimal prime divisor of the ideal  $(\pi, \mathfrak{P}_\alpha)$  in  $\mathfrak{Q}(S/\alpha)$ , and in this way all the minimal prime divisors of  $(\pi, \mathfrak{P}_\alpha)$  can be obtained.

LEMMA 2.  $\mu(\Omega_0, \tilde{\mathfrak{X}}) = e((\pi, \mathfrak{P}_\alpha)\mathfrak{Q}(S/\alpha)_{\mathfrak{M}}/\mathfrak{P}_\alpha\mathfrak{Q}(S/\alpha)_{\mathfrak{M}})$ .

*Proof.* Let  ${}_i w_j$ ,  $i = 1, \dots, r$ , and  $j = 0, 1, \dots, m$ , be  $r(m+1)$  independent variables over  $K_v$ ; we set  ${}_i w = ({}_i w_0, {}_i w_1, \dots, {}_i w_m)$  and denote by  $w$  the set of all  $r(m+1)$  variables  ${}_i w_j$ . We set  $F({}_0 U, w) = F({}_0 U, {}_1 w, \dots, {}_r w)$ , and similarly for other associated forms; we denote by  $K_{v,w}$  the quotient ring of  $K_v[w]$  with respect to the prime ideal generated in it by  $\pi$ , so that  $K_{v,w}$  is a real discrete valuation ring; and we observe that if we denote by  $\bar{w}$  the image of  $w$  in the residue field of  $K_{v,w}$ , then  $\bar{w}$  is a set of independent variable over  $\bar{K}$  and  $\bar{K}(\bar{w})$  is this residue field. We set  $L_i = {}_i w_0 + \sum_{j=1}^m {}_i w_j X_j$  and denote by  $\mathfrak{Q}(S/\alpha, K_{v,w})$  the local ring defined in the same way as  $\mathfrak{Q}(S/\alpha)$  but with  $K_{v,w}$  replacing  $K_v$ ; we set

$$M = \mathfrak{Q}(S/\alpha) \mathfrak{P}_\alpha, \quad \mathfrak{g} = \mathfrak{N}/\mathfrak{P}_\alpha, \quad N = \mathfrak{Q}(S/\alpha, K_{v,w})/(\mathfrak{P}_\alpha, L_1, \dots, L_r),$$

and

$$\mathfrak{h} = \mathfrak{N}\mathfrak{Q}(S/\alpha, K_{v,w})/(\mathfrak{P}_\alpha, L_1, \dots, L_r).$$

Then it is easily seen that  $N_{\mathfrak{h}}$  can be obtained from  $M_{\mathfrak{g}}$  by first adjoining the  $rm$  variables  ${}_i w_j$ ,  $j \neq 0$ , and then forming the quotient ring of the so obtained ring with respect to the ideal generated in it by  $\mathfrak{g}$ . It follows that  $e(\pi N_{\mathfrak{h}}) = e(\pi M_{\mathfrak{g}}) = e((\pi, \mathfrak{P}_\alpha) \mathfrak{Q}(S/\alpha)_{\mathfrak{P}_\alpha} / \mathfrak{P}_\alpha \mathfrak{Q}(S/\alpha)_{\mathfrak{P}_\alpha})$ . Since  $N_{\mathfrak{h}}^*$  is a finite module over  $K_{v,w}^*$ , it follows by a well-known argument<sup>7</sup> using the decomposition of the ideal  $\mathfrak{h}N_{\mathfrak{h}}^*$  that

$$e(\pi N_{\mathfrak{h}}) = e(\pi N_{\mathfrak{h}}^*) = l(N_{\mathfrak{h}}^* / \pi N_{\mathfrak{h}}^*) = [N_{\mathfrak{h}}^* : K_{v,w}^*] / [N_{\mathfrak{h}} / \mathfrak{h} : \bar{K}(\bar{w})].$$

It is easily verified that  $[N_{\mathfrak{h}} / \mathfrak{h} : \bar{K}(\bar{w})]$  is equal to the degree of the form  $\Phi_0({}_0 U, \bar{w})$  and hence is equal to  $d_0$ . In order to complete the proof of our lemma, it remains to show that  $[N_{\mathfrak{h}}^* : K_{v,w}^*] = d_1$ . This can be done by generalizing the usual argument in the ramification theory, first observing that<sup>8</sup> there exists a factorization  $F({}_0 U, w) = F_1({}_0 U)F_2({}_0 U)$  in  $K_{v,w}^*[{}_0 U]$  corresponding to the factorization  $\bar{F}({}_0 U, \bar{w}) = \Phi_1({}_0 U, \bar{w})\Phi_2({}_0 U, \bar{w})$ , and then showing that the number  $[N_{\mathfrak{h}}^* : K_{v,w}^*]$  is equal to the degree of the form  $F_1({}_0 U)$ , which is the same as the degree  $d_1$  of  $\Phi_1({}_0 U, \bar{w})$ .

Let  $A_i$  be the components in the positive cycle  $\tilde{\mathfrak{X}}$  which contain the point  $\alpha$ , let  $\tilde{\mathfrak{N}}_i$  be the prime ideals in  $\mathfrak{Q}(\bar{S}/\alpha)$  associated with  $A_i$ , and let  $\mathfrak{N}_i = E^{-1}(\tilde{\mathfrak{N}}_i)$  be the corresponding minimal prime divisor of the ideal  $(\pi, \mathfrak{P}_\alpha)$  in  $\mathfrak{Q}(S/\alpha)$ . If we denote by  $\alpha_i$  the image ideals of  $\mathfrak{N}_i$  in  $R$  under the homomorphism  $G$ , then it is easily seen that the ideals  $\alpha_i$  are the

<sup>7</sup> See e.g. C. Chevalley, "On the Theory of Local Rings," *Annals of Mathematics*, vol. 44 (1943), pp. 690-708, § IV, Lemma 2.

<sup>8</sup> See Lemma 1 in our recent paper "On the Principle of Degeneration in Algebraic Geometry," *Annals of Mathematics*, vol. 66 (1957), pp. 70-79.

minimal prime divisors of the ideal  $(\pi, \mathfrak{p})R$ , and we have the equality  $e((\pi, \mathfrak{p})R_{\alpha_i}/\mathfrak{p}R_{\alpha_i}) = e((\pi, \mathfrak{P}_\alpha)\mathfrak{Q}(S/\alpha)_{\mathfrak{M}_i}/\mathfrak{P}_\alpha\mathfrak{Q}(S/\alpha)_{\mathfrak{M}_i})$ . It follows then from Lemma 2 that  $\mu(A_i, \tilde{\mathfrak{X}}) = e((\pi, \mathfrak{p})R_{\alpha_i}/\mathfrak{p}R_{\alpha_i})$ . For each  $i$ , let  $\alpha_{ij}$  be the minimal prime divisors of  $\alpha_i R^*$ ; since  $\alpha_i$  is analytically unramified (for  $R/\alpha_i$  is a "geometrical" local ring), we have the equality

$$e((\pi, \mathfrak{p})R^*_{\alpha_{ij}}/\mathfrak{p}R^*_{\alpha_{ij}}) = e((\pi, \mathfrak{p})R_{\alpha_i}/\mathfrak{p}R_{\alpha_i}).$$

We consider now, as in Section 3, two positive cycles  $\mathfrak{X}^r$  and  $\mathfrak{Y}^s$  in a variety  $V^n$  ( $n=r+s$ ), and we shall use the same notation as developed there; while  $\bar{V}$  is still assumed to be a variety which is simple at the point  $\alpha$ , we shall assume no longer that  $\tilde{\mathfrak{X}}$  or  $\tilde{\mathfrak{Y}}$  is simple at  $\alpha$ , nor that they are transversal to each other. Instead, we shall only assume that  $\tilde{\mathfrak{X}}$  and  $\tilde{\mathfrak{Y}}$  intersect properly at  $\alpha$ , which is equivalent to assuming that the ideal  $(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}})$  is primary for  $\tilde{\mathfrak{m}}$ ; this latter assertion follows from the fact that the radical of the ideal  $\tilde{\mathfrak{p}}$  (or  $\tilde{\mathfrak{q}}$ ) is the ideal associated with the support of  $\tilde{\mathfrak{X}}$  (or  $\tilde{\mathfrak{Y}}$ ), and this fact can be easily proved by a method similar to that used in the proof of Lemma 1. Since  $(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}})$  has the rank  $n$ , the ideal  $(\mathfrak{p}, \mathfrak{q})$  must have at least the same rank  $n$ , and since  $R$  is an unramified regular local ring, the rank of  $(\mathfrak{p}, \mathfrak{q})$  must be equal to  $n$ . Let  $e_i$  be the minimal prime divisors of the ideal  $(\mathfrak{p}, \mathfrak{q})R^*$ . Consider the residue ring  $T_i = R^*/e_i$ ; it is clear that  $K_v^*$  can be embedded canonically in  $T_i$  as a subring. In fact,  $T_i$  is a finite module over  $K_v^*$ , and we have the equality  $[T_i:K_v^*] = e((\pi, e_i)R^*/e_i)$ . If we denote by  $H_i$  the canonical homomorphism of  $R^*$  onto  $R^*/e_i$  and set  $a^{(i)}_j = H_i(x_j)$ , then it can be shown exactly as in Section 3 that the point  $a^{(i)} = (a^{(i)}_1, \dots, a^{(i)}_m)$  is an intersection point of  $\mathfrak{X}$  and  $\mathfrak{Y}$ ; and if we denote by  $\mathfrak{Z}_i$  the 0-cycle consisting of the complete set of conjugates of the point  $a^{(i)}$  over  $K_v^*$ , then  $\mathfrak{Z}_i$  is evidently rational over  $K_v^*$  and its specialization  $\bar{\mathfrak{Z}}_i$  is the cycle consisting of the point  $\alpha$  with the coefficient  $e((\pi, e_i)R^*/e_i)$ . We set  $\mathfrak{Z} = \sum_i i(\mathfrak{Z}_i; \mathfrak{X} \cdot \mathfrak{Y}) \cdot \mathfrak{Z}_i$ ; then a point  $a$  in  $|\mathfrak{X}| \cap |\mathfrak{Y}|$  will specialize to  $\alpha$  over the specialization  $(\mathfrak{X}, \mathfrak{Y}) \rightarrow (\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}})$  if and only if the coefficient  $\mu(a, \mathfrak{Z})$  of  $a$  in  $\mathfrak{Z}$  is positive. We shall now show that  $\deg \mathfrak{Z} = i(\alpha; \tilde{\mathfrak{X}} \cdot \tilde{\mathfrak{Y}})$ .

Let  $R'$  be another copy of the local ring  $R$ , but with the same basic ring  $K_v$ , and let a prime be used to denote generally the corresponding entities in  $R'$ ; we consider the complete tensor product  $J$  of  $R$  and  $R'$  (or of  $R^*$  and  $R'^*$ ) over  $K_v^*$  and denote by  $\mathfrak{d}$  the ideal in  $J$  generated by the  $n$  elements  $x_1 - x'_1, \dots, x_n - x'_n$ . It is well-known that the prime ideals  $(\mathfrak{d}, e_i)J$  are precisely the minimal prime divisors of the ideal  $(\mathfrak{d}, \mathfrak{p}, \mathfrak{q}')J$ , and hence the ideals  $(\mathfrak{d}, e_i)J/(\mathfrak{p}, \mathfrak{q}')J$  are the minimal prime divisors of the ideal  $(\mathfrak{d}, \mathfrak{p}, \mathfrak{q}')J/(\mathfrak{p}, \mathfrak{q}')J$  in the local ring  $J/(\mathfrak{p}, \mathfrak{q}')J$ . Applying the Associativity Formula, we obtain the equation

$$\begin{aligned}
 & e((\mathfrak{d}, \pi, \mathfrak{p}, \mathfrak{q}')J/(\mathfrak{p}, \mathfrak{q}')J) \\
 (1) \quad &= \sum_i e((\mathfrak{d}, \mathfrak{p}, \mathfrak{q}')J_{(\mathfrak{d}, \mathfrak{e}_i)}/(\mathfrak{p}, \mathfrak{q}')J_{(\mathfrak{d}, \mathfrak{e}_i)}) \cdot e((\mathfrak{d}, \pi, \mathfrak{e}_i)J/(\mathfrak{d}, \mathfrak{e}_i)J) \\
 &= \sum_i e((\mathfrak{d}, \mathfrak{p}, \mathfrak{q}')J_{(\mathfrak{d}, \mathfrak{e}_i)}/(\mathfrak{p}, \mathfrak{q}')J_{(\mathfrak{d}, \mathfrak{e}_i)}) \cdot e((\pi, \mathfrak{e}_i)R^*/\mathfrak{e}_i R^*) \\
 &= \sum_i i(\mathfrak{Z}_i; \mathfrak{X} \cdot \mathfrak{Y}) \cdot \deg \mathfrak{Z}_i = \deg \mathfrak{Z}.
 \end{aligned}$$

Let  $B_k$  be the components in  $\bar{V}$  which contain the point  $\alpha$ , and let  $\mathfrak{b}_k$  and  $\mathfrak{b}_{kl}$  have the same meanings with respect to  $B_k$  as  $\mathfrak{a}_i$  and  $\mathfrak{a}_{ij}$  have with respect to  $A_i$ ; then it can be easily seen that the ideals  $\mathfrak{c}_{ij,kl} = (\mathfrak{a}_{ij}, \mathfrak{b}_{kl}')J$  are the minimal prime divisors of the ideal  $(\pi, \mathfrak{p}, \mathfrak{q}')J$ , and we have the relation

$$\begin{aligned}
 e((\pi, \mathfrak{p}, \mathfrak{q}')J_{\mathfrak{c}_{ij,kl}}/(\mathfrak{p}, \mathfrak{q}')J_{\mathfrak{c}_{ij,kl}}) &= e((\pi, \mathfrak{p})R_{\mathfrak{a}_{ij}}^*/\mathfrak{p}R_{\mathfrak{a}_{ij}}^*) \cdot e((\pi, \mathfrak{q})R_{\mathfrak{b}_{kl}}^*/\mathfrak{q}R_{\mathfrak{b}_{kl}}^*) \\
 &= e((\pi, \mathfrak{p})R_{\mathfrak{a}_i}/\mathfrak{p}R_{\mathfrak{a}_i}) \cdot e((\pi, \mathfrak{q})R_{\mathfrak{b}_k}/\mathfrak{q}R_{\mathfrak{b}_k}).
 \end{aligned}$$

Applying again the Associativity Formula, with the ideal  $\mathfrak{d}$  and  $\pi J$  interchanged, we obtain the equation

$$\begin{aligned}
 & e((\mathfrak{d}, \pi, \mathfrak{p}, \mathfrak{q}')J/(\mathfrak{p}, \mathfrak{q}')J) \\
 (2) \quad &= \sum_{i,j,k,l} e((\pi, \mathfrak{p}, \mathfrak{q}')J_{\mathfrak{c}_{ij,kl}}/(\mathfrak{p}, \mathfrak{q}')J_{\mathfrak{c}_{ij,kl}}) \cdot e((\mathfrak{d}, \mathfrak{c}_{ij,kl})J/\mathfrak{c}_{ij,kl}) \\
 &= \sum_{i,k} e((\pi, \mathfrak{p})R_{\mathfrak{a}_i}/\mathfrak{p}R_{\mathfrak{a}_i}) \cdot e((\pi, \mathfrak{q})R_{\mathfrak{b}_k}/\mathfrak{q}R_{\mathfrak{b}_k}) \cdot \left( \sum_{j,l} e((\mathfrak{d}, \mathfrak{c}_{ij,kl})J/\mathfrak{c}_{ij,kl}) \right) \\
 &= \sum_{i,k} \mu(A_i, \bar{\mathfrak{X}}) \cdot \mu(B_k, \bar{\mathfrak{Y}}) \cdot i(\alpha; A_i \cdot B_k) = i(\alpha; \bar{\mathfrak{X}} \cdot \bar{\mathfrak{Y}}).
 \end{aligned}$$

Combining the equations (1) and (2), we obtain the equation  $\deg \mathfrak{Z} = i(\alpha; \bar{\mathfrak{X}} \cdot \bar{\mathfrak{Y}})$ , which proves our assertion.

We can summarize our results in the following theorem:

**THEOREM 4.** *Let  $V^n$  be a variety in  $S^m$ , defined over a field  $K$  which is complete with respect to a real discrete valuation  $v$ , and let  $\mathfrak{X}^r$  and  $\mathfrak{Y}^s$  ( $r+s=n$ ) be two positive cycles in  $V$ , rational over  $K$ ; let  $\bar{V}$ ,  $\bar{\mathfrak{X}}$ , and  $\bar{\mathfrak{Y}}$  be the specializations of  $V$ ,  $\mathfrak{X}$ , and  $\mathfrak{Y}$  respectively over  $v$ . If  $\alpha$  is a rational point over  $\bar{K}$  in  $\bar{V}$  such that  $\bar{V}$  is simple at  $\alpha$  and that  $\bar{\mathfrak{X}}$  and  $\bar{\mathfrak{Y}}$  intersect properly at  $\alpha$ , then there exists exactly one positive 0-cycle  $\mathfrak{Z}$  in  $V$ , with support in  $|\mathfrak{X}| \cap |\mathfrak{Y}|$ , such that  $\mathfrak{Z}$  is rational over  $K$  and its specialization  $\bar{\mathfrak{Z}}$  over  $v$  consists of the point  $\alpha$  with the coefficient  $i(\alpha; \bar{\mathfrak{X}} \cdot \bar{\mathfrak{Y}})$ ; furthermore, a point  $a$  in  $|\mathfrak{X}| \cap |\mathfrak{Y}|$  specializes to  $\alpha$  over the specialization  $(\mathfrak{X}, \mathfrak{Y}) \rightarrow (\bar{\mathfrak{X}}, \bar{\mathfrak{Y}})$  if and only if  $\mu(\alpha, \mathfrak{Z}) > 0$ , and when such is the case, we have  $i(a; \mathfrak{X} \cdot \mathfrak{Y}) = \mu(a, \mathfrak{Z})$ .*

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## ON THE PREPARATION OF MANUSCRIPTS.

The following instructions are suggested or dictated by the necessities of the technical production of the *American Journal of Mathematics*. Authors are urged to comply with these instructions, which have been prepared in their interests.

Manuscripts not complying with the standards usually have to be returned to the authors for typographic explanation or revisions and the resulting delay often necessitates the deferment of the publication of the paper to a later issue of the *Journal*.

Horizontal fraction signs should be avoided. Instead of them use either solidus signs / or negative exponents.

Neither a solidus nor a negative exponent is needed in the symbols  $\frac{1}{2}$ ,  $\frac{1}{\pi}$ ,  $\frac{1}{2\pi}$ ,  $\frac{1}{2\pi i}$ , which are available in regular size type.

Binomial coefficients should be denoted by  $C_k^n$  and not by parentheses. Correspondingly, for symbols of the type of a quadratic residue character the use of some non-vertical arrangement is usually imperative.

For square roots use either the exponent  $\frac{1}{2}$  or the sign  $\sqrt{\quad}$  without the top line, as in  $\sqrt{-1}$  or  $\sqrt{a+b}$ .

Replace  $e^{(\quad)}$  by  $\exp(\quad)$  if the expression in the parenthesis is complicated.

By an appropriate choice of notations, avoid unnecessary displays.

Simple formulae, such as  $A + iB = \frac{1}{2}C^*$  or  $s_n = a_1 + \dots + a_n$ , should not be displayed (unless they need a formula number).

Use ' or  $d/dx$ , possibly  $D$ , but preferably not a dot, in order to denote ordinary differentiation and, as far as possible, a subscript in order to denote partial differentiation (when the symbol  $\partial$  cannot be avoided, it should be used as  $\partial/\partial x$ ).

Commas between indices are usually superfluous and should be avoided if possible.

In a determinant use a notation which reduces it to the form  $a_{ik}$ .

Subscripts and superscripts cannot be printed in the same vertical column, hence the manuscript should be clear on whether  $a_i^*$  or  $a^{*i}$  is preferred. (Correspondingly, the limits of summation must not be typed after the  $\Sigma$ -sign, unless either  $\Sigma_i^m$  or  $\Sigma^m_i$  is desired.) If a letter carrying a subscript has a prime, indicate whether  $a_i'$  or  $a'_i$  is desired.

Experience shows that a tilde or anything else over a letter is very unsatisfactory. Such symbols often drop out of the type after proof-reading and, when they do not, they usually appear uneven in print. For these reasons we advise against their use. This advice applies also to a bar over a Greek or German letter (for the symbol of complex conjugation an asterisk is often allowed by the context). Type carrying bars over ordinary size italic letters of the Latin alphabet is available.

Bars reaching over several letters should in any case be avoided (in particular, type  $\limsup$  and  $\liminf$  instead of  $\lim$  with upper and lower bars).

Repeated subscripts and superscripts should be used only when they cannot be avoided, since the index of the principal index usually appears about as large as the principal index. Bars and other devices over indices cannot be supplied. On the other hand, an asterisk or a prime (to be printed after the subscript) is possible on a subscript. The same holds true for superscripts.

Distinguish carefully between l. c. "oh," cap. "oh" and zero. One way of distinguishing them is by underlining one or two of them in different colors and explaining the meaning of the colors.

Distinguish between  $\epsilon$  (epsilon) and  $\epsilon$  or  $\varepsilon$  (symbol), between  $\omega$  (eks) and  $\times$  (multiplication sign), between l. c. and cap. phi, between l. c. and cap. psi, between l. c.  $k$  and kappa and between "ell" and "one" (for the latter, use  $l$  and  $l$  respectively).

Avoid unnecessary footnotes. For instance, references can be incorporated into the text (parenthetically, when necessary) by quoting the number in the bibliographic list, which appear at the end of the paper. Thus: "[3], pp. 261-266."

Except when informality in referring to papers or books is called for by the context, the following form is preferred:

[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Société Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

In any case, the references should be precise, unambiguous and intelligible.

Usually sections numbers and section titles are printed in bold face, the titles "Theorem," "Lemma" and "Corollary" are in caps and small caps, "Proof," "Remark" and "Definition" are in italics. This (or a corresponding preference) should be marked in the manuscript. Use a period, and not a colon, after the titles Theorem, Lemma, etc.

German, script and bold face letters should be underlined in various colors and the meaning of the colors should be explained. The same device is needed for Greek letters if there is a chance of ambiguity. In general, mark all cap. Greek letters.

All instructions and explanations for the printer can conveniently be collected on a separate sheet, to be attached to the manuscript.

In case of doubt, recent issues of the *Journal* may be consulted.

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